HW 2 Solutions (mostly adapted from Abbott’s Instructor’s Manual)

1.3.8. Let $\epsilon = \sup B - \sup A > 0$. Then there exists an element $b \in B$ satisfying $\sup B - \epsilon < b$, which implies $\sup A < b$. Further, for every $a \in A$, $a \leq \sup A \leq b$ so $b$ is an upper bound for $A$.

1.4.3. We have to show the existence of an irrational number between any two real numbers $a$ and $b$ with $a < b$. Apply Theorem 1.4.3 for the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$ to find a rational number $r$ satisfying $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$ and $r + \sqrt{2}$ is irrational by Exercise 1.4.2(b).

1.4.4. Clearly, 0 is a lower bound for $A = \{1/n : n \in \mathbb{N}\}$, as $0 < 1/n$ for all $n \in \mathbb{N}$. To show that 0 is the greatest lower bound of $A$, we need to show that any $\epsilon > 0$ is not a lower bound for $A$. This follows from the fact that $1/n < \epsilon$ for some $n \in \mathbb{N}$.

2.2.1. (b) Let $a_n = \frac{3n+1}{2n+3}$. Pick an $\epsilon > 0$. We must produce an $N \in \mathbb{N}$ so that $n \geq N$ implies $|a_n - \frac{3}{2}| < \epsilon$. After a bit of algebra,

$$|a_n - \frac{3}{2}| = \frac{13}{4n+10}.$$ 

So $|a_n - \frac{3}{2}| < \epsilon$ as soon as $n > (13/4)\epsilon^{-1} - 5/2$. One can take, say, $N$ to be the smaller integer larger that $4\epsilon^{-1}$.

2.2.5. (b) The sequence $a_n$ satisfies $a_n = 0$ for $n \geq 11$, and so the limit is 0. For any $\epsilon > 0$ we can take $N = 11$ to guarantee $|a_n - 0| < \epsilon$ for $n \geq N$. (Here, $N$ is independent of $\epsilon$, which is almost never true!)

2.2.8. (a) Frequently yes. (In fact, frequently means that $a_n \in A$ for infinitely many $n$. This is true for all even $n$ in this case.) Eventually no. (In fact, eventually means that $a_n \in A$ for all but finitely many $n$. This is true in this case, as $a_n \notin A$ for infinitely many $n$, that is, for all odd $n$.) (b) Eventually implies frequently but not the reverse. See the above discussion for (a).