Math 25, Fall 2014.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, first name first): ____________________________

NAME(sign): ______________________________________________________________________

ID#: ____________________________

Instructions: Each of the $\frac{4}{5}$ problems has equal worth. Read each question carefully and answer it in the space provided. You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1

2

3

4

TOTAL
1. (a) Assume \((a_n)\) is a sequence of real numbers, and \(a \in \mathbb{R}\). Define precisely what this statement means: \(\lim_{n \to \infty} a_n = a\).

\[
(\forall \varepsilon > 0) \ (\exists N \in \mathbb{N}) \ (\forall n \geq N) \ (|a_n - a| < \varepsilon)
\]

(b) Assume \((a_n)\) is a sequence of real numbers. Define precisely what this statement means: \((a_n)\) is bounded.

\[
(\exists M \geq 0) \ (\forall n \in \mathbb{N}) \ (|a_n| \leq M)
\]

(c) Let \(a_n = \frac{2\sqrt{n} + 1}{2\sqrt{n} - 3}\). Using only the definition in (a), prove that \(\lim_{n \to \infty} a_n = a\) for some \(a \in \mathbb{R}\.

(Identify a first.)

\[a = 1.\]

Fix an \(\varepsilon > 0\). We need to find an \(N\) so that \(n \geq N\) will guarantee that \(|a_n - 1| < \varepsilon\).

\[
\left| \frac{2\sqrt{n} + 1}{2\sqrt{n} - 3} - 1 \right| < \varepsilon
\]

\[
\left| \frac{4}{2\sqrt{n} - 3} \right| < \varepsilon
\]

\[n \geq 4 \Rightarrow \frac{4}{2\sqrt{n} - 3} < \varepsilon\]

\[
\frac{2\sqrt{n} - 3}{4} < \varepsilon \Rightarrow 2\sqrt{n} > \frac{4}{\varepsilon} - 3 \Rightarrow n > \left(\frac{4}{\varepsilon} - 3\right)^2
\]

(d) Is the sequence \((a_n)\) from (c) bounded?

Yes, every convergent sequence is bounded.

Take \(N = \left(\frac{4}{\varepsilon} - 3\right)^2 + 1\).
2. (a) Recall that \( n! = 1 \cdot 2 \cdots n \) is the product of first \( n \) natural numbers. Prove by induction that for all \( n \in \mathbb{N}, n \geq 4 \) implies that \( 2^n < 3(n-1)! \).

\[
(n=4) \quad 2^4 < 3 \cdot 3! \quad , \quad 16 < 18 \ \checkmark
\]

\( (n \rightarrow n+1) \) Assume \( 2^n < 3(n-1)! \) and \( n \geq 4 \). Then

\[
2^{n+1} = 2 \cdot 2^n < 2 \cdot 3(n-1)! \quad \leq \quad n \cdot 3(n-1)! = 3n!
\]

So
\[
2^{n+1} < 3 \left( (n+1) - 1 \right)!
\]

(b) Use (a), or any other method, to prove that \( \lim_{n \to \infty} \frac{2^n}{n!} = 0 \).

By (a):

\[
0 < \frac{2^n}{n!} < \frac{3 \cdot (n-1)!}{n!} = \frac{3}{n}
\]

As \( \frac{3}{n} \to 0 \) as \( n \to \infty \), the same must be true for \( \frac{2^n}{n!} \) (by order theorems)

3
3. Find 
\[ \bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right). \]

Carefully prove your assertion.

This intersection equals \([1,2]\).

Clearly \(x \in [1,2]\) implies (that \(x \in \left( -\frac{1}{n}, 2 + \frac{1}{n} \right)\)

for some \(N\), so that \([1,2] \subseteq \bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)\). We now show that \(x \notin [1,2]\) implies
\[ x \notin \bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right), \]

we look at two cases:

- If \(x > 2\), \(\frac{1}{n} < x - 2\) if \(n\) is large enough, and so \(2 + \frac{1}{n} < x\) for large enough \(n\), \(x \notin \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)\).

- If \(x < 1\), \(\frac{1}{n} < 1 - x\) if \(n\) is large enough, and so \(x < 1 - \frac{1}{n}\), and \(x \notin \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)\).

In either case, there exists an \(n\) so that \(x \notin \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)\), so \(x \notin \bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)\).

This proves that \(\bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right) \subseteq [1,2]\).

The two inclusions show that
\[ \bigcap_{n=1}^{\infty} \left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right) = [1,2], \]
4. (a) State the definition of sup $A$ for a set $A \subseteq \mathbb{R}$.

\[ b = \sup A \text{ if } \]
\[ \text{(i) } b \text{ is an upper bound for } A, \text{ i.e., } a \leq b \text{ for every } a \in A; \text{ and } \]
\[ \text{(ii) } \text{for every upper bound } c \text{ for } A, \ b \leq c. \]

(b) What do you need to assume about a set $A \subseteq \mathbb{R}$ to ensure that sup $A$ exists?

$A$ has to be bounded above (exists $b \in \mathbb{R}$ so that $a \leq b$ for every $a \in A$), and $A \neq \emptyset$.

(c) Let $A = (0, 2)$ and $B = A \cap \mathbb{Q}$. Find sup $A$ and sup $B$. Carefully prove your assertions.

\[ \text{sup } A = \text{sup } B = 2. \]

Clearly, sup $A \leq 2$ and sup $B \leq 2$.

We claim that for $\varepsilon > 0$, \exists \, r \in B$ so that $r > 2 - \varepsilon$. This is true as there exists an $r \in \mathbb{Q}$ in the interval $(2 - \varepsilon, 2)$ by density of rational numbers. This implies that sup $B = 2$ and that sup $A = 2$ (as such $r$ is also in $A$).

(d) Assume that $A \subseteq \mathbb{R}$. Assume also that $A$ is bounded above and that $A \cap \mathbb{Q}$ is nonempty. Is it necessarily true that sup $A =$ sup($A \cap \mathbb{Q}$)?

No. Take $A = (0, 2) \cup \{1 + \sqrt{2}, 3\}$. Then $A \cap \mathbb{Q} = (0, 2) \cap \mathbb{Q} \neq \emptyset$, $1 + \sqrt{2} \notin \mathbb{Q}$.

\[ \text{Sup } A = \max A = 1 + \sqrt{2} \]

\[ \text{sup } (A \cap \mathbb{Q}) = 2. \]