LECTURE 3:

An exactly solvable growth model

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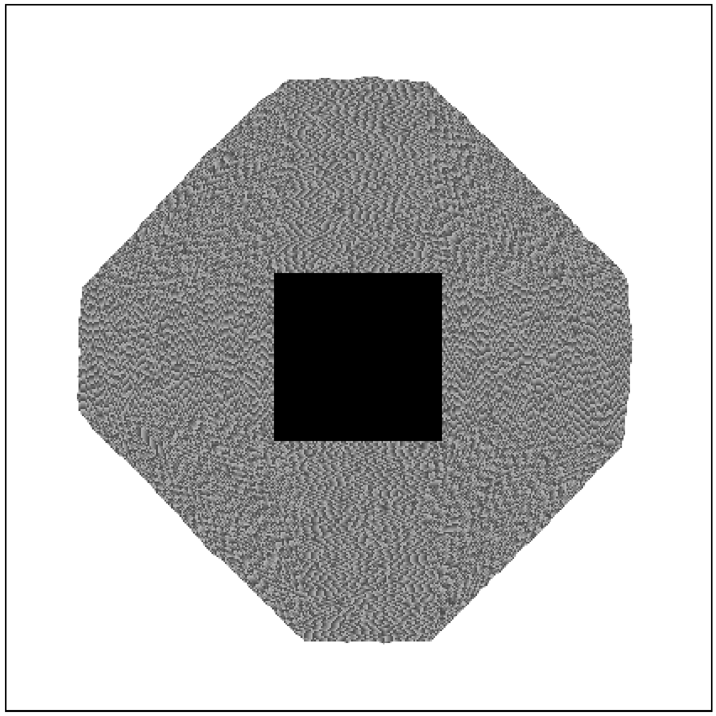
Approximations for small update probabilities.

Consider range 1 box threshold 3 TGM, with $p_i = p$, $i \geq 3$. What happens with the shape L_p as $p \to 0$? It turns out that $p^{-1}L_p$ converges to the shape of the continuous time dynamics, about which little is known.

So instead we take $p_3 = p$, and $p_4 = 1$. The exact stability near $p = 1$ obviously still holds. What is now the geometry for $p \to 0$? (A similar problem was introduced by Kesten–Schonmann, 1995.)

Coordinate and diagonal directions u are the only ones for which $w_p(u) \to 0$. Those are, therefore, the only ones likely to be "seen" in the limit. In fact, the following precise result can be proved.

Theorem. *As* $p \to 0$, $p^{-1/2}L_p \to \{x \in \mathbf{R}^2, ||x||_1 \leq \sqrt{2}\}$.



TGM from the previous page, with $p_3 = 0.01$, started from 100×100 square, run until time 1050.

Why can the limiting set be precisely identified? Consider $A_0 = H_{e_2}^-$. We wish to determine the height $h_t = h_t(0)$ above the origin. As this rule generates a coherent interface, the advance above a site x from time t to $t+1$ may happen deterministically (as $p_4 = 1$) or based on the outcome of a Bernoulli (p) r.v. $\epsilon_{x,t}$. Then call any space-time site (x,t) with $\epsilon_{x,t} = 1$ *marked*. Only marked sites inside the triangle with vertices $(0, t-1)$, $(\pm(t-1), 0)$ influence $h_t(0)$. If $t = \tau/\sqrt{p}$ and the triangle is rescaled by $\sqrt{p}$, then the position of the marked sites converges to the Poisson point location $\mathcal{P}$ with intensity 1. In the limiting dynamics, a point above $x \in \mathbf{R}$ advances by 1 at times t such that $(x,t) \in \mathcal{P}$, creating a thin stalk which expands in both directions at speed 1. This is a version of the *Hammersley's process*.

Under this rule, $h_t(0)$ is given exactly by the length of the *longest path*, i.e., the piecewise linear path which connects $(0,t)$ and the x–axis, on which all line segments have absolute slopes at least 1, and connect the most points in $\mathcal{P}$. The longest such path which connects $(0,t)$ and $(0,0)$ is, conditioned on the number of points in $\mathcal{P}$ inside the square being m, exactly the length of the longest increasing subsequence in a random permutation of $1,\ldots,m$. This length has a.s. asymptotics $2\sqrt{m}$ (Logan–Shepp, Vershik–Kerov, 1977; Aldous–Diaconis, 1995). It follows that $h_t(0) \sim 2\sqrt{t^2/2} = \sqrt{2}t$.

Consequently, $w_p(e_2) \sim \sqrt{2}\sqrt{p}$ as $p \to 0$ and a similar argument computes the asymptotics in the diagonal direction.

Also, note that K_{1/w_p} is non–convex for small p, thus the shape L_p must have *at least one corner* for small p. (A flat edge for small p is very unlikely, but not ruled out at this point.)

An exactly solvable TGM

Consider the TGM with

$$x + \mathcal{N} = \begin{array}{cc} \bullet & x \\ & \bullet \end{array}$$

$\theta = 1$, $p_1 = p$, and $p_2 = 1$. This TGM is called *oriented digital boiling (ODB)*.

This model (Seppäläinen 1998; Johansson, 1999; G–Tracy–Widom, 2000) is exactly solvable when $A_0 = \{(x, y) \in \mathbf{Z}^2 : x \geq 0, y \leq -x\}$.

As before, let $h_t(x)$ be the height above x so that $A_t = \{y \leq h_t(x)\}$. Also, toss the p–coins in advance to get independent Bernoulli random variables $\epsilon_{x,t}$, $x \geq 0, t \geq 0$ and mark the points (x, t) for which $e_{x,t} = 1$. Then

$$h_t(x) = \max\{h_{t-1}(x-1), h_{t-1}(x) + \epsilon_{x,t-1}\}.$$

Path description.

For a space–time point $(x, t), x \leq t$, its backwards light-cone is

$$\mathcal{L}(x,t) = \{(x',t') : 0 \leq x' \leq x, x' \leq t' < x' + t - x\}$$

and H is the longest sequence $(x_1, t_1), \ldots, (x_k, t_k)$ which

(1) consists of marked points,

(2) $x_{i-1} \leq x_i$, and

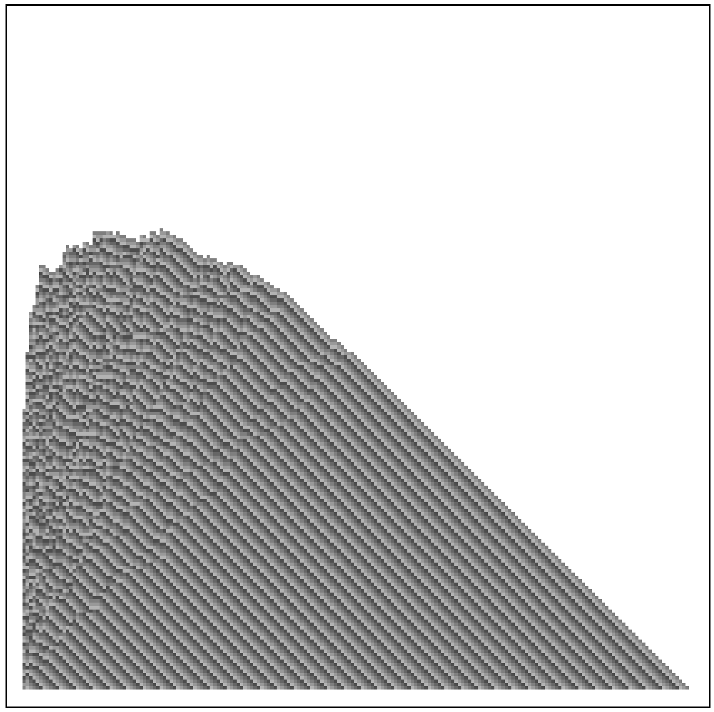
(3) $x_i - x_{i-1} + 1 \leq t_i - t_{i-1}$.

Alternatively, let $m = t - x$ and $n = x + 1$, and A a random $m \times n$ matrix with independent Bernoulli (p) entries. Label columns as usual, but rows started at the bottom. Then $H = H(m,n)$ is the largest sequence of 1's in A, with

(1) column index non–decreasing,

(2) row index strictly increasing.

Lemma. *For $x < t$, $h_t(x) = H(m,n)$.*



ODB, with $p = 0.5$, started from the wedge.

Shape and fluctuations.

Let A_t be given by $\{y \le h_t(x)\}$. There are four asymptotic regimes.

(1) *Square–root regime.* Keep x fixed and let $t \to \infty$. Then

$$\frac{h_t(x) - pt}{\sqrt{p(1-p)t}} \xrightarrow{d} M_x,$$

a functional of the $n = x + 1$ dimensional Brownian motion $(B_0, \ldots, B_x)$:

$$M_x = \max\{B_0(t_0) + B_1(t_1) - B_1(t_0) + \cdots + B_x(t_x) - B_x(t_{x-1}) : \\ 0 \le t_0 \le t_1 \le \cdots \le t_x = 1\}.$$

Its distribution is given by

$$P(M_x \le s) = c_n^{-1} \int_{y \in (-\infty, s]^n} \Delta(y)^2 e^{-\frac{1}{2}||y||_2^2} \, dy,$$

where

$$c_n = 1!2!\ldots n!(2\pi)^{n/2}, \quad \Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

(2) *Universal regime.* Assume $\gamma = x/t < 1 - p$ and let

$$
\begin{aligned}
c_1 =& 2(1-\gamma)p - p + 2\sqrt{p\gamma(1-p)(1-\gamma)},\\
c_2 =& \left(\gamma(1-\gamma)\right)^{1/6}\left(p(1-p)\right)^{1/2}\\
& \left(1 + \sqrt{(1-p)\gamma p^{-1}(1-\gamma)^{-1}}\right)^{2/3}\\
& \left(\sqrt{(1-\gamma)\gamma^{-1}} - \sqrt{p(1-p)^{-1}}\right)^{2/3}.
\end{aligned}
$$

Then

$$
P\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} \leq s\right) \to F_2(s),
$$

where

$$
F_2(s) = \exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right)
$$

and q solves

$$
q'' = sq + 2q^3, \quad q \sim Ai(s) \text{ as } s \to \infty.
$$

(3) *Critical regime.* If $x/t = (1-p) + o(t^{-1/2})$, then $h_t(x) - (t-x)$ converges in distribution: for any fixed $k \in \mathbf{Z}_+$,

$$P(h_t(x) - (t-x) \leq -k)$$

converges to a $k \times k$ determinant. The interface is *tight* in this direction.

(4) *Deterministic regime.* If $\gamma = x/t > 1-p$ then $P(h_t(x) = t-x)$ converges to 1 exponentially fast.

This is simply because the leftmost point of the -45 degree edge does a random walk: it moves up with prob. p and to the right with prob. $1-p$.

Main steps in proving the theorem.

Step 1. Combinatorics and algebra. The dual RSK algorithm, Gessel's theorem (1990) and Borodin–Okounkov identity (1999) establishes a connection between 01–matrices and determinants of operators, the final result being

$$P(h_t(x) \leq h) = \det(I - K_h),$$

where $K_h : \ell^2 \to \ell^2$ is the product of two matrices, given by (j,k)–entries

$$a^+_{jk}(h) = \frac{1}{2\pi i} \int (1+rz)^n (z-1)^m z^{-m+h+j+k}\, dz,$$
$$a^-_{jk}(h) = \frac{1}{2\pi i} \int (1+rz)^{-n} (z-1)^{-m} z^{m-h-j-k-2}\, dz.$$

The contours for both integrals go around the origin once counterclockwise; in the second integral 1 is inside and $-1/r$ is outside.

Step 2. Analysis. In the universal regime, we take $h = cm + sm^{1/3}$, $j = m^{1/3}x$, $k = m^{1/3}y$. Then, e.g.,

$$a^+_{jk}(h) = \frac{1}{2\pi i}\int \psi(z)(-z)^{m^{1/3}(x+y+s)}\,dz.$$

The asymptotics of the integrals are computed by the steepest descent method. To get a nontrivial limit, we need to choose $c = c_1$ so that the $\frac{d}{dz}\log\psi(z)$ has a double 0, the third derivative of $\log\psi(z)$ then determines c_2. The limit is another Fredholm determinant, of an operator on $L^2[0,1]$ with the Airy kernel.

The main technical effort is in establishing trace–class convergence of the approximations.

Connections with random matrices.

The Gaussian Unitary Ensemble (GUE) is a random $n \times n$ Hermitian matrix where entries are i.i.d. normal, real and complex part of the entries above the diagonal have variance 1, and diagonal entries have variance 2. Such random matrices form a unique measure on Hermitian matrices invariant under unitary transformations. (Introduced by Wigner, 1950s). It turns out that:

(1) There exists an explicit formula for the distribution of the largest eigenvalue $\lambda_{\max}$ of such matrix: it equals the distribution of M_x!

(2) The largest eigenvalue $\lambda_{\max}$ obeys the limit law

$$P((\lambda_{\max} - 2\sqrt{n}) \cdot n^{1/6} \leq s) \to F_2(s),$$

as $s \to \infty$.

There does not seem to be any intuitive connection between largest eigenvalues and increasing paths – the formulas just turn out the same. The convergence theorems and explicit formulas are results of Tracy–Widom, established through many papers in 1990s.

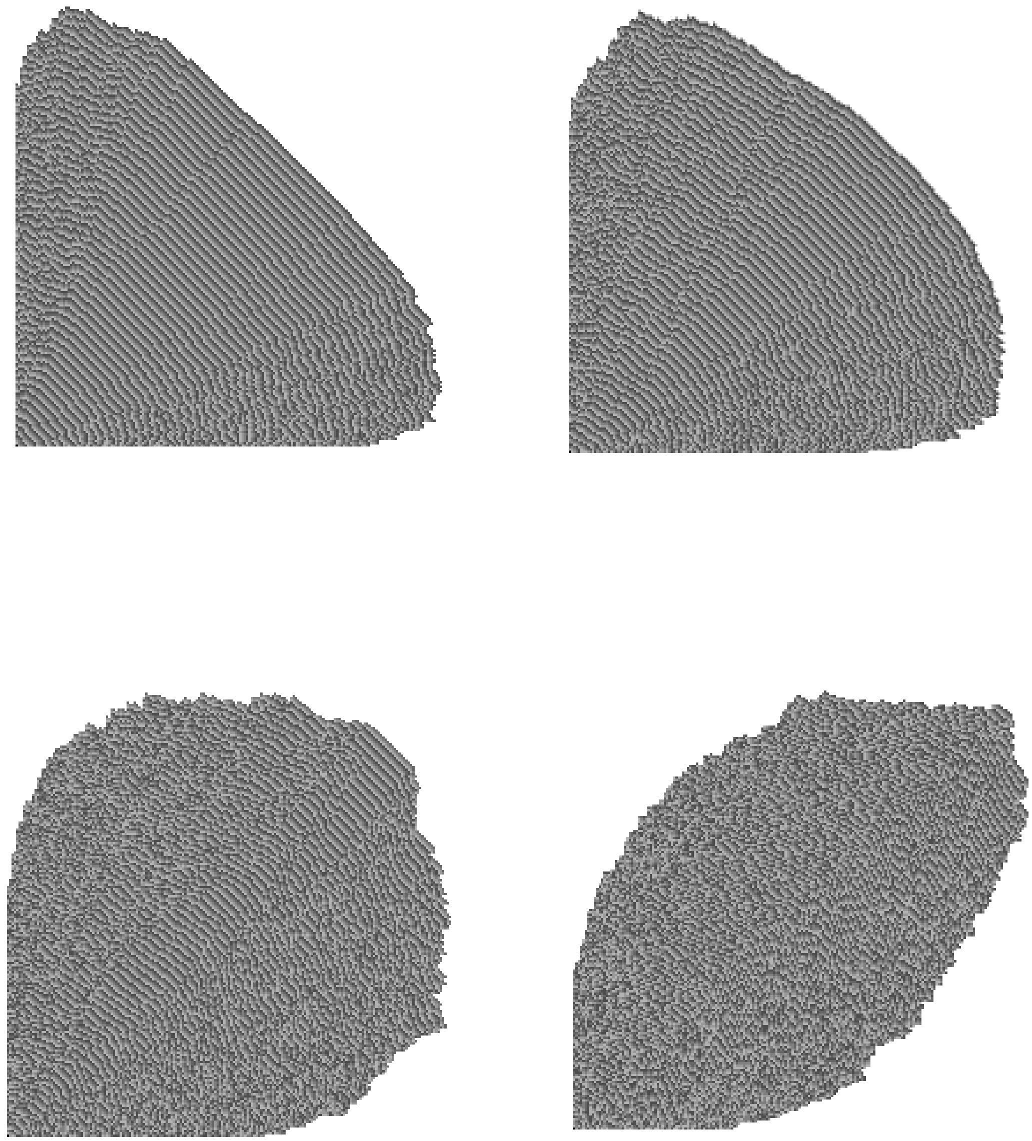
Shape of ODB from finite sets.

Assume (without loss of generality) that $A_0 = \{0\}$.

Then the shape L_p is simply given as the intersection of L'_p and L'_p reflected over the line $y = x$. Properties of L_p inside the the first quadrant:

- ∂L_p has a flat edge, but no corners, when $p > 1/2$,
- ∂L_p is smooth and strictly convex when $p = 1/2$,
- ∂L_p has a corner (in the diagonal direction), but no flat edge when $p < 1/2$,
- $p^{-1/2} L_p \to$
  $$\{(x, y) \in \mathbf{R}^2 : x^2 \leq 4y(1-y), y^2 \leq 4x(1-x)\}$$
  as $p \to 0$.

The fluctuations in any direction $\gamma = x/t$ can be completely described, *except in the diagonal direction when* $p \leq 1/2$.



ODB shapes for $p = 0.6, 0.5, 0.3, 0.1$.

Open problems.

1. Describe the fluctuations on the boundary of A_t in the diagonal direction. Such fluctuations are not known for *any* local growth model in a direction where the asymptotic shape has a kink.

2. Can the asymptotic shapes be explicitly computed for any non–oriented growth model, such as our first one?