# Slow Convergence in Bootstrap Percolation

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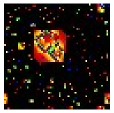
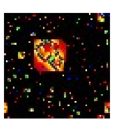


# Outline

- Introduction to bootstrap percolation as a model of metastability and nucleation.
- Phase transition and slow convergence to the exponential rate.



# Bootstrap percolation

This is a growth process on two-dimensional lattice $\mathbb{Z}^2$, in which points with two occupied nearest neighbors join the already occupied set.

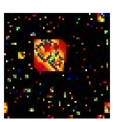
More formally, let the neighborhood $x + \mathcal{N}$ of a site $x \in \mathbb{Z}^2$ consist of itself and nearest four sites

$$\mathcal{N} = \begin{array}{ccc} & \bullet & \\ \bullet & x & \bullet \\ & \bullet & \end{array},$$

then for any set $A \subset \mathbb{Z}^2$, define

$$\mathcal{B}(A) = A \cup \{x \in \mathbb{Z}^2 : |A \cap (x + \mathcal{N})| \geq 2\}.$$

The occupied set from $A$ at time $t$ is the iterate $\mathcal{B}^t(A)$, and the *final set* from $A$ is $\langle A \rangle = \cup_{t \geq 0} \mathcal{B}^t(A)$.



# Example

$$
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot
\end{array}
$$

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
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| · | 1 | ● | 1 | · | · | · |
| · | · | · | · | · | · | · |
| · | · | · | · | · | · | · |
| ● | 1 | · | · | · | · | · |
| 1 | ● | ● | 1 | ● | · | 1 |
| ● | 1 | 1 | ● | 1 | · | · |

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
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| · | · | · | · | · | · | · |
| 1 | 1 | ● | · | · | · | · |
| 1 | 1 | 1 | 1 | 1 | ● | 1 |
| 1 | 1 | 1 | 1 | 1 | · | · |

# Example

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|---|---|---|---|---|---|---|
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| 1 | 1 | 1 | ● | · | · | · |
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# Example

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|---|---|---|---|---|---|---|
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| . | 1 | 1 | 1 | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| 1 | 1 | 1 | 1 | ● | . | . |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | ● |

# Example

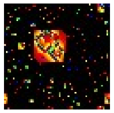
| | | | | | | |
|---|---|---|---|---|---|---|
| · | 1 | 1 | 1 | · | · | · |
| · | 1 | 1 | 1 | · | · | · |
| · | · | · | · | · | · | · |
| · | · | · | · | · | · | · |
| 1 | 1 | 1 | 1 | 1 | ● | · |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
| · | 1 | 1 | 1 | · | · | · |
| · | 1 | 1 | 1 | · | · | · |
| · | · | · | · | · | · | · |
| · | · | · | · | · | · | · |
| 1 | 1 | 1 | 1 | 1 | 1 | ● |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Example

| | | | | | | |
|---|---|---|---|---|---|---|
| · | 1 | 1 | 1 | · | · | · |
| · | 1 | 1 | 1 | · | · | · |
| · | · | · | · | · | · | · |
| · | · | · | · | · | · | · |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |



# Modifications

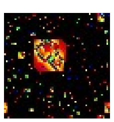
Occupied points = 1’s, non-occupied points = 0’s.

*Modified b. p.* A 0 in becomes 1 iff it encounters two 1’s in one of the following four configurations

$$\begin{array}{cc} 1 & \\ 0 & 1 \end{array}, \quad \begin{array}{cc} 1 & 0 \\ & 1 \end{array}, \quad \begin{array}{cc} 0 & 1 \\ 1 & \end{array}, \quad \begin{array}{cc} & 1 \\ 1 & 0 \end{array}.$$

*Froböse b. p.* A 0 in becomes 1 iff it encounters three 1’s in one of the following four configurations

$$\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}, \quad \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}, \quad \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}.$$



# Random initial states

Let $\Pi(p) \subset \mathbb{Z}^2$ be the product measure with density $p > 0$. For set $K \subset \mathbb{Z}^2$, usually the $L \times L$ square $R(L) = [1, L]^2 \cap \mathbb{Z}^2$, let $K_p = \Pi(p) \cap K$ and call $K$ *internally spanned* if $\langle K_p \rangle = K$.

Main object of interest are the quantities

$$I(L, p) = P_p(R(L) \text{is internally spanned}),$$

$$T(p) = \inf\{t \geq 0 : 0 \in \mathcal{B}^t(\Pi(p))\}.$$

*Puzzle*. Is it possible that $R(L)$ is internally spanned if $|R(L)_p| < L$?

# Exponential metastability

Let $\lambda = \pi^2/18 \approx 0.548$ for b. p. and $\lambda = \pi^2/6 \approx 1.645$ for the modified and Froböse models. Choose a square of exponential size in $1/p$, i.e., for some $\alpha > 0$,

$$L = e^{\alpha/p}.$$

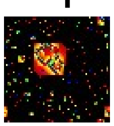
**Theorem 1.** *(Aizenman-Lebowitz, 1988; Holroyd, 2003)*

- *If* $\alpha > \lambda$*, then* $I(L,p) \to 1$*.*
- *If* $\alpha < \lambda$*, then* $I(L,p) \to 0$*.*

A rescaling step implies that

$$p \cdot \log T(p) \to \lambda,$$

in probability, as $p \to 0$.

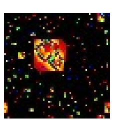


# Regular ring growth

A large occupied set in the Froböse b p. results from the following configuration

$$\begin{array}{cccccc}
\cdot & 1 & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \cdot & 1 & \bullet & \cdot \\
\cdot & 1 & \cdot & \bullet & \cdot & \cdot \\
\cdot & 1 & \bullet & \cdot & \cdot & 1 \\
1 & \bullet & \cdot & 1 & \cdot & \cdot \\
1 & 1 & 1 & \cdot & 1 & \cdot
\end{array}$$

This set is a *nucleus* for further growth.



# Nucleation probability

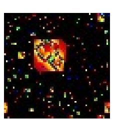
Regular ring growth happens from some corner $x$ with prob.

$$\pi_p = \prod_{k=1}^{\infty} \left(1-(1-p)^k\right)^2 \approx \prod_{k=1}^{\infty} \left(1-e^{-pk}\right)^2$$

and so

$$-p \log \pi_p \to -2\int_0^{\infty} \log\left(1-e^{-x}\right)\, dx = \frac{\pi^2}{3} = 2\lambda.$$

If possible number of initial corners in $R(L)$, which is about $L^2$, is larger than $1/\pi_p \approx \exp(2\lambda/p)$, then a large unstoppable occupied set is likely to nucleate. This proves one half of Theorem 1 for the Froböse b. p.



# Problem

For a fixed $L$, define $p_a = p_a(L)$ by

$$I(L, p_a) = a.$$

By Theorem 1,

$$p_{1/2} \cdot \log L \to \lambda, \quad \text{as } L \to \infty.$$

Instead, computer experiments (Adler, Stauffer, Aharony, 1989 and on) typically give estimates which are off by about factor 2: $p_{1/2} \cdot \log L$ is for $L \approx 20,000$ about $0.25$ for b. p. and about $0.75$ for the other two.

The regular ring growth is for squares of realistic size not the only scenario.

# Use entropy?

There are other possibilities which are about as likely as the regular ring growth, which may look like this, again for Froböse b. p.:

$$
\begin{array}{ccccccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & {\color{red}\bullet} \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & {\color{red}\bullet} & {\color{red}1} & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & {\color{red}\bullet} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {\color{red}1} & \cdot & \cdot \\
\cdot & 1 & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}1} & {\color{red}\bullet} & \cdot & 1 \\
1 & {\color{red}\bullet} & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
{\color{red}1} & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Do these make nucleation more likely?

# Markov Chain for Froböse b. p.

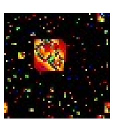
The nucleation problem can be formulated as a *local b. p.*, a Markov Chain on 4-tuples (plus an extra "cemetary" state) $(\ell^H, \ell^V, a^H, a^V)$, where $\ell^H, \ell^V \geq 1$ are lengths of edges and $a^H$, $a^V$ indicate the number of active (unexamined) sites outside the edges (either 1 or all=2).

Probabilities of reaching any state starting from $(1, 1, 2, 2)$ are given recursively. These stabilize at distances $\gg 1/p$.

Moreover, one can show that paths which deviate $\mathcal{O}(1/\sqrt{p})$ from the main diagonal incur the reduction in probability ("energy cost") by the factor $\mathcal{O}(\sqrt{p})$. But the number of choices ("entropy gain") multiplies the probability by

$$\sum_{m=0}^{C/\sqrt{p}} \binom{n}{m} (c\sqrt{p})^m \geq e^{c/\sqrt{p}},$$

thus the prob. that local b. p. grows is at least $\exp(c/\sqrt{p} - 2\lambda/p)$.



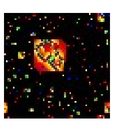
# Slow convergence

**Theorem 2.** *(G-Holroyd, 2007)*

- *If* $p \log L > \lambda - c_1\sqrt{p}$, *then* $I(L, p) \to 1$.
- *If* $p \log L > \lambda - c_2/\sqrt{\log L}$, *then* $I(L, p) \to 1$.

Remarks:

- $1/\sqrt{\log 20,000} \approx 0.32$.
- To halve the error in approximating $\lambda$ one would need to replace $L$ by $L^4$. For modified b. p., one can prove that $L = 10^{500}$ is necessary for 2% error.



# Transition window

**Theorem 3.** *(Balogh, Bollobás, 2003) For any fixed* $\epsilon > 0$*,*

$$(p_{1-\epsilon} - p_\epsilon) \log L = \mathcal{O}\left(p_{1/2} \cdot \log \frac{1}{p_{1/2}}\right)$$

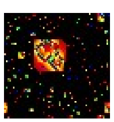
So

$$\lambda - p_{1/2} \log L \geq \frac{c_2}{\sqrt{\log L}}$$

while

$$(p_{1-\epsilon} - p_\epsilon) \log L \leq \frac{C \log \log L}{\log L}.$$

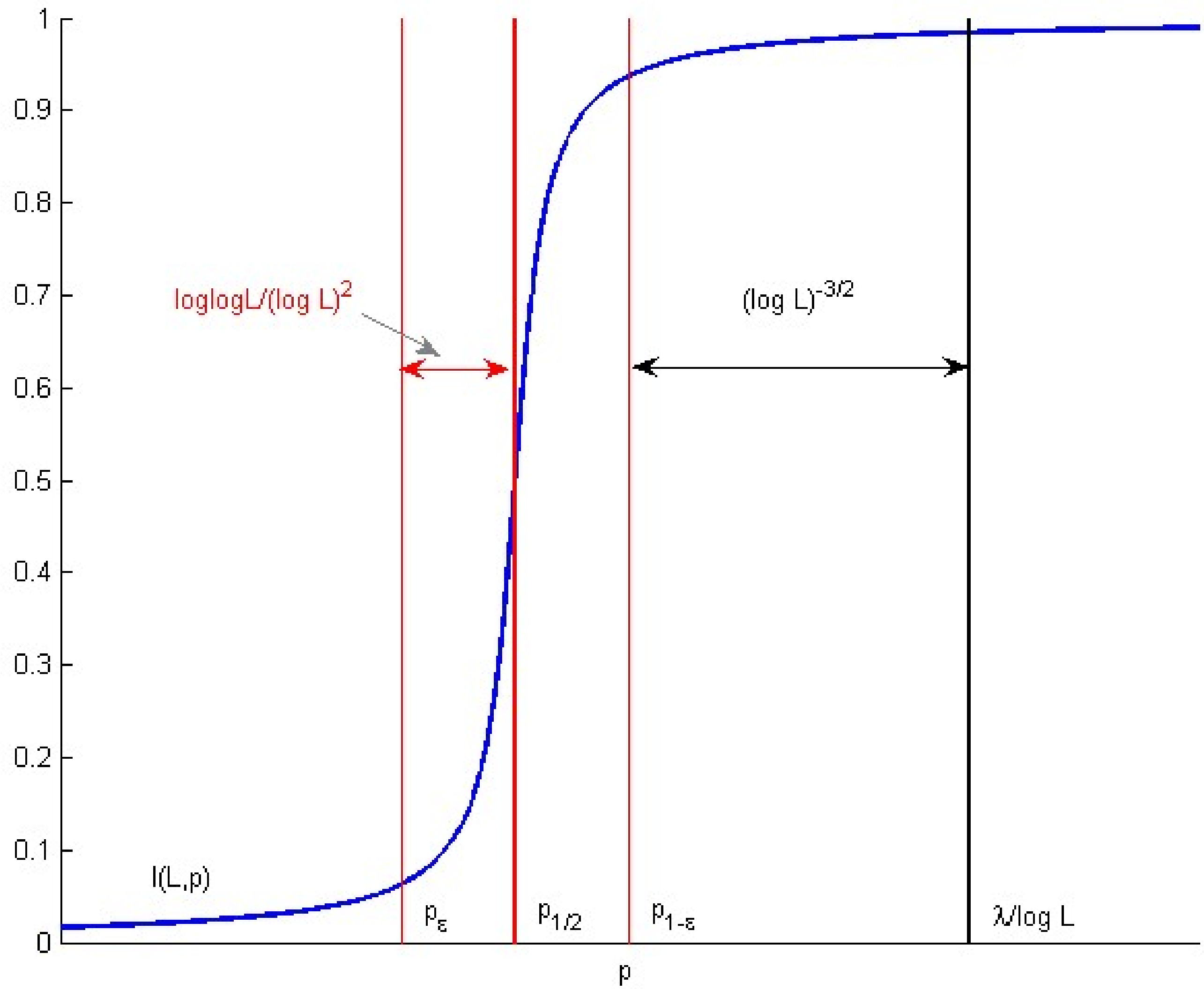
This is an instance of a *sharp transition* phenomenon. General methods exist for proving this, particularly for monotone events.



# Transition window vs. convergence

Width of the transition window is much narrower than distance from its asymptotic location.

Although this is conjectured to be a common phenomenon, only one other result in this direction is known, on an integer partitioning problem. (Borgs, Chayes, Pittel, 2001).


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
loglogL/(log L)^2
(log L)^-3/2
I(L,p)
p_ε
p_1/2
p_1-ε
λ/log L
p

# Continuous nucleation

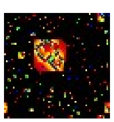
Now, let $A_0 = \emptyset$ and $A_{t+1} = \mathcal{B}(A_t) \cup \Pi_t(p)$, with independent product measures $\Pi_t(p)$. Denote $p_n = 1 - (1-p)^n \sim np$, for small $np$. Let $T'(p)$ be the first passage time. By monotonicity, for any $\alpha > 0$,

$$P(T(p_{\alpha n}) \leq (1-\alpha)n) \leq P(T'(p) \leq n) \leq P(T(p_n) \leq n).$$

Therefore $P(T'(p) \leq n) \to 0$ if $p_n \log n \leq \lambda - \epsilon$, i.e., $np \log(1/p) \leq \lambda - \epsilon$. Also, $P(T'(p) \leq n) \to 1$ if $p_{\alpha n} \log((1-\alpha)n) \geq \lambda + \epsilon$, i.e., $np \log(1/p) \geq (\lambda + \epsilon)/\alpha$. Conclusion:

$$p \log(1/p) \cdot T'(p) \to \lambda,$$

in probability.



# Open problems

- Is it true that, for some large $C$, $p \log L < \lambda - C/\sqrt{\log L}$ implies that $I(L, p) \to 0$?
- Give the lower bound for the length of the transition window.
- Determine the speed of convergence for the continuous nucleation case.