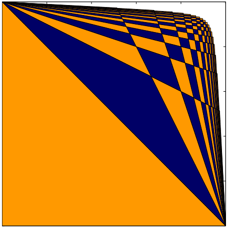
Long-range bootstrap percolation



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Bootstrap percolation

Select a graph $G = (V, E)$. (In fact, a sequence of graphs, say, G_n.)
Fix an integer *threshold* $\theta \geq 1$.

Choose a set ω_0 of initially **occupied** ("infected," 1) vertices of G, and call the remaining sites **empty** ("susceptible," 0).
Typically, the sites are initially occupied independently with probability $p \in (0, 1)$ (which depends on n).

Bootstrap percolation: for $t = 0, 1, \ldots$, is the increasing sequence $(\omega_t)_{t=0}^{\infty}$ of configurations $\omega_t \in \{0, 1\}^V$. At time $t + 1$, an empty site x becomes occupied iff:

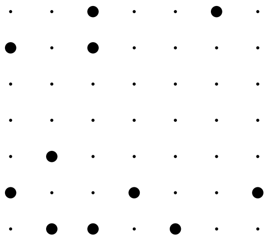
at least θ of the neighbors of x are occupied at time t.

Once occupied, a site never changes state.

The "final" configuration ω_∞ is the configuration of eventually occupied sites. The event $\{\omega_\infty \equiv 1\}$ is called spanning.

Example with $V = \mathbb{Z}^2$

Start with the finite occupied set below, on the infinite 2-dimensional nearest-neighbor lattice with $\theta = 2$:



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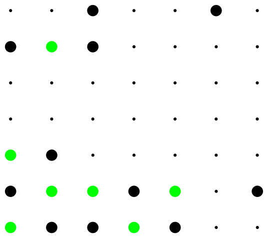

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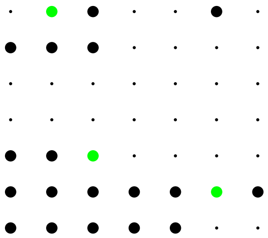

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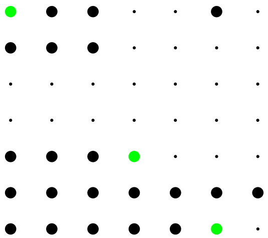

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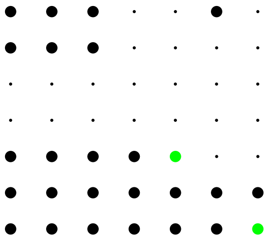

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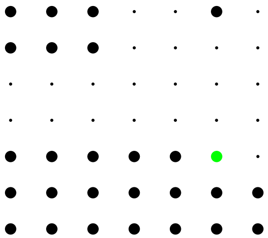

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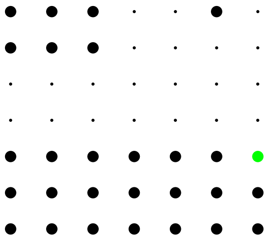

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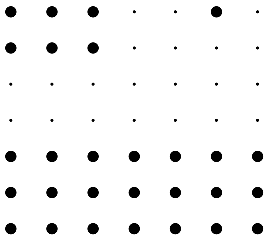
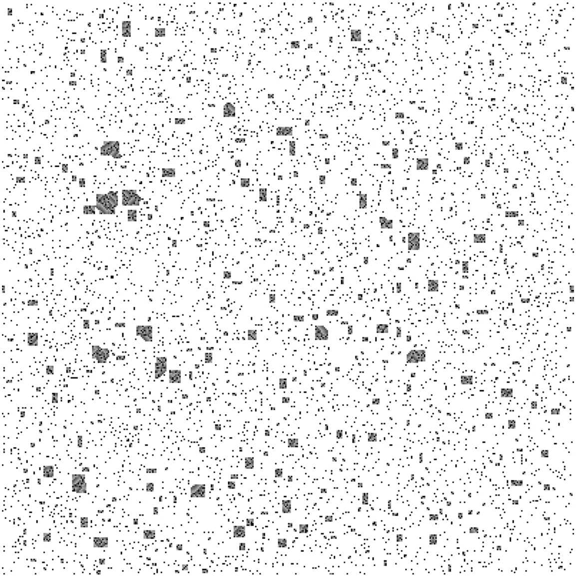


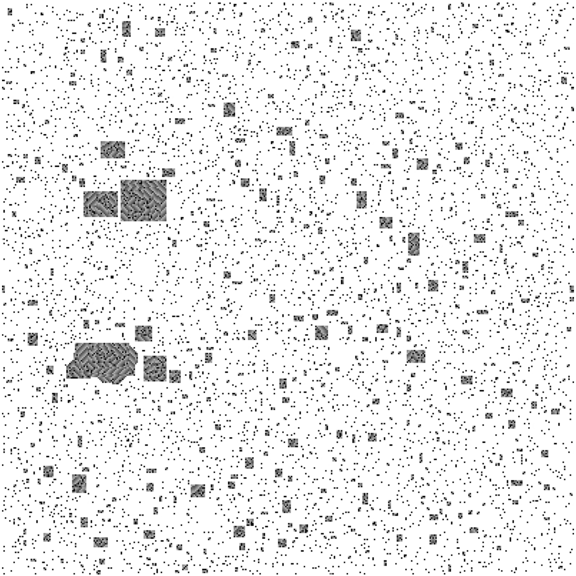
Nucleation in bootstrap percolation

$V = \mathbb{Z}^2$, $\theta = 2$,
$p = 0.045$,
400×400 square with periodic boundary,
$t = 20$



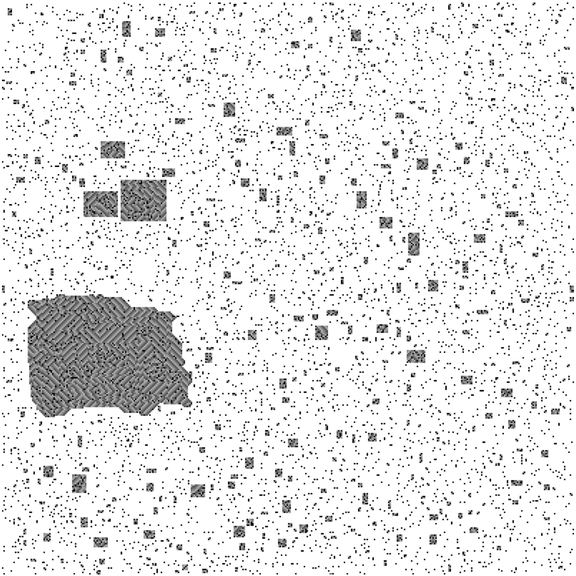
Nucleation in bootstrap percolation

$V = \mathbb{Z}^2$, $\theta = 2$,
$p = 0.045$,
400×400 square
with periodic
boundary,
$t = 100$



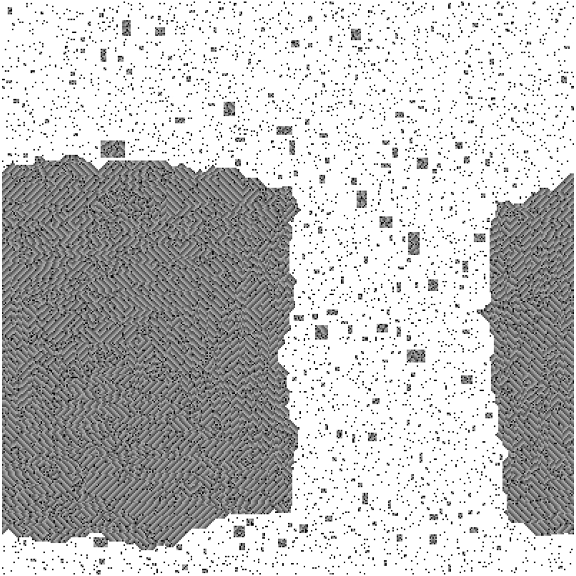
Nucleation in bootstrap percolation

$V = \mathbb{Z}^2$, $\theta = 2$,
$p = 0.045$,
400×400 square
with periodic
boundary,
$t = 200$



Nucleation in bootstrap percolation

$V = \mathbb{Z}^2$, $\theta = 2$,
$p = 0.045$,
400×400 square
with periodic
boundary,
$t = 500$



Bootstrap percolation: brief history

First rigorous result is due to van Enter (1987) for $d = 2$, extended to $d \geq 2$ by Schonmann (1992).

Theorem (Nearest-neighbor infinite lattice $\mathbb{Z}^d$)

If $\theta \leq d$, then $\mathbb{P}_p(\omega_\infty \equiv 1) = 1$ for all $p > 0$. If $\theta > d$, then $\mathbb{P}_p(\omega_\infty \equiv 1) = 0$ for all $p < 1$.

For finite graphs, the critical value p_c is defined as

$$\mathbb{P}_{p_c}(\omega_\infty \equiv 1) = 1/2.$$

Many deep and surprising results have been proved for bootstrap percolation on finite subboxes of $\mathbb{Z}^d$, starting with Aizenman and Lebowitz (1988), who proved that for $V = \mathbb{Z}_n^d$ with nearest neighbor edges, and $\theta = 2$, $p_c \asymp (\log n)^{-(d-1)}$.

Bootstrap percolation: brief history

Discrete tori: $V = \mathbb{Z}_n^2$, nearest neighbor edges, $\theta = 2$:

- $p_c \asymp \frac{1}{\log n}$ [Aizenman, Lebowitz, 1988].
- $p_c \sim \frac{\pi^2}{18 \log n}$ [Holroyd, 2003]
- $p_c = \frac{\pi^2}{18 \log n} - \frac{1}{(\log n)^{3/2 - o(1)}}$ [G., Holroyd, Morris, 2012]
- $p_c = \frac{\pi^2}{18 \log n} - \frac{\Theta(1)}{(\log n)^{3/2}}$ [Hartarsky, Morris, 2018]

Discrete tori: $V = \mathbb{Z}_n^d$, nearest neighbor edges, $2 \le \theta \le d$:

- $p_c \asymp (\log^{\theta-1} n)^{-(d-\theta+1)}$ [Cerf & Cirillo, 1999; Cerf & Manzo, 2002]
- $p_c \sim \lambda(d, \theta)\,(\log^{\theta-1} n)^{-(d-\theta+1)}$ [Balogh, Bollobás, Duminil-Copin, Morris, 2012]

Hypercube: $V = \{0, 1\}^n$, nearest neighbor edges, $\theta = 2$.

- $p_c \sim \lambda\, n^{-2} 2^{-2\sqrt{n}}$ [Balogh, Bollobás, Morris, 2010]

Bootstrap percolation: brief history

In all these cases, the thresholds are sharp: for any $\epsilon > 0$,

$$\mathbb{P}_{(1-\epsilon)p_c}(\omega_\infty \equiv 1) \to 0, \mathbb{P}_{(1+\epsilon)p_c}(\omega_\infty \equiv 1) \to 1.$$

Sharpness of a transitions is in fact often easier to establish than its location, due to the general results of Friedgut and Kalai. For $\mathbb{Z}_n^2$, the above is valid for $\epsilon = (\log n)^{-1+o(1)}$, so the "tightening" is faster than convergence.

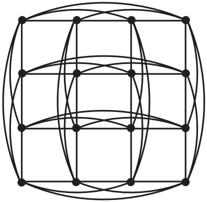
- For $V = \mathbb{Z}_n^2$, sharp thresholds have been identified for other neighborhoods:
  - 'cross' neighborhood [Holroyd, Liggett, Romik, 2004]
  - anisotropic [Duminil-Copin, Van Enter, 2013]
  - balanced [Duminil-Copin, Holroyd, 2015]
  - drift/oriented [Bollobás, Duminil-Copin, Morris, Smith, 2016]

The Hamming graph

The Hamming graph with side length n and dimension d is the graph with the following vertex set V, and edge set E:

$$V = \{1, 2, \ldots, n\}^d$$
$$E = \{(x, y) \in V \times V : d_{\mathrm{H}}(x, y) = 1\},$$

where $d_{\mathrm{H}}(x, y)$ is the Hamming distance between x and y (number of coordinates at which they differ). This is the Cartesian product of d complete graphs K_n, so we denote it by $K_n^d = K_n \times \cdots \times K_n$.



$G = K_n^2$, $\theta \geq 2$

Precise result for the Hamming square:

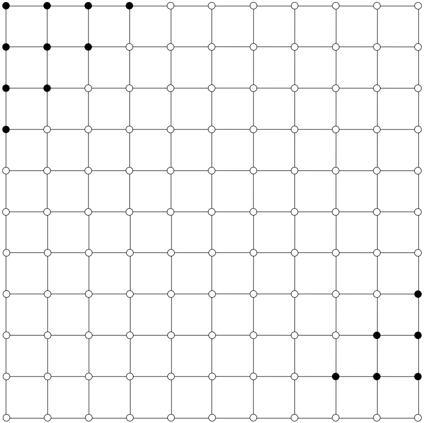
Theorem (Gravner, Hoffman, Pfeiffer, S., 2015)

Let $\ell \geq 2$ and $p = a \cdot n^{-(\ell+1)/\ell}$.

- *If $\theta = 2\ell - 1$, then $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - \exp(-2a^\ell/\ell!)$.*
- *If $\theta = 2\ell$, then $\mathbb{P}_p(\omega_\infty \equiv 1) \to \left[1 - \exp(-a^\ell/\ell!)\right]^2$.*
- *If $\theta = 2$ and $p = an^{-2}$, then $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - (1+a)e^{-a}$.*

Note that the threshold p_c is not sharp.

$G = K_n^2$, odd θ ($\theta = 7$)



$$\mathbb{P}(\omega_\infty \equiv 1) \approx$$
$$\mathbb{P}(\exists \text{ line with} \geq 4 \text{ points}) \quad \times \prod_{k<\theta/2} \mathbb{P}(\exists \text{ row with} \geq k \text{ points})^2$$

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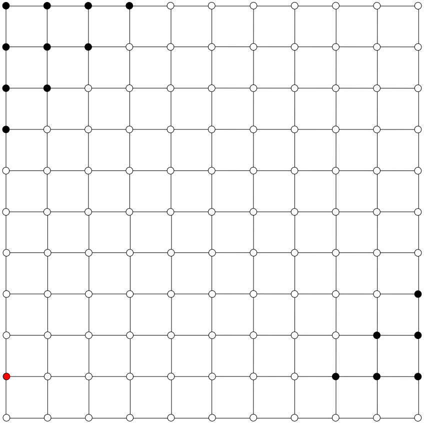

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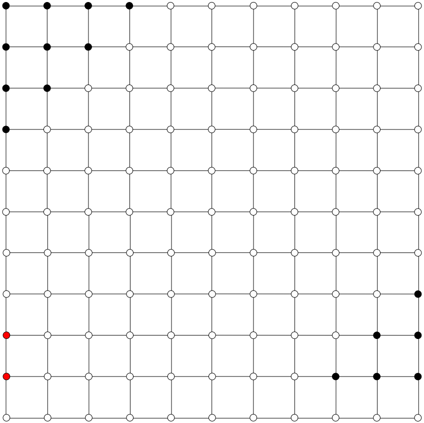

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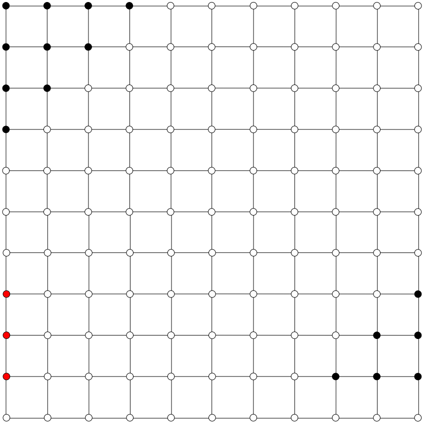

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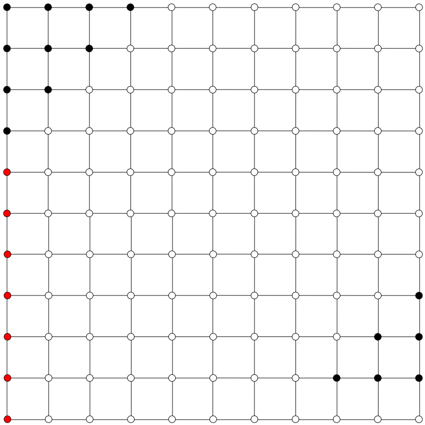

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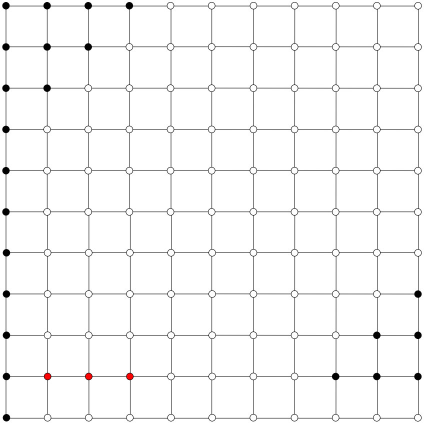

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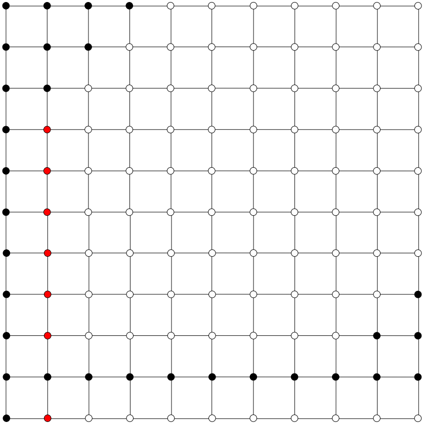

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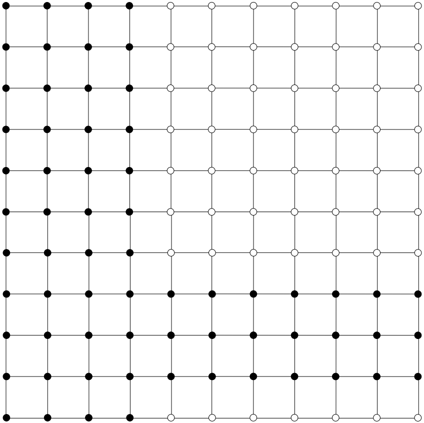

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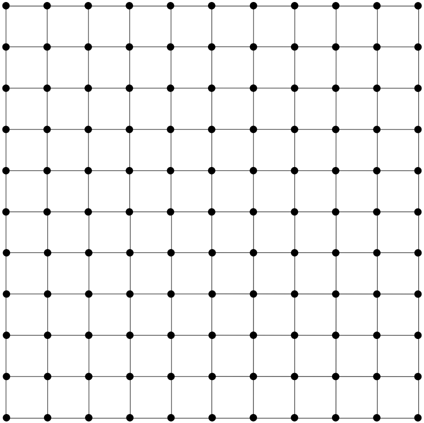

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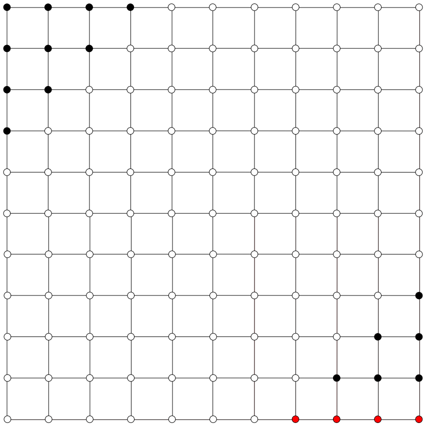
$G = K_n^2$, Even $\theta = 8$



$$\mathbb{P}(\omega_\infty \equiv 1) \approx$$
$$\mathbb{P}(\text{row with} \geq 4 \text{ points})^2 \times \prod_{k<\theta/2} \mathbb{P}(\text{row with} \geq k \text{ points})^2$$

$G = K_n^2$, $\theta \geq 3$

Choose p so that the number of rows with $\ell = \lceil \theta/2 \rceil$ points is $\Theta(1)$:

$$n \cdot \binom{n}{\ell} p^{\ell} = \mathit{Constant} \qquad \Rightarrow \qquad p \asymp n^{-(\ell+1)/\ell}.$$

Then rows with $\geq \ell - 1$ points are plentiful, and for even $\theta = 2\ell$

$$\begin{aligned}\mathbb{P}(\omega_\infty \equiv 1) &\approx \mathbb{P}(\text{row with} \geq \ell \text{ points})^2 \\ &\approx \mathbb{P}(\text{Poisson}(n^{\ell+1} p^{\ell}/\ell!) \geq 1)^2.\end{aligned}$$

Large deviations

Assume that the initial configuration with density $p > 0$ on the Hamming rectangle $K_n \times K_m$. We want to say that

$$n \approx p^{-\alpha}, \;\; m \approx p^{-\beta} \implies \mathbb{P}_p(\omega_\infty \equiv 1) \approx p^{\mathrm{I}}.$$

To be precise, we assume that, as $p \to 0$, $n, m \to \infty$ and

$$\log n \sim -\alpha \log p, \quad \log m \sim -\beta \log p,$$

and call the quantity

$$\mathrm{I}(\alpha, \beta) = \lim_{p \to 0} \frac{\log \mathbb{P}_p(\omega_\infty \equiv 1)}{\log p}$$

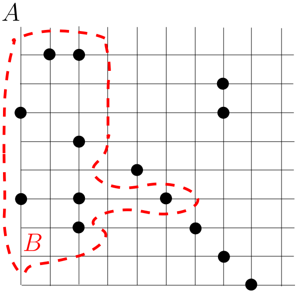
the large deviation rate for the event $\{\omega_\infty \equiv 1\}$, provided it exists.

The energy-entropy functional

For a finite set $B \subseteq \mathbb{Z}_+^2$, let $\pi_x(B)$ and $\pi_y(B)$ be projections of B on the x-axis and y-axis, respectively. Then, for a finite set $A \subset \mathbb{Z}_+^2$, let

$$\rho(\alpha, \beta, A) = \max_{B \subseteq A} \big(|B| - \alpha|\pi_x(B)| - \beta|\pi_y(B)|\big)$$
$$= \max_{B \subseteq A} \left((\text{energy of } B) - (\text{entropy of } B)\right).$$

For any $B \subseteq A$, $\mathbb{P}_p(\omega_0 \text{ contains } A) \leq \mathbb{P}_p(\omega_0 \text{ contains } B)$
$$\leq C_B n^{|\pi_x(B)|} m^{|\pi_y(B)|} p^{|B|} = C_B p^{|B| - \alpha|\pi_x(B)| - \beta|\pi_y(B)| + o(1)}$$



$$|B| - \alpha|\pi_x(B)| - \beta|\pi_y(B)|$$
$$= 8 - 4\alpha - 5\beta$$

Variational principle for I

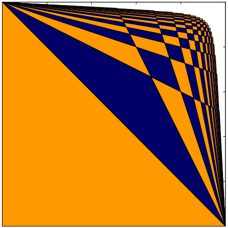
Let $\mathcal{A}$ be the set of finite spanning sets. Any set $A \in \mathcal{A}$ gives a lower bound on $\mathbb{P}_p(\omega_\infty \equiv 1)$ of about $p^{\rho(\alpha,\beta,A)}$; we select the "best" such set.

Theorem (G., Sivakoff, Slivken, 2017)

The large deviation rate $\mathrm{I}(\alpha, \beta)$ exists. Moreover, there exists a finite set $\mathcal{A}_0 \subseteq \mathcal{A}$, independent of α and β, so that

$$\mathrm{I}(\alpha,\beta) = \inf\{\rho(\alpha,\beta,A) : A \in \mathcal{A}\} = \min\{\rho(\alpha,\beta,A) : A \in \mathcal{A}_0\}.$$

Moreover, $\operatorname{supp} \mathrm{I} \subset [0,1]^2$ can be determined and $\mathbb{P}_p(\omega_\infty \equiv 1) \to 1$ if $(\alpha, \beta) \notin \operatorname{supp} \mathrm{I}$. However, explicit formula for I is known only on the diagonal $\alpha = \beta$. More general growth dynamics on Hamming rectangle can be considered.



Hamming graphs K_n^d of dimension $d \geq 3$

The problems are *much* harder, even for $d = 3$.

Theorem (Slivken, 2015)

Assume $d \geq 3$ *and* $\theta = 2$, $J(J+1) < d < (J+1)(J+2)$ *for some* $J \geq 1$, *and* $p = an^{-d/(J+1)-J}$, *then*

$$\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - \exp\left[-\binom{d}{2J}(2J)!2^{-J-1}a^{J+1}\right].$$

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Theorem (G., Hoffman, Pfeiffer, Sivakoff, 2015)

Assume $d = 3$ and $\theta = 3$. Let $p = an^{-2}$. As $n \to \infty$,

$$\mathbb{P}_p(\omega_\infty \equiv 1) \to 1 - e^{-a^3-(3/2)a^2(1-e^{-2a})} \times \left[\frac{3}{2}a^2\left(\left(e^{-a} + ae^{-3a}\right)^2 - e^{-2a}\right)e^{-a^2e^{-2a}} + e^{a^3e^{-3a}}\right].$$

Hamming graphs K_n^d of dimension $d \geq 3$

Critical exponents for large θ (GHPS, 2015)

For fixed $d \geq 3$, θ sufficiently large depending on d, and n sufficiently large depending on d, θ,

$$1 + \frac{2}{\theta} + \frac{\sqrt{7}}{\theta^{3/2}} \leq \frac{-\log p_c}{\log n} \leq 1 + \frac{2}{\theta} + \frac{4(d^2+1)}{\theta^{3/2}}.$$

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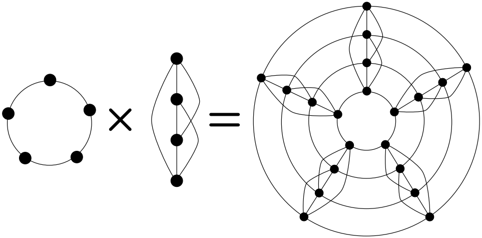
Gradual transition is expected for all d and θ, but for most larger d and θ, the critical scaling, i.e., determining $\gamma = \gamma(d, \theta)$ so that $p_c \approx n^{-\gamma}$, is an open problem.

Graphs with community structure

Consider the Cartesian product of complete graphs with a lattice.



We will consider Cartesian products $\mathbb{Z}^d \times K_n$ and $\mathbb{Z}^2 \times K_n^2$.

Bootstrap percolation on $\mathbb{Z}^d \times K_n$

If $\theta \leq d$, then by theorems of van Enter and Schonman, $\mathbb{P}_p(\omega_\infty \equiv 1) = 1$ for all $p > 0$. This is not true if $\theta > d$: e.g., when $d = 2$ and $\theta = 3$ a 2×2 box of empty "communities," e.g., $\{0,1\}^2 \times K_n^2$, cannot be invaded. We are thus interested in the final occupation density, i.e., $\mathbb{P}_p(\omega_\infty(v) = 1)$ for a fixed vertex v.

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Theorem (G., Sivakoff, 2018)

Assume $\theta > d$. *Let* $p = a \cdot n^{-1}$. *Then both* $\liminf_n \mathbb{P}_p(\omega_\infty(v) = 1)$ *and* $\limsup_n \mathbb{P}_p(\omega_\infty(v) = 1)$

- *are in* $(0, 1)$*;*
- *converge to* 0 *as* $a \to 0$*; and*
- *converge to* 1 *as* $a \to \infty$*.*

Moreover, if θ *is sufficiently large, then* $\lim_n \mathbb{P}_p(\omega_\infty(v_0) = 1)$ *exists and is continuous in* a*.*

Bootstrap percolation on $\mathbb{Z}^d \times K_n$

If $\theta \leq d$, then by theorems of van Enter and Schonman, $\mathbb{P}_p(\omega_\infty \equiv 1) = 1$ for all $p > 0$. This is not true if $\theta > d$: e.g., when $d = 2$ and $\theta = 3$ a 2×2 box of empty "communities," e.g., $\{0,1\}^2 \times K_n^2$, cannot be invaded. We are thus interested in the final occupation density, i.e., $\mathbb{P}_p(\omega_\infty(v) = 1)$ for a fixed vertex v.

Theorem (G., Sivakoff, 2018)

Assume $\theta > d$. Let $p = a \cdot n^{-1}$. Then both $\liminf_n \mathbb{P}_p(\omega_\infty(v) = 1)$ *and* $\limsup_n \mathbb{P}_p(\omega_\infty(v) = 1)$

- *are in $(0,1)$;*
- *converge to 0 as $a \to 0$; and*
- *converge to 1 as $a \to \infty$.*

Moreover, if θ is sufficiently large, then $\lim_n \mathbb{P}_p(\omega_\infty(v_0) = 1)$ exists and is continuous in a.

Question: Is the last statement true for all $\theta > d$?

Idea of the proof:
On each fiber $\{x\} \times K_n$, the positions of initially occupied sites converge to a Poisson point location on $\mathbb{R}_+$ of intensity *a*, as $n \to \infty$. Thus there is a limiting dynamics on $\mathbb{Z}^d \times \mathbb{R}_+$, which can be appropriately coupled to the one for finite *n*.

$G = \mathbb{Z}^2 \times K_n^2$ and $\theta = 2\ell + 1$

Assume $p = a \cdot \dfrac{1}{n^{1+1/\ell}}$.

Theorem (G., Sivakoff, 2018)

For $\ell \geq 1$, there exists $a_c = a_c(\ell) \in (0, \infty)$ such that the following hold.

- *If $a < a_c$, then $\lim_n \mathbb{P}_p(\omega_\infty(v) = 1) = 0$.*
- *If $a > a_c$, then $\mathbb{P}_p(\omega_\infty(v) = 1)$ is bounded away from 0 and 1. Furthermore,*

$$\liminf_n \mathbb{P}_p(\omega_\infty(v) = 1) \to 1 \qquad \text{as } a \to \infty.$$

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$$\liminf_n \mathbb{P}_p(\omega_\infty(v) = 1) \to 1 \qquad \text{as } a \to \infty.$$

Question: existence of limit, and continuity in a.

$G = \mathbb{Z}^2 \times K_n^2$ and $\theta = 2\ell + 2$

Assume $p = a \cdot \dfrac{(\log n)^{1/\ell}}{n^{1+1/\ell}}$.

Theorem (G., Sivakoff, 2018)

For $\ell \geq 1$, let $a_c = [2(\ell-1)!]^{1/\ell}$.

- *If $a < a_c$, then $\mathbb{P}_p(\omega_\infty(v) = 1) = n^{-2/\ell + o(1)}$.*
- *If $a \geq a_c$, then $\lim_n \mathbb{P}_p(\omega_\infty(v) = 1) = 1$.*
  Moreover, if $a > a_c$, then

$$\mathbb{P}_p(\omega_\infty(v) = 0) = \begin{cases} n^{4/\ell - 4a^\ell/\ell! + o(1)} & \ell \geq 2 \\ n^{-2a + o(1)} & \ell = 1 \end{cases} \qquad \text{as } n \to \infty,$$

Question: determine $f(n)$ so that the critical scaling $p = a_c \cdot (\log n)^{1/\ell} n^{-(1+1/\ell)} + f(n)$ makes $\lim_n \mathbb{P}_p(\omega_\infty(v) = 1)$ nontrivial.

Heterogeneous bootstrap percolation on $\mathbb{Z}^2$

Main idea for all proofs: Couple with a heterogeneous bootstrap percolation process on $\mathbb{Z}^2$, then analyze this process.

Given an initial state $\xi_0 \in \{0, 1, 2, 3, 4, 5\}^{\mathbb{Z}^2}$, the HBP dynamics follow the rule

$$\xi_{t+1}(x) = \begin{cases} 0 & \text{if } \#\{y : y \sim x \text{ and } \xi_t(y) = 0\} \geq \xi_t(x) \\ \xi_t(x) & \text{otherwise.} \end{cases}$$

If $\xi_0(x) = i$, then x needs the help of i 0s in its neighborhood to turn to 0. Thus a 5 will never turn to 0. (We now think of 0s as advancing occupation. Bootstrap percolation with $\theta = 2$: only 0s and 2s.) For the next example, we start with a configuration on a 5×5 square, will 5s outside.

Heterogeneous bootstrap percolation on $\mathbb{Z}^2$

1	1	0	1	2
5	4	1	2	3
1	2	3	2	1
0	2	4	3	1
3	1	1	1	0

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Coupling with HBP: domination from below

Call a Hamming plane $\{x\} \times K_n^2$ r-internally spanned (r-IS) iff bootstrap percolation with threshold r and initial configuration $\omega_0|_{\{x\}\times K_n^2}$ spans $\{x\} \times K_n^2$.

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Initialize ξ_0 as follows:

- $\xi_0(x) = 0$ if $\{x\} \times K_n^2$ is θ-IS;
- $\xi_0(x) = k \in \{1, 2, 3, 4\}$ if $\{x\} \times K_n^2$ is $(\theta - k)$-IS, but not $(\theta - k + 1)$-IS; and
- $\xi_0(x) = 5$ if $\{x\} \times K_n^2$ is not $(\theta - 4)$-IS.

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Then

$$\bigcup \{\{x\} \times K_n^2 : \xi_\infty(x) = 0\} \subset \omega_\infty.$$

1	1	0	1	2
5	4	1	2	3
1	2	3	2	1
0	2	4	3	1
3	1	1	1	0

Domination from above

The domination in the other direction is similar, except that we have to allow help from nearest neighbors, which introduces small correlations into the initial state.

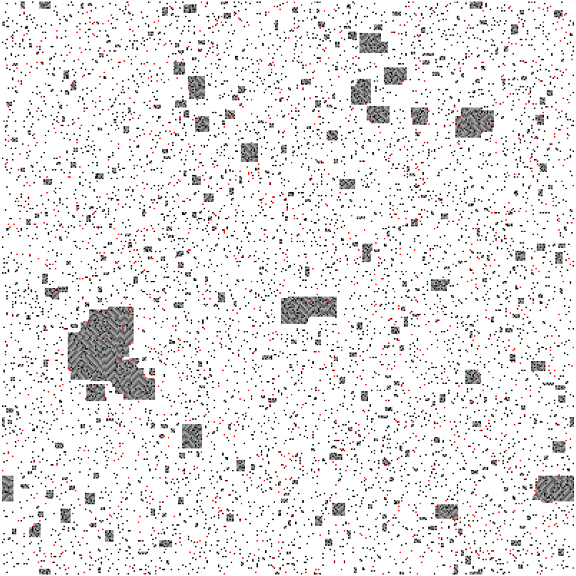
Roughly, 1s perform bootstrap percolation with $\theta = 2$ on 2's, after the obstacles 3s, 4s, and 5s are permanently removed from $\mathbb{Z}^2$. This is poluted bootstrap percolation [G., McDonald, 1997]. Provided the following two densities are *both small*, the transition between large and small final density of 0s occurs when:

$$(\mathbb{P}(\xi_0 = 1))^2 \asymp \mathbb{P}(\xi_0 \geq 3).$$

This is the result of [G., McDonald, 1997], but it does not quite apply to our case, due to initial correlations. Instead, we use the random surface approach of [G., Holroyd, Sivakoff, 2018].

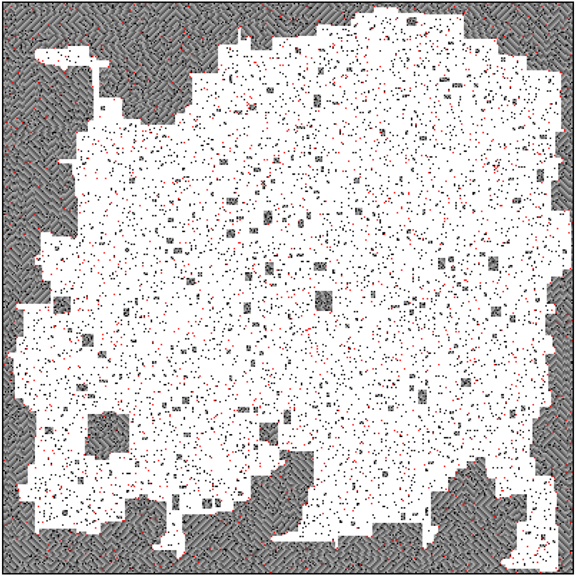
Polluted bootstrap percolation

obst. dens.$=0.01$,
occ. dens.$=0.045$,
final state on
400×400 square
with periodic
boundary



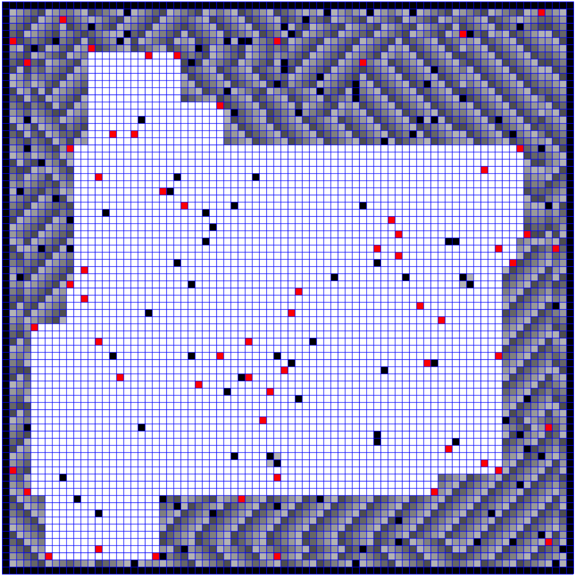
Polluted bootstrap percolation

obst. dens.$= 0.01$,
occ. dens.$= 0.045$,
final state on
400×400 square
with occupied
boundary



Polluted bootstrap percolation

obst. dens.$= 0.01$,
occ. dens.$= 0.02$,
final state on
80×80 square
with occupied
boundary



Initial densities for $\theta = 2\ell + 1$

When $p = a \cdot \frac{1}{n^{1+1/\ell}}$:

$$\mathbb{P}(\xi_0(x) = 0) \asymp \mathbb{P}_p(\{x\} \times K_n^2 \text{ is } \theta\text{-IS}) \qquad \asymp n^{-1/\ell}$$

$$\mathbb{P}(\xi_0(x) = 1) \sim \mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - 1)\text{-IS}) \qquad \sim \left(1 - e^{-a^\ell/\ell!}\right)^2$$

$$\mathbb{P}(\xi_0(x) \geq 3) \sim \mathbb{P}_p(\{x\} \times K_n^2 \text{ is not } (\theta - 2)\text{-IS}) \qquad \sim \exp\left[-\frac{2a^\ell}{\ell!}\right]$$

Recall: K_n^2 is $(2\ell - 1)$-IS about when there is a line with ℓ occupied sites, and expected no. of such lines is about $2n\, n^\ell p^\ell/\ell! \to 2a^\ell/\ell!$; and K_n^2 is (2ℓ)-IS about when there are both horizontal and vertical lines with ℓ occupied sites.

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Recall: K_n^2 is $(2\ell - 1)$-IS about when there is a line with ℓ occupied sites, and expected no. of such lines is about $2n\, n^\ell p^\ell/\ell! \to 2a^\ell/\ell!$; and K_n^2 is (2ℓ)-IS about when there are both horizontal and vertical lines with ℓ occupied sites.

Start HBP ξ_t with states 0, 1, 2, 3, with the above asymptotic probabilities. Essentially, $a_c = \inf\{a : \mathbb{P}(\xi_\infty(0) = 0) > 0\}$. By percolation arguments (as 0s are now rare, the origin only turns to 0 if it is connected to a 0 by a long path of 1s), $a_c \in (0, \infty)$. By monotonicity in a, $\lim_n \mathbb{P}(\omega_\infty(v) = 1) = 0$ for all $a < a_c$.

Initial densities for $\theta = 2\ell + 2$

When $p = a \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}}$.

$$\mathbb{P}(\xi_0(x) = 0) \asymp \mathbb{P}_p(\{x\} \times K_n^2 \text{ is } \theta\text{-IS}) \asymp \frac{(\log n)^{2+2/\ell}}{n^{2/\ell}}$$

$$\mathbb{P}(\xi_0(x) = 1) \asymp \mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta-1)\text{-IS}) \asymp \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}$$

$$\mathbb{P}(\xi_0(x) \geq 3) \sim \mathbb{P}_p(\{x\} \times K_n^2 \text{ is not } (\theta-2)\text{-IS}) \sim 2n^{-a^\ell/\ell!}$$

Expected no. of lines with $\ell + 1$ occ. sites is $\asymp n\, n^{\ell+1} p^{\ell+1}$.
Expected no. of horizontal lines with ℓ occ. sites is $\sim n\, n^\ell p^\ell/\ell! = (a^\ell/\ell!) \log n$.

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$$\mathbb{P}(\xi_0(x) = 1) \asymp \mathbb{P}_p(\{x\} \times K_n^2 \text{ is } (\theta - 1)\text{-IS}) \qquad \asymp \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}$$

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Expected no. of horizontal lines with ℓ occ. sites is $\sim n\, n^\ell p^\ell / \ell! = (a^\ell/\ell!) \log n$.
Phase transition occurs when $\mathbb{P}(\xi_0(x) \geq 3) \approx \mathbb{P}(\xi_0(x) = 1)^2$, giving

$$\frac{a_c^\ell}{\ell!} = \frac{2}{\ell}.$$

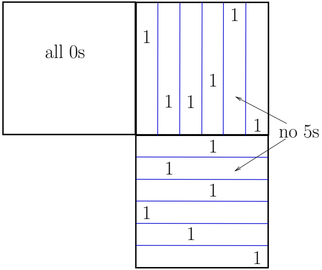
Let $a > a_c$ and switch all initial 3s and 4s to 5s, so

$$\mathbb{P}(\xi_0(x) = 0) > 0$$
$$\mathbb{P}(\xi_0(x) = 1) = \mathrm{p}$$
$$\mathbb{P}(\xi_0(x) = 5) = \mathrm{q}$$
$$\mathbb{P}(\xi_0(x) = 2) = 1 - \mathbb{P}(\xi_0(x) = 0) - \mathbb{P}(\xi_0(x) = 1) - \mathbb{P}(\xi_0(x) = 5),$$

where $\mathrm{q} \leq \mathrm{p}^{2+\epsilon}$ as $(\mathrm{p}, \mathrm{q}) \to (0, 0)$.

Let $N = \frac{2}{\mathrm{p}} \log(1/\mathrm{p})$. An $N \times N$ box is: *good* if it has no 5s and a 1 (or 0) on every row and column;
and *very good* if it has only 0s.

A good box is likely to connect to a very good box:



$$\mathbb{P}(\text{box is not good}) \leq N^2\mathrm{q} + 2N(1-\mathrm{p})^N$$
$$\leq 4\mathrm{p}^{\epsilon}(\log(1/\mathrm{p}))^2 + 2\mathrm{p}\log(1/\mathrm{p})$$

Subcritical regime for $\theta = 2\ell + 2$

Let $a < a_c$ and switch all initial 4s and 5s to 3s and 1s to 0s, so

$$\begin{aligned}
\mathbb{P}(\xi_0(x) = 0) &= \mathrm{p} \\
\mathbb{P}(\xi_0(x) = 3) &= \mathrm{q} \\
\mathbb{P}(\xi_0(x) = 2) &= 1 - \mathrm{p} - \mathrm{q},
\end{aligned}$$

where $\mathrm{q} \gg \mathrm{p}^2$.

Subcritical regime for $\theta = 2\ell + 2$

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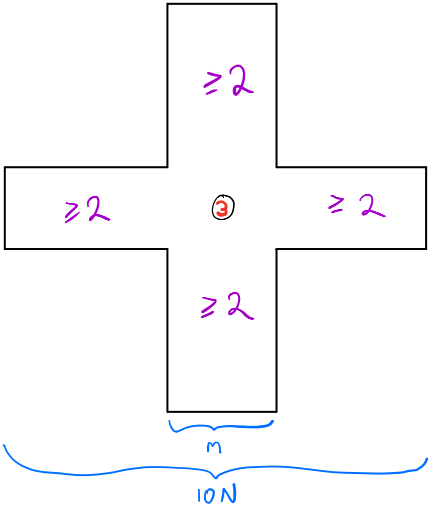
$$\mathbb{P}(\xi_0(x) = 3) = \mathrm{q}$$
$$\mathbb{P}(\xi_0(x) = 2) = 1 - \mathrm{p} - \mathrm{q},$$

where $\mathrm{q} \gg \mathrm{p}^2$.
We show that, with high probability, the origin is in a "protected set," which is never entered by 0s from the outside, and is small enough that 0s inside quickly stop growing.

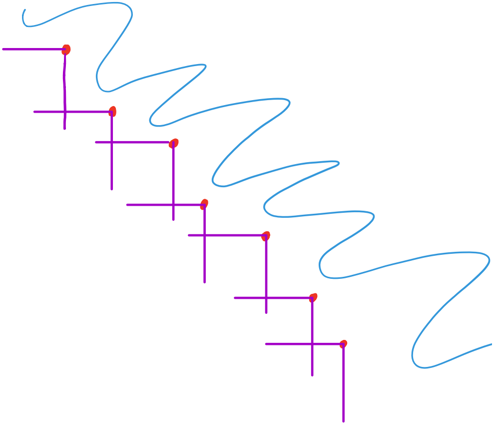
Subcritical regime for $\theta = 2\ell + 2$

The boundary of the protected set consists of nice sites, which are 3s with no 0 nearby (m large constant, $N = \delta/(mp)$):



Subcritical regime for $\theta = 2\ell + 2$

Well-positioned nice sites prevent 0s from spreading.



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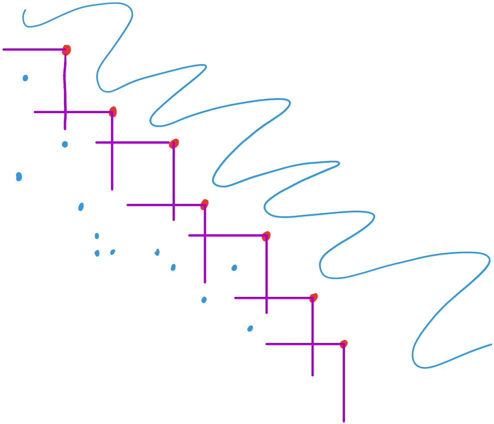
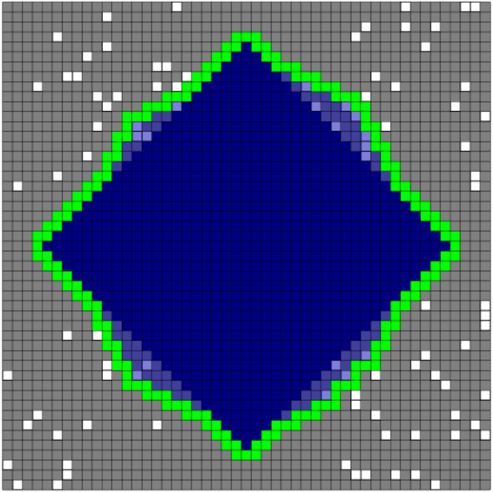
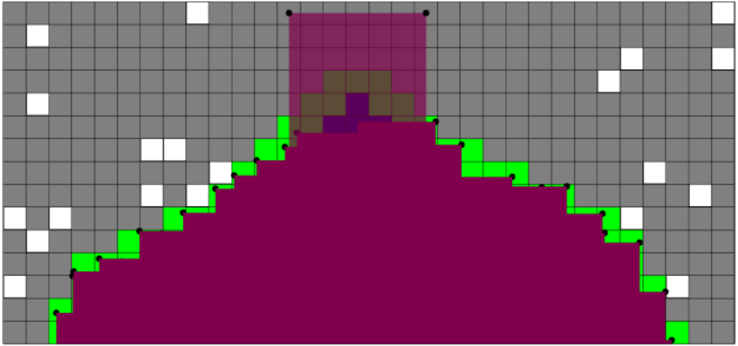


Shell of Good boxes

An $N \times N$ box is *good* if it contains a nice site.

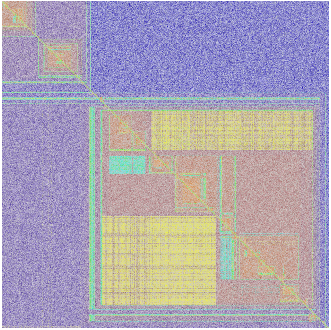


Add “fortresses” of nice sites to protect the corners from “leaking.”



Another long-range growth process: graph bootstrap percolation

Add edges of the complete graph according to some monotone rule. These dynamics were introduced by [Balogh, Bollobás, Morris, 2012]. Subsequent work by [Kolesnik, 2018], [Andjel, Kolesnik, 2018], [G., Kolesnik, 2019].



Thank you!

More on my web page
https://www.math.ucdavis.edu/∼gravner/papers/