Slow Convergence in Bootstrap Percolation

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Outline

• Introduction to bootstrap percolation as a model of metastability and nucleation.
• Phase transition and slow convergence to the exponential rate.
Bootstrap percolation

This is a growth process on two-dimensional lattice $\mathbb{Z}^2$, in which points with two occupied nearest neighbors join the already occupied set.

More formally, let the neighborhood $x + \mathcal{N}$ of a site $x \in \mathbb{Z}^2$ consist of itself and nearest four sites

\[ \mathcal{N} = \bullet \ x \ \bullet \ \bullet, \]

then for any set $A \subset \mathbb{Z}^2$, define

\[ \mathcal{B}(A) = A \cup \{ x \in \mathbb{Z}^2 : |A \cap (x + \mathcal{N})| \geq 2 \}. \]

The occupied set from $A$ at time $t$ is the iterate $\mathcal{B}^t(A)$, and the final set from $A$ is $\langle A \rangle = \bigcup_{t \geq 0} \mathcal{B}^t(A)$.
Example

. . . 1 . . .
. 1 1 . . .
. . 1 . . .
. . . . .
. . 1 . . .
1 1 1 1 .
. 1 1 1 . .
Example

\[ \begin{array}{ccccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \bullet & 1 & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\bullet & 1 & \cdot & \cdot & \cdot & \cdot & \\
1 & \bullet & \bullet & 1 & \bullet & \cdot & 1 \\
\bullet & 1 & 1 & \bullet & 1 & \cdot & \cdot \\
\end{array} \]
Example

. . . • 1 . . .
. 1 1 1 . . .
. . . . . . .
. . . . . . .
. . . . . . .
1 1 . . . .
1 1 1 1 1 • 1
1 1 1 1 1 . .
Example

```
. 1 1 . . .
. 1 1 1 . . .
. . . . . . .
. . . . . . .
1 1 1 . . .
1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
```

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Example

. 1 1 1 . . .
. 1 1 1 . . .
. . . . . . .
. . . . . . .
1 1 1 1 . . .
1 1 1 1 1 1 1
1 1 1 1 1 1 1
Example

```
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

```
Example

1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1

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Example

. 1 1 1 . . .
. 1 1 1 . . .
. . . . . . .
. . . . . . .
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
Modifications

Occupied points = 1’s, non-occupied points = 0’s.

*Modified b. p.* A 0 in becomes 1 iff it encounters two 1’s in one of the following four configurations

\[
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & ' & 1 & ' & 1 & 0 \\
\end{array}
\]

*Froböse b. p.* A 0 in becomes 1 iff it encounters three 1’s in one of the following four configurations

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & ' & 0 & 1 & ' & 1 & 0 & ' & 1 & 1 \\
\end{array}
\]
Random initial states

Let $\Pi(p) \subset \mathbb{Z}^2$ be the product measure with density $p > 0$. For set $K \subset \mathbb{Z}^2$, usually the $L \times L$ square $R(L) = [1, L]^2 \cap \mathbb{Z}^2$, let $K_p = \Pi(p) \cap K$ and call $K$ internally spanned if $\langle K_p \rangle = K$.

Main object of interest are the quantities

$$I(L, p) = P_p(R(L) \text{ is internally spanned}),$$
$$T(p) = \inf\{t \geq 0 : 0 \in \mathcal{B}^t(\Pi(p))\}.$$  

**Puzzle.** Is it possible that $R(L)$ is internally spanned if $|R(L)_p| < L$?

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Exponential metastability

Let $\lambda = \pi^2/18 \approx 0.548$ for b. p. and $\lambda = \pi^2/6 \approx 1.645$ for the modified and Froboese models. Choose a square of exponential size in $1/p$, i.e., for some $\alpha > 0$,

$$L = e^{\alpha/p}.$$

**Theorem 1.** (Aizenman-Lebowitz, 1988; Holroyd, 2003)

- If $\alpha > \lambda$, then $I(L, p) \to 1$.
- If $\alpha < \lambda$, then $I(L, p) \to 0$.

A rescaling step implies that

$$p \cdot \log T(p) \to \lambda,$$

in probability, as $p \to 0$.

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Regular ring growth

A large occupied set in the Froböse b p. results from the following configuration

```
  . 1 . . . . 1
  . . . 1 . . .
  . 1 . . . . .
  . 1 . . . . 1
  1 . . . 1 . .
  1 1 1 . 1 . .
```

This set is a *nucleus* for further growth.
Nucleation probability

Regular ring growth happens from some corner $x$ with prob.

$$\pi_p = \prod_{k=1}^{\infty} (1 - (1 - p)^k)^2 \approx \prod_{k=1}^{\infty} (1 - e^{-pk})^2$$

and so

$$-p \log \pi_p \to -2 \int_0^\infty \log (1 - e^{-x}) \, dx = \frac{\pi^2}{3} = 2\lambda.$$

If possible number of initial corners in $R(L)$, which is about $L^2$, is larger than $1/\pi_p \approx \exp(2\lambda/p)$, then a large unstoppable occupied set is likely to nucleate. This proves one half of Theorem 1 for the Frohse b. p.
Problem

For a fixed $L$, define $p_a = p_a(L)$ by

$$I(L, p_a) = a.$$  

By Theorem 1,

$$p_{1/2} \cdot \log L \to \lambda, \quad \text{as } L \to \infty.$$  

Instead, computer experiments (Adler, Stauffer, Aharony, 1989 and on) typically give estimates which are off by about factor 2: $p_{1/2} \cdot \log L$ is for $L \approx 20,000$ about 0.25 for b. p. and about 0.75 for the other two.

The regular ring growth is for squares of realistic size not the only scenario.
Use entropy?

There are other possibilities which are about as likely as the regular ring growth, which may look like this, again for Froböse b. p.:

```
. . . . 1 . . . . . .
. 1 . . . . . . . .
. . . 1 . . . . .
. . . . . . . . . .
. . . . . . . . .
. . . . . . . . .
. 1 . . . . . . .
. 1 . . . . . . .
1 . 1 . . . . .
1 . 1 . . . . .
```

Do these make nucleation more likely?

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Markov Chain for Froböse b. p.

The nucleation problem can be formulated as a *local b. p.*, a Markov Chain on 4-tuples (plus an extra “cemetary” state) \((\ell^H, \ell^V, a^H, a^V)\), where \(\ell^H, \ell^V \geq 1\) are lengths of edges and \(a^H, a^V\) indicate the number of active (unexamined) sites outside the edges (either 1 or all=2).

Probabilities of reaching any state starting from \((1, 1, 2, 2)\) are given recursively. These stabilize at distances \(\gg 1/p\).

Moreover, one can show that paths which deviate \(O(1/\sqrt{p})\) from the main diagonal incur the reduction in probability (“energy cost”) by the factor \(O(\sqrt{p})\). But the number of choices (“entropy gain”) multiplies the probability by

\[
\sum_{m=0}^{C/\sqrt{p}} \binom{n}{m}(c\sqrt{p})^m \geq e^{c/\sqrt{p}},
\]

thus the prob. that local b. p. grows is at least \(\exp(c/\sqrt{p} - 2\lambda/p)\).
Slow convergence

Theorem 2. (G-Holroyd, 2007)

- If $p \log L > \lambda - c_1 \sqrt{p}$, then $I(L, p) \to 1$.
- If $p \log L > \lambda - c_2/\sqrt{\log L}$, then $I(L, p) \to 1$.

Remarks:

- $1/\sqrt{\log 20,000} \approx 0.32$.
- To halve the error in approximating $\lambda$ one would need to replace $L$ by $L^4$. For modified b. p., one can prove that $L = 10^{500}$ is necessary for 2% error.
Transition window

Theorem 3. (Balogh, Bollobás, 2003) For any fixed $\epsilon > 0$,

$$(p_{1-\epsilon} - p_\epsilon) \log L = O \left( p_{1/2} \cdot \log \frac{1}{p_{1/2}} \right)$$

So

$$\lambda - p_{1/2} \log L \geq \frac{c_2}{\sqrt{\log L}}$$

while

$$(p_{1-\epsilon} - p_\epsilon) \log L \leq \frac{C \log \log L}{\log L}.$$ 

This is an instance of a sharp transition phenomenon. General methods exist for proving this, particularly for monotone events.
Transition window vs. convergence

Width of the transition window is much narrower than distance from its asymptotic location.

Although this is conjectured to be a common phenomenon, only one other result in this direction is known, on an integer partitioning problem. (Borgs, Chayes, Pittel, 2001).
loglogL/(\log L)^2

\( \lambda / \log L \)
Continuous nucleation

Now, let $A_0 = \emptyset$ and $A_{t+1} = B(A_t) \cup \Pi_t(p)$, with independent product measures $\Pi_t(p)$. Denote $p_n = 1 - (1 - p)^n \sim np$, for small $np$. Let $T'(p)$ be the first passage time. By monotonicity, for any $\alpha > 0$,

$$P(T(p_{\alpha n}) \leq (1 - \alpha)n) \leq P(T'(p) \leq n) \leq P(T(p_n) \leq n).$$

Therefore $P(T'(p) \leq n) \to 0$ if $p_n \log n \leq \lambda - \epsilon$, i.e., $np \log(1/p) \leq \lambda - \epsilon$. Also, $P(T'(p) \leq n) \to 1$ if $p_{\alpha n} \log ((1 - \alpha)n) \geq \lambda + \epsilon$, i.e., $np \log(1/p) \geq (\lambda + \epsilon)/\alpha$. Conclusion:

$$p \log(1/p) \cdot T'(p) \to \lambda,$$

in probability.

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Open problems

• Is it true that, for some large $C$, $p \log L < \lambda - C/\sqrt{\log L}$ implies that $I(L, p) \to 0$?
• Give the lower bound for the length of the transition window.
• Determine the speed of convergence for the continuous nucleation case.