HW 4 Solutions (mostly adapted from Abbott’s Instructor’s Manual)

2.4.2. First proof. First check with a bit of algebra that for any real number \( x < 4 \),
\[
x \geq 2 - \sqrt{3} \iff \frac{1}{4 - x} \geq 2 - \sqrt{3} \quad \text{and} \quad x \leq 2 + \sqrt{3} \iff \frac{1}{4 - x} \leq 2 + \sqrt{3}.
\]
It follows by induction that for every \( n \), \( 2 - \sqrt{3} < x_n \leq 2 + \sqrt{3} \). Next, we check that \( x_{n+1} \leq x_n \), which follows from
\[
x_{n+1} \leq x_n \iff 1 + x_n(x_n - 4) \leq 0 \iff (x_n - 2)^2 \leq 3
\]
and the last inequality says exactly that \(|x_n - 2| \leq \sqrt{3}\), which we have already proved. Thus \( x_n \) is a decreasing bounded sequence, hence convergent; denote the limit by \( x \). As \( x = 1/(4 - x) \) and \( x \leq 3 \), \( x = 2 - \sqrt{3} \).

Second proof. Another way (a bit more straightforward) is to begin by showing that \( x_{n+1} \leq x_n \) for all \( n \in \mathbb{N} \), using induction. This is true for \( n = 1 \), as \( x_1 = 3, x_2 = 1 \). The \((n \to n+1)\) step follows by this chain of implications:
\[
x_{n+1} \leq x_n \implies 4 - x_{n+1} \geq 4 - x_n \implies \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n} \implies x_{n+2} \leq x_{n+1}.
\]
So we know that \( x_n \) decreases, and so \( x_n \leq 3 \), and so \( x_{n+1} = 1/(4 - x_n) > 0 \). So the sequence is bounded and decreasing and therefore has a limit \( x \), which satisfies \( x = 1/(4 - x) \) and \( x \leq 3 \), so \( x = 2 - \sqrt{3} \).

2.4.4. This is the sequence defined by \( x_1 = \sqrt{2} \) and \( x_{n+1} = \sqrt{2}x_n \). If \( 0 < x_n < 2 \), then \( 0 < x_{n+1} < 2 \), thus \( 0 < x_n < 2 \) for all \( n \), the sequence is bounded. Then \( 2x_n - x_n^2 = x_n(2 - x_n) > 0 \), which implies that \( x_n < \sqrt{2x_n} = x_{n+1} \), the sequence is increasing. Thus the limit \( x = \lim x_n \) exists, and, as \( x_{n+1} = 2x_n \), satisfies \( x^2 = 2x \). As \( x \neq 0 \), \( x = 2 \).

2.4.5. Observe that \( x_n > 0 \) for all \( n \) (by simple induction). Also, we claim that for any \( x > 0 \), \( x + 2/x \geq 2\sqrt{2} \). Since we cannot use calculus yet, we use algebra to transform the inequality to \( x^2 - 2\sqrt{2}x + 2 \geq 0 \), which is true as \( x^2 - 2\sqrt{2}x + 2 = (x - \sqrt{2})^2 \). Thus \( x_n \geq \sqrt{2} \) for every \( n \) and
\[
x_n - x_{n+1} = \frac{1}{2} \left( x_n - \frac{2}{x_n} \right) = \frac{x_n^2 - 2}{2x_n} \geq 0
\]
and the sequence decreases. The equation \( x = \frac{1}{2}(x + 2/x) \) has the only positive solution \( x = \sqrt{2} \).

Therefore \( \lim x_n = \sqrt{2} \).

The sequence defined by \( x_1 = c \), and \( x_n = \frac{1}{2} (x_n + c/x_n) \) converges to \( \sqrt{c} \), by a very similar proof.

2.4.6. This (and more) was done in discussion.

2.5.3. (a) Let \( a_n = 1/(n+1) \) for odd \( n \) and \( a_n = 1 - 1/n \) for even \( n \).
(b) Impossible. Assume that an increasing sequence \((a_n)\) has a subsequence \((a_{n_k})\) that is bounded above by a real number \( M \). Then for every \( n \) there exists a \( k \) so that \( n \leq n_k \) and so \( a_n \leq a_{n_k} \leq M \),
so that \((a_n)\) is bounded as well.

(c) Let \(p_1, p_2, \ldots\) be the sequence of primes. Let \(a_{p_i^k} = 1/i\) for every \(k\), that is at each power of the \(i^{\text{th}}\) prime the element is \(1/i\). Set the rest of terms of \((a_n)\) to be 0. (We use prime powers because they are never equal to each other and so ambiguity does not arise.)

(d) Let \(a_n = 0\) for odd \(n\) and \(a_n = n\) for even \(n\).

(e) Impossible. The bounded subsequence must have a convergent subsequences (a sub-subsequence, so to speak), which is also a subsequence of the original sequence.

2.5.4. One way to do this is to use Problem 2 on Discussion 4, but a direct argument is perhaps easier. Assume \((a_n)\) does not converge to \(a\). Then there exists an \(\epsilon > 0\) so that for every \(N \in \mathbb{N}\) there exists an \(n \geq N\) so that \(|a_n - a| \geq \epsilon\). This is equivalent to saying that there exists a subsequence \((a_{n_k})\) so that \(|a_{n_k} - a| \geq \epsilon\). As \((a_n)\) is bounded, so is \((a_{n_k})\), and so it has a convergent subsequence \((a_{n_{k\ell}})\). This is also a subsequence of \((a_n)\) and so \(\lim \ell a_{n_{k\ell}} = a\). One the other hand, \(|a_{n_{k\ell}} - a| \geq \epsilon\) for every \(\ell\). This is a contradiction.

2.5.5. Let \(a_n = b^n\). We need to check that \(|a_n| \to 0\). However, \(|a_n| = |b|^n\) and \(0 < |b| < 1\), so \(|a_n| \to 0\) by what we proved in class.

2.6.1. (a) Let \(a_n = (-1)^n/n\) (which converges to 0 and is therefore Cauchy, but is not monotone).
(b) Let \(a_n = n\) (which is unbounded and is therefore not Cauchy, but is monotone).
(c) Impossible. A Cauchy sequence converges and therefore so does every one of its subsequences.
(d) Let \(a_n = 0\) for even \(n\) and \(a_n = 1\) for odd \(n\) (the even terms form a constant, hence convergent, hence Cauchy, subsequence).

2.6.2. (a) The Cauchy criterion requires that \(|s_m - s_n| < \epsilon\) for all \(m, n \geq N\) not just when \(m\) and \(n\) differ by 1. In fact, pseudo-Cauchy requirement is that \(\lim_n (s_{n+1} - s_n) = 0\) while the Cauchy requirement is that \(\lim_n \sup_{m \geq n} |s_m - s_n| = 0\).
(b) Take \(s_n = 1 + 1/2 + 1/3 + \ldots + 1/n\). Then \(s_{n+1} - s_n = 1/(n+1) \to 0\), but \(s_n\) diverges as the harmonic series diverges.