MIDTERM EXAM 2

NAME(print in CAPITAL letters, first name first): ________________________________

NAME(sign): _____________________________________________

ID#: __________________________________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1
2
3
4
TOTAL
1. Assume $x_1 = 1$ and 

$$x_{n+1} = \sqrt{6 + x_n} \quad \text{for } n \in \mathbb{N}.$$ 

Show that the sequence is increasing, and that $\lim_{n \to \infty} x_n$ exists. Compute the limit.

Prove by induction that $x_n < x_{n+1}$ for every $n \in \mathbb{N}$.

- $(n=1)$: $x_2 = \sqrt{7} > 1 = x_1$.

- $(n \to n+1)$: By the induction hypothesis, $x_n < x_{n+1}$.

Then $6 + x_n < 6 + x_{n+1}$ and $\sqrt{6 + x_n} < \sqrt{6 + x_{n+1}}$, i.e., $x_{n+1} < x_{n+2}$.

Thus, $(x_n)$ is increasing. We need to prove that it is bounded.

Claim: $x_n \leq 3$ for every $n$.

- $(n=1)$: True $x_1 = 1 \leq 3$.

- $(n \to n+1)$: If $x_n \leq 3$, then $x_{n+1} \leq \sqrt{6 + 3} = 3$.

The claim is proved.

As the sequence is monotone and bounded ($0 \leq x_n \leq 3$ for all $n$), the limit $x = \lim_{n \to \infty} x_n$ exists and satisfies

$$x = \sqrt{6 + x}$$

$$x^2 = 6 + x$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

As $x > 0$, $x = 3$.

$$\lim_{n \to \infty} x_n = 3.$$
2. For part (a), you may use without definition the concept of limit of a sequence. Then, for each of the series in (b), (c), (d), determine (with proof) whether it converges absolutely, converges conditionally, or diverges.

(a) Assume \( a_n, n \in \mathbb{N}, \) are real numbers. Define precisely what these two statement mean: \( \sum_{k=1}^{\infty} a_k \) converges absolutely; \( \sum_{k=1}^{\infty} a_k \) converges conditionally.

\[
\begin{align*}
\text{a} & \text{ converges absolutely: } \sum_{k=1}^{\infty} |a_k| \text{ converges} \\
\text{a} & \text{ converges conditionally: } \sum_{k=1}^{\infty} a_k \text{ converges, but } \sum_{k=1}^{\infty} |a_k| \text{ does not.}
\end{align*}
\]

(b) \( \sum_{k=1}^{\infty} \left( \frac{1}{k} + \frac{7}{k!} \right) \)

As \( \frac{1}{k} + \frac{1}{k!} > \frac{1}{k} \) and \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges (harmonic series), this series diverges.

(c) \( \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{3k}{4k+1} \right)^k \)

As \( \sqrt[k]{|a_k|} = \frac{3k}{4k+1} \rightarrow \frac{3}{4} < 1 \) as \( k \rightarrow \infty \), the series converges absolutely by the root test.

(d) \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}} \)

As \( \frac{1}{\sqrt{k}} \) is decreasing and converges to 0 as \( k \rightarrow \infty \), the series converges by the alternating series test. As \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \) diverges (p-series with \( p = \frac{1}{2} < 1 \)), the series converges conditionally.
3. Assume that $a_k > 0$ for all $k \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $a_k = 1$ for all even $k$, then $\sum_{k=1}^{\infty} a_k$ diverges.

\[ \text{Yes, } a_k \to 0 \text{ as } k \to \infty, \text{ so the terms diverge by the } n\text{-th term test.} \]

(b) If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (a_k + a_k^2)$ converges.

We know that $\lim_{k \to \infty} a_k = 0$, and so

\[ \lim_{k \to \infty} \frac{a_k + a_k^2}{a_k} = \lim_{k \to \infty} (1 + a_k) = 1, \]

so the series $\sum_{k=1}^{\infty} (a_k + a_k^2)$ converges by the limit comparison test. \textbf{Yes}.

(c) If $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges, then $\sum_{k=1}^{\infty} a_k^2$ converges.

\[ \text{No. Take } a_k = \frac{1}{\sqrt{k}}. \text{ We know that } \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges by } 2(d), \text{ but } \sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges (as it is the harmonic series).} \]
4. Assume \((a_n)\) is a sequence of real numbers.

(a) Is the following statement true: if \(0 \leq a_n \leq 7\) for all \(n \geq 10\), then \((a_n)\) has a convergent subsequence? (Justify your assertion.)

Yes. The sequence is bounded to the end since it follows from the Bolzano-Weierstrass theorem.

(b) Prove: if \(\lim \inf (na_n) = 2\), then \(\sum_{n=1}^{\infty} a_n\) is a divergent series.

If \(\lim \inf (na_n) = 2\), then there exists an \(N \in \mathbb{N}\) such that \(na_n \geq 1\) for \(n \geq N\). Then \(a_n \geq \frac{1}{n}\) for \(n \geq N\) and so \(\sum \frac{1}{n}\) diverges by comparison with the harmonic series.