Lecture 2:
Toom's method and applications

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Toom's method.

Take a deterministic CA $\xi_t \in \{0,1\}^{\mathbf{Z}^2}$, 1's=occupied sites, 0's=empty sites. Assume that $\xi_0 \equiv 1$ implies $\xi_1 \equiv 1$. The simplest assumption is the following:

(T1) There exists a fixed line $\ell \subset \mathbf{R}^2$ so that $\xi_t(0) = 0$ implies that there exist $x_1, x_2 \in \mathbf{Z}^2$ strictly separated by ℓ so that $\xi_{t-1}(x_1) = \xi_{t-1}(x_2) = 0$.

Introduce random errors into this rule: after ξ_t is computed according to the deterministic rule, each site x is independently turned to 0 with probability $1-p$. We say that (x,t) is an *error site*.

Toom's Theorem. *Assume (T1) holds. If $\xi_t(x) = 0$, then there exists a graph $G = G(x,t)$, whose vertex set is included in $\{(z,s) : s \le t, z \in \mathbf{Z}^2\}$, and which satisfies the following properties, uniformly in (x,t):*

(1) Number of possible graphs G with m edges is bounded by C^m,

(2) For a graph G with m edges, at least m/C vertices are error sites.

The classic application.

Consider the threshold growth model with von-Neumann neighborhood, started from all 1's with error probability $1-p$. Is the initial state *stable* in the sense that

$$P(\xi_t(x) = 1) \to 1, \qquad \text{as } p \to 1,$$

uniformly in t?

If $\theta = 1$, this is the oriented percolation dynamics. If $\xi_t(x) = 0$, then either (x, t) is an error site or else all neighbors of x are 0 at time $t-1$. So the standard Peierls argument on (a one-dimensional slice of) space$\times$ time proves stability. The same is true for any supercritical growth model. (This is the case when $w > 0$ or, equivalently, when a finite set can grow.)

If $\theta = 3$, then a 2×2 block of 0's will never be destroyed and so

$$\lim_{t\to\infty} P(\xi_t(x) = 1) = 0$$

and stability does not hold. This is the property of subcritical growth models, when $w \equiv 0$ and a finite hole cannot be filled.

Neither of these two techniques applies to the $\theta = 2$ case. This is a *critical* rule, in which no finite set of 1's can grow (because $w(u) =$ for some u) but any finite hole can be filled (because $w(\pm u) > 0$ for some u – recall that we assume in general that the rule is symmetric). However, in the deterministic (error–free) case, $\xi_t(0) = 0$ implies that there is a 0 on either side of some line ℓ, in this example given by $\ell = \{(x, y) : y = x\}$. Thus stability follows.

Perturbation of deterministic shapes.

Now consider a random TGM, with parameters $\mathcal{N}$ and θ, in which A_t is a randomly growing set ($A_t \subset A_{t+1}$) and $x \notin A_t$ joins A_{t+1} with probability $p \cdot 1_{\{|(x+\mathcal{N})\cap A_t|\geq\theta\}}$.

Let

$$L_p = \lim_{t\to\infty} \frac{A_t}{t} = K^*_{1/w_p}$$

be the asymptotic shape started from a large enough finite set. What are the properties of L_p for p close to 1?

A classic result (Durrett–Liggett, 1981) states that, for the *additive* (that is, $\theta = 1$) nearest neighbor TGM,

(1) $L_p \to L_1$ as $p \to 1$ and has a flat edge in the diagonal direction as soon as p is close enough to 1.

(2) However, L_p is for $p < 1$ *not* equal to L_1 due to the fact that its extent in coordinate direction is below 1.

The reason for (1) is that the growth at the boundary of tL_1 does exactly 1d oriented percolation, which survives for large enough p. This part can be generalized to arbitrary supercritical TGM, by rescaling: from a single copy of the invariant shape τL (τ fixed), will produce $2\tau L$ after time τ with large probability. It follows that a supercritical 1-dependent oriented percolation can be embedded into the edge dynamics. For p close enough to 1, then, L_p retains a portion of every flat edge of L_1.

Is it possible that $L_p = L_1$? We call such property *exact stability*.

Exact stability.

This holds if the TGM is as far from additive as possible (G–Griffeath, 2002). Note that for additive TGM $K_{1/w} = \mathcal{N}^*$ is convex. Rules other than additive have convex $K_{1/w}$; this property simplifies many interactions. Accordingly, such rules are called *quasi–additive*. An example: Moore neighborhood with $\theta = 2$.

Let $\partial K' = \partial(K_{1/w}) \cap \partial(\mathrm{co}K_{1/w})$.

Theorem. *Consider a supercritical TGM. There are three possibilities:*

Case 1. *$\partial K'$ consists of isolated points, no three of which are on the same line.*

Then, for p close enough to 1, $L_p = L_1$. Moreover, convergence to L_1 is tight: for any $\epsilon > 0$, there exists an M so that any boundary point of tL_1 is within M of A_t with probability $1-\epsilon$, uniformly in t. Finally, with prob. 1, $(t - C\log t)L_1 \cap \mathbf{Z}^2 \subset A_t$ eventually, for large enough C.

Case 2. *$\partial K'$ consists of isolated points, three of which lie on some line.*

The deterministic shape L_1 is still strictly stable, the a.s. deviations are still logarithmic. However, tightness no longer holds: a corner of tL_1 is at distance at least $c \log t$ from A_t, a.s.

Case 3. *$\partial K'$ includes a line segment.*

Then $L_p \neq L_1$ for every $p < 1$.

Moore neighborhood TGM:

Case 1: $\theta = 3$.

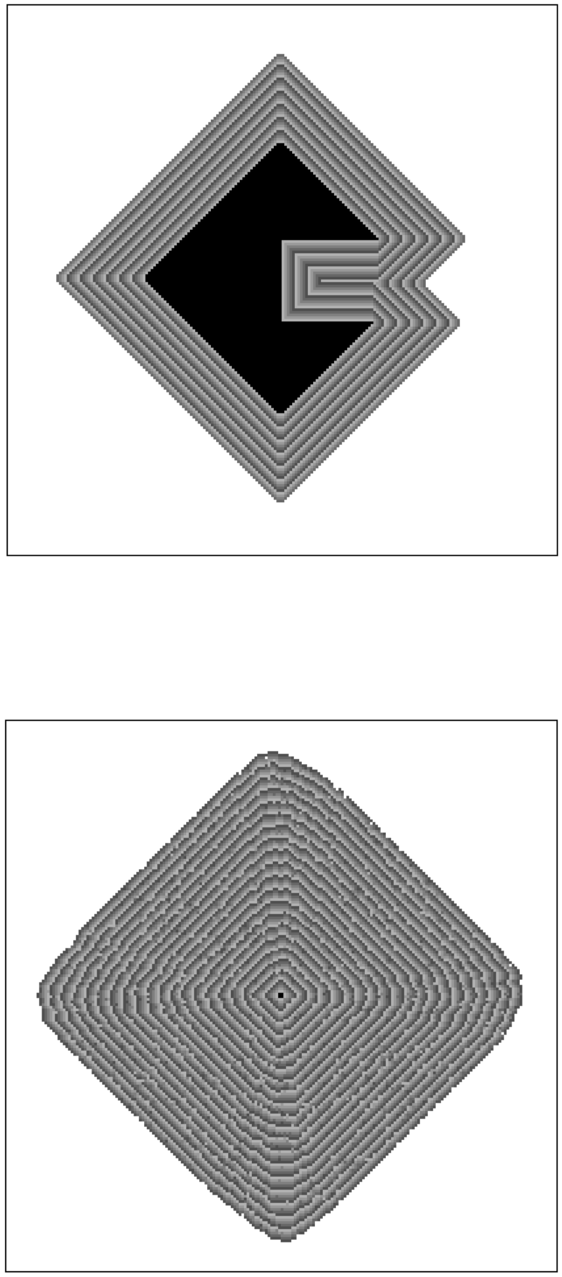
Case 3: $\theta = 1, 2$.

Range 2 TGM:

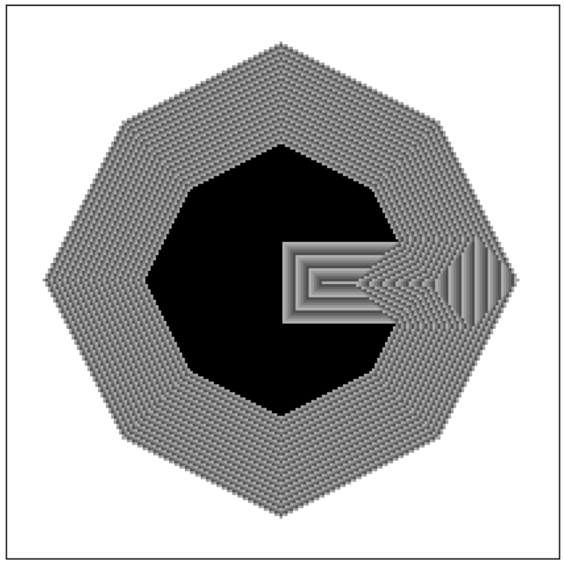
Case 1: $\theta = 4, 6$.

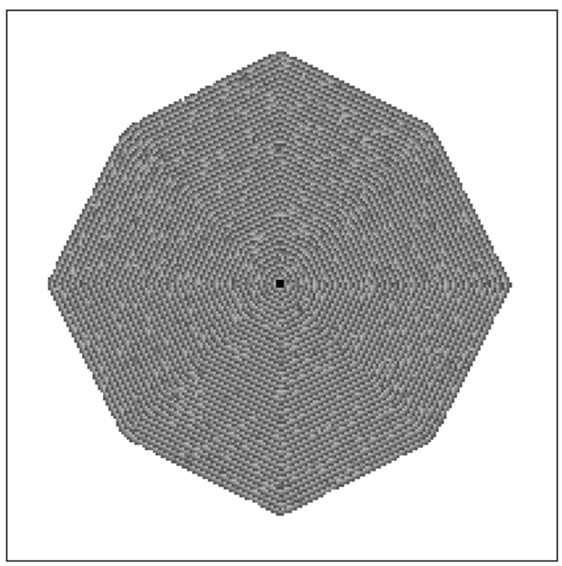
Case 2: $\theta = 7, 9, 10$.

Case 3: $\theta = 1, 2, 3, 5, 8$.



Moore neighborhood, $\theta = 2$. Top figure: perturbation of the invariant set in the deterministic TGM. Bottom figure: random TGM with $p = 0.9$.





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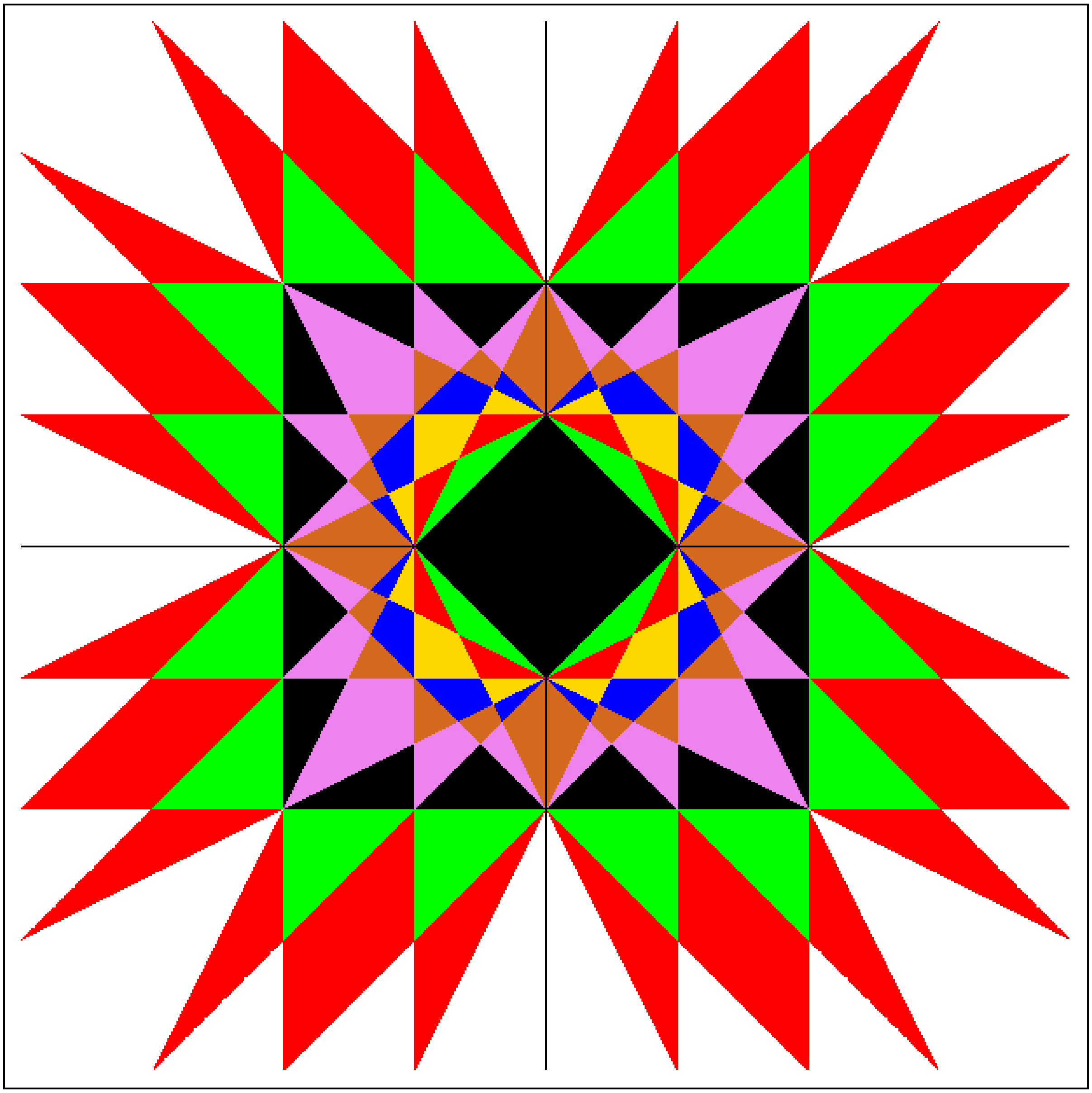
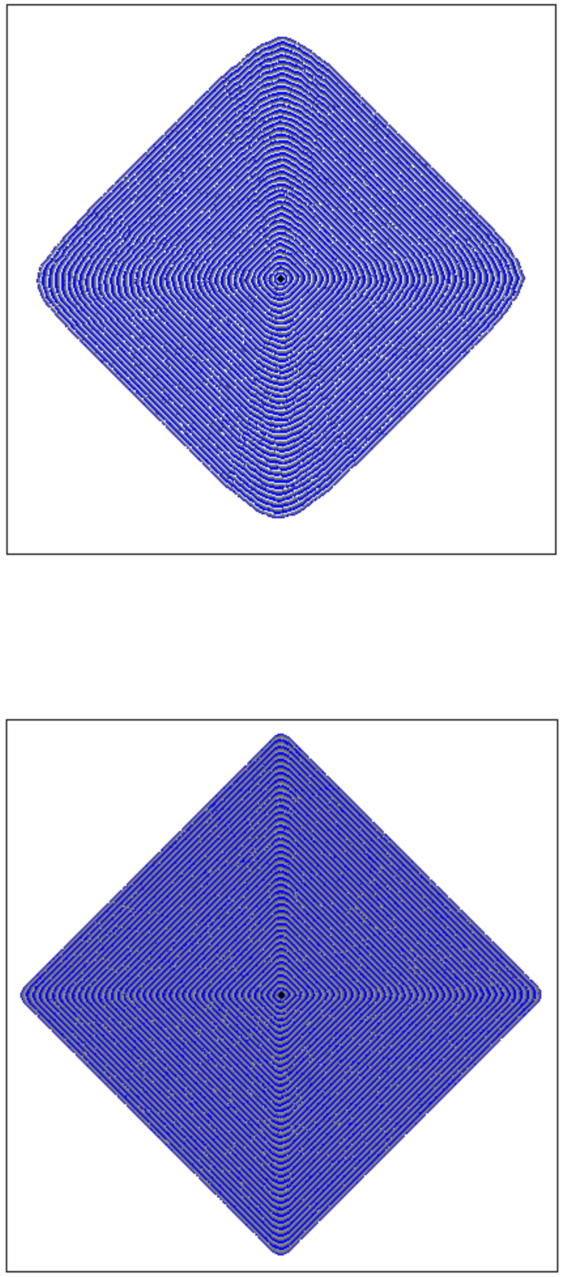


$K_{1/w}$ for range 2 TGM, $\theta = 1, 2, \ldots, 10$.



A comparison of two range 2 random TGM, with $p = 0.95$, at time 200. Top figure: $\theta = 8$. Bottom figure: $\theta = 7$.

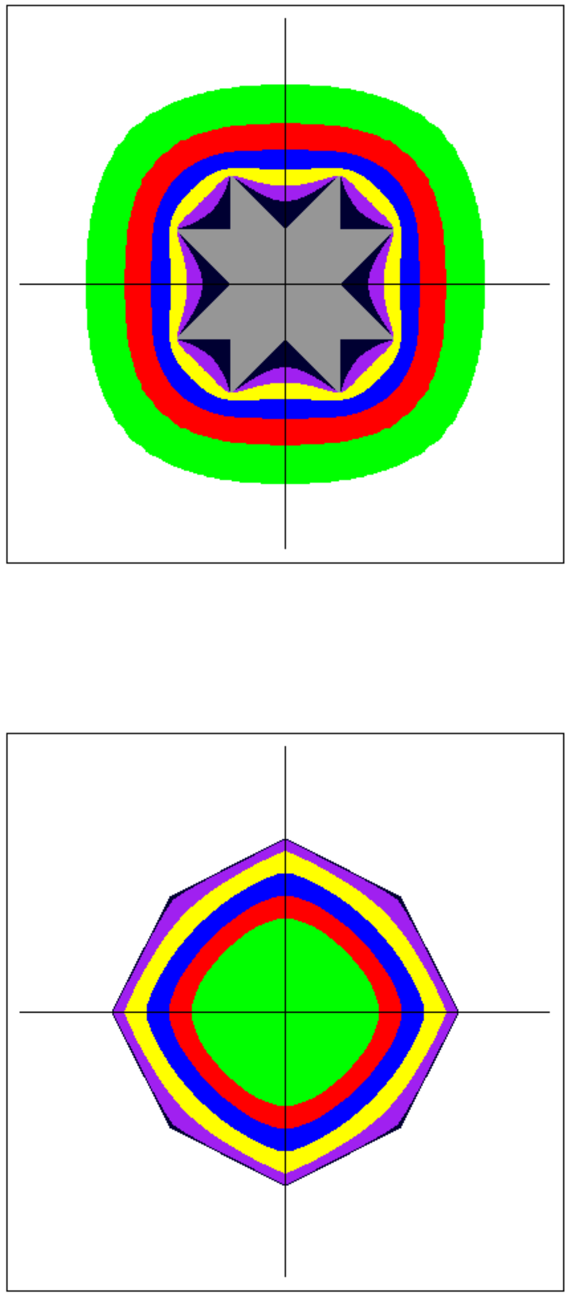
Reasons for exact stability.

In Case 1, the deterministic TGM is able to fix (or "erode") any finite perturbation. Therefore, the random dynamics on infinite wedges determined by corners of L_1 can be favorably compared to a Toom rule.

The corners can be patched together by an oriented percolation comparison in the middle of the edges.

In Case 2, the mistake–fixing property still holds, but for wider wedges than in Case 1. So, the corners must be rounded off before a Toom comparison can work.

In Case 3, a corner must lag behind the deterministic growth, because an appropriate half–space does so.

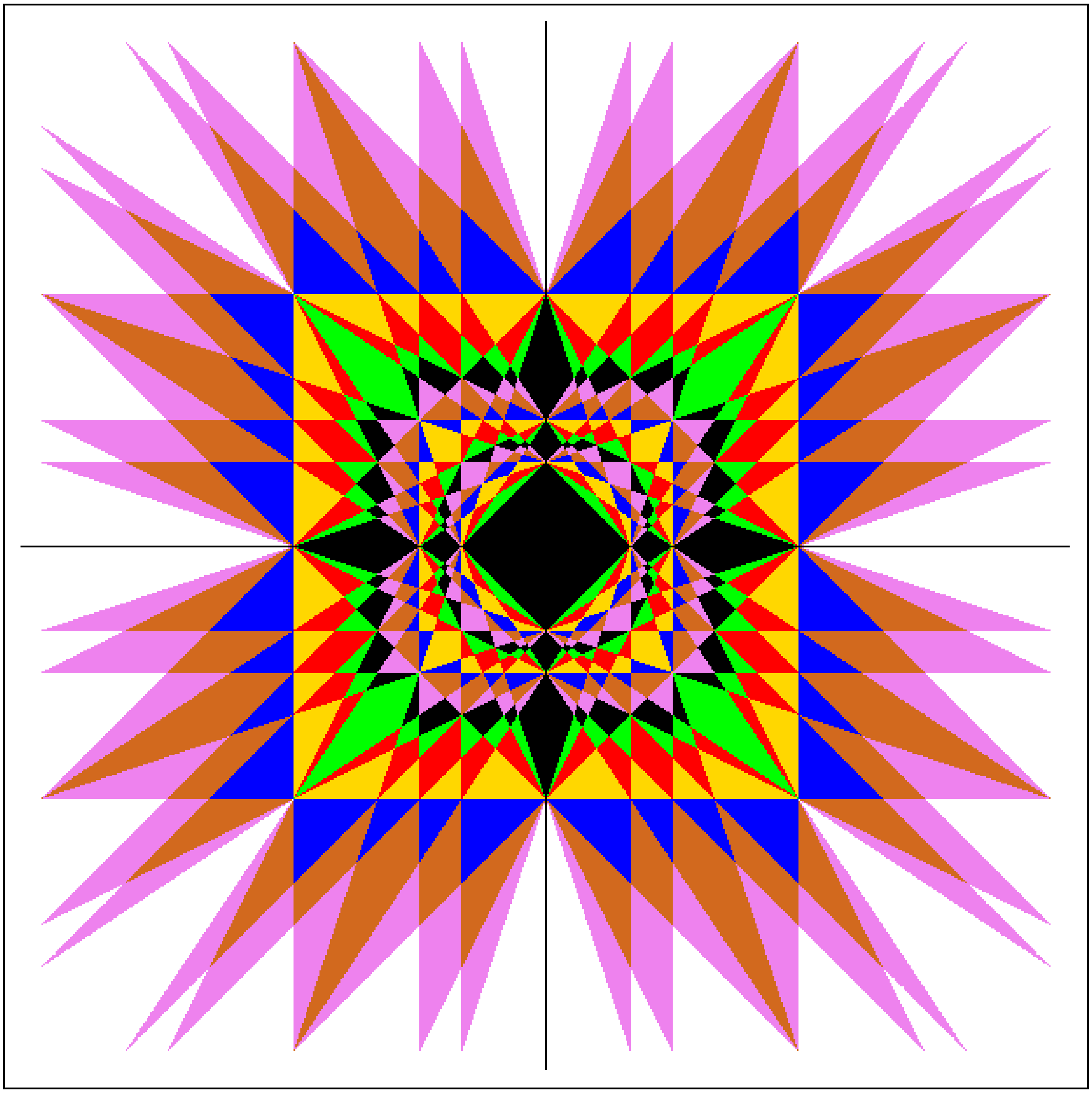


K_{1/w_p} and L_p for $p = 1, \ldots, 0.4$.

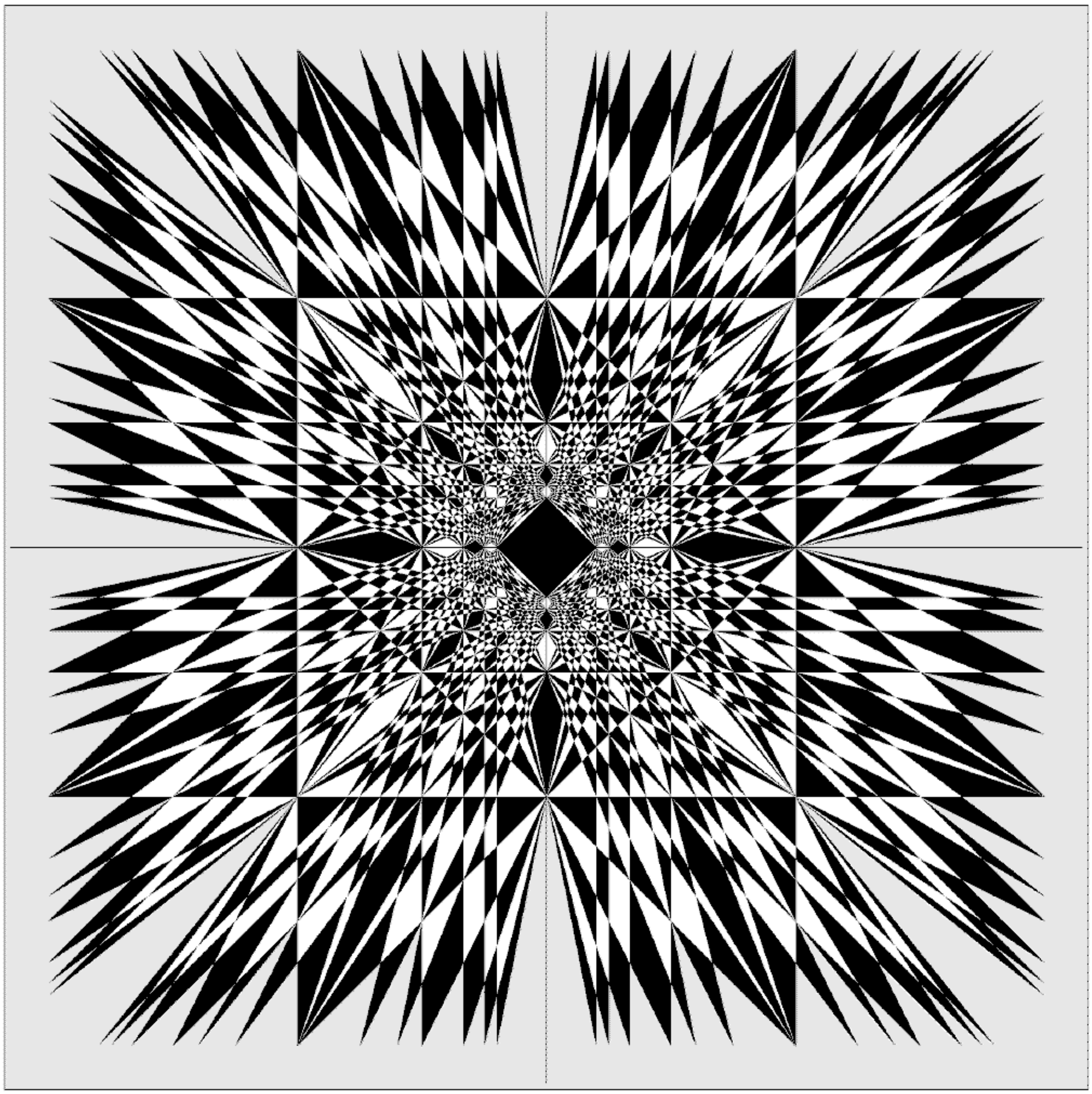
Preponderance of exact stability.

For large neighborhoods $\mathcal{N}_\rho$, we have $\rho(2\rho+1)$ shapes (as this is the largest supercritical θ). How many are exactly stable?

It turns out that *exact stability holds with asymptotic proportion 1*, in fact, the number of cases which are not exactly stable is $O(\rho^2/\log^c \rho)$, for a small $c \approx 0.18$. At the heart of this is a result of Erdös, which states that the cardinality of $\{ab : 1 \leq a, b \leq \rho\}$, is of the above order.



$K_{1/w}$ for range 3 TGM, $\theta = 1, 2, \ldots, 21$.



$K_{1/w}$ for range 5 TGM, $\theta = 1, 2, \ldots, 55$.

Open problems.

1. Consider a growth model with *imposed* interface coherence. Assume A_0 are the sites on or below the x–axis. First, every occupied site at time t (i.e., a member of A_t) with a vacant nearest neighbor becomes vacant independently with probability p. Second, to maintain coherence, remove any occupied sites with a vacant site anywhere directly below them. The third step fills in any site with 2 or more occupied neighbors.

Is the asymptotic speed of this interface exactly 0 for p close to 1? Note that this is not a local rule, so Toom's theorem does not apply, but a modification due to Bramson and Gray might.

2. Does arbitrary TGM have convex K_{1/w_p} when p is small? As of now, there is no technique for demonstrating convexity of K_{1/w_p} in any non–additive case.