LECTURE 4:

# Critical growth in random environment

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## Definition.

A set of *occupied points* is a subset of two–dimensional lattice $\mathbf{Z}^2$. Points in this set, which increases over time, will be labeled as 1's. *Obstacles*, labeled as 2's, form another subset of $\mathbf{Z}^2$. The remaining *empty* sites are labeled as 0's.

Set–up: 1's grow according to some local cellular automaton (CA) rule, but only on 0's. Once a point is occupied, it remains so forever. The set of obstacles never changes.

For concreteness, we assume that the CA rule is a nearest neighbor TGM. The state of this process will be denoted by $\xi_t \in \{0,1,2\}^{\mathbf{Z}^2}$. The rule:

- If $\xi_t(x) > 0$, then $\xi_{t+1}(x) = \xi_t(x)$.
- If $\xi_t(x) = 0$, and $|\{\xi_t = 1\} \cap (x + \mathcal{N})| \geq \theta$, then $\xi_{t+1}(x) = 1$.
- Otherwise, $\xi_{t+1}(x) = 0$.

The initial state $\xi_0$ is a product measure with small $p = P(\xi_0(x) = 1)$ and $q = P(\xi_0(x) = 2)$. The final state, defined pointwise, is $\xi_\infty$.

## The main question.

What are the relative sizes of $p$ and $q$ if most sites become 1 with high probability as $p, q \to 0$? Presumably, $q$ would have to decrease sufficiently fast, compared to $p$.

If $\theta = 1$, then, if $q$ is small, $0's$ and $1's$ together form an infinite connected (through nearest neighbor paths) cluster. Even a single 1 in this cluster will eventually paint it all 1. Thus

$$\lim_{q \to 0} P(\xi_\infty(x) = 1) = 1,$$

as long as $p = p(q) > 0$, so there is no scaling between $p$ and $q$. This is true for all supercritical growth CA.

If $\theta = 3$, no $2 \times 2$ block without a 1 can be changed and so, even if $q = 0$,

$$\lim_{p \to 0} P(\xi_\infty(x) = 1) = 0.$$

## Bootstrap percolation (BP).

If $\theta = 2$, no finite set of 1's can grow, but we know that in the absence of 2's any hole in a sea of 1's can be filled.

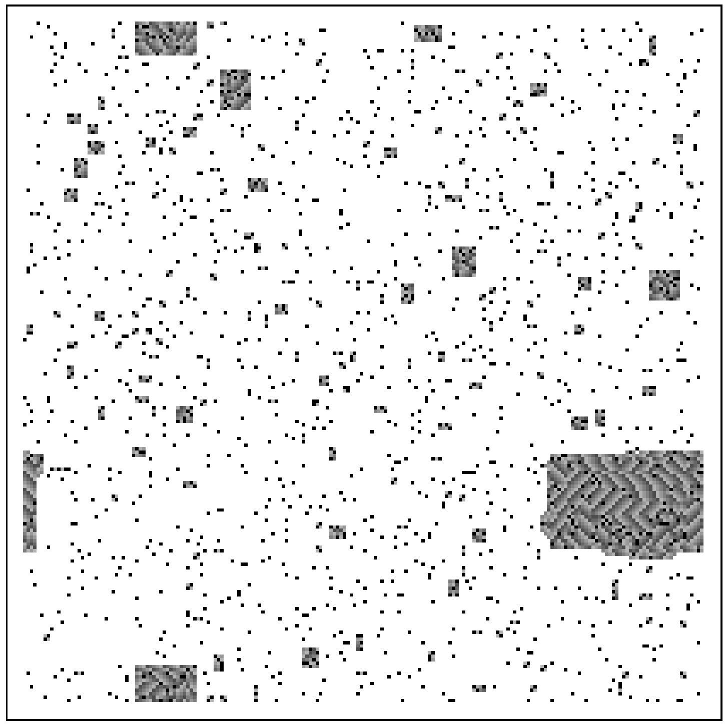
First question: for $q = 0$ and $p > 0$ small, do 1's grow substantially? The configuration below (centered, say, at $x$) fills the circumscribed $7 \times 7$ square.

$$
\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & & & 1 & & \\
 & 1 & & 1 & & & \\
 & & 1 & 1 & 1 & 1 & \\
1 & & & 1 & & & 1 \\
 & & 1 & & & & \\
 & & & 1 & & &
\end{array}
$$

Call any configuration centered at $x$, such that there is a 1 in every face of every ring, a *nucleus* at $x$. A nucleus anywhere fills $\mathbf{Z}^2$ with 1's. The probability that this happens at a fixed $x$ is

$$p \cdot p^4 \cdot (1 - (1-p)^3)^4 \cdot (1 - (1-p)^5)^4 \cdot \dots > 0,$$

therefore $P(\xi_\infty \equiv 1) > 0$ and by the ergodic theorem $P(\xi_\infty \equiv 1) = 1$ (van Enter, 1988).



Bootstrap percolation without obstacles, on $200 \times 200$ box with periodic boundary and $p = 0.04$.

## Finite boxes.

Consider the BP CA on an $L \times L$ box with free (i.e., 0) boundary, started with density $p$ of 1's. This box is *internally spanned* if every site eventually becomes occupied, i.e. $\xi_\infty \equiv 1$.

A celebrated result be Aizenman and Lebowitz (1988) is that there exist constants $0 < c_1 \le c_2 < \infty$ so that

If $L > e^{c_2/p}$, then $P(\xi_\infty \equiv 1) \to 1$ as $p \to 0$.

If $L < e^{c_1/p}$, then, for every $x$, $P(\xi_\infty(x) = 1) \to 0$ as $p \to 0$.

Sketch of the proof:

$$\lim_{p\to 0} p \cdot \log P(x \text{ is a nucleus}) = \lim_{p\to 0} 4p \sum_{k=0}^{\infty} \log(1 - e^{-2kp})$$

$$= \lim_{p\to 0} 4 \int_0^{\infty} \log(1 - e^{-2x}) dx = -\frac{\pi^2}{3}$$

After a rescaling argument, this demonstrates that $c_2$ can be taken to be (anything larger than) $\pi^2/3 \approx 3.3$.

If a rectangle $R$ of size $a \times b$, $a < b$, is internally spanned, then for every $b' < b$ there is an internally spanned sub-rectangle with longest side between $b'$ and $2b'$.

To show this, start with a collection of occupied sites. Given a collection of rectangles, find two which are separated by less than 2 sites, and combine them into a new rectangle. This way, the longest side in the collection is $\leq 2\times$(previous longest side)+1.

$$
\begin{aligned}
&P(\xi_\infty(x) = 1)\\
&\leq P(\text{at least one 1 in } B_\infty(x, 10))\\
&\ + P(\text{at least one i.s. rectangle with } b \in [5, 10] \text{ in } B_\infty(x, 2/p))\\
&\ + P(\text{at least one i.s. rectangle with } b \in [1/p, 2/p])\\
&\leq Cp + Cp^{-2} P(\text{at least 3 1's in at most 100 places})\\
&\ + Cp^{-2} L^2 (1 - (1-p)^{2/p})^{1/(2p)},
\end{aligned}
$$

and this shows that $c_1$ can be chosen to be (anything smaller than) $-\log(1 - e^{-2})/2 \approx 0.073$.

A remarkable argument by Holroyd (2002), demonstrates that $c_1 = c_2 = \pi^2/18 \approx 0.55$.

**Bootstrap percolation with obstacles.**

Example:

$$
\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & & 1 & \\
 & & 2 & 1 & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccccc}
 & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 1 \\
 & & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

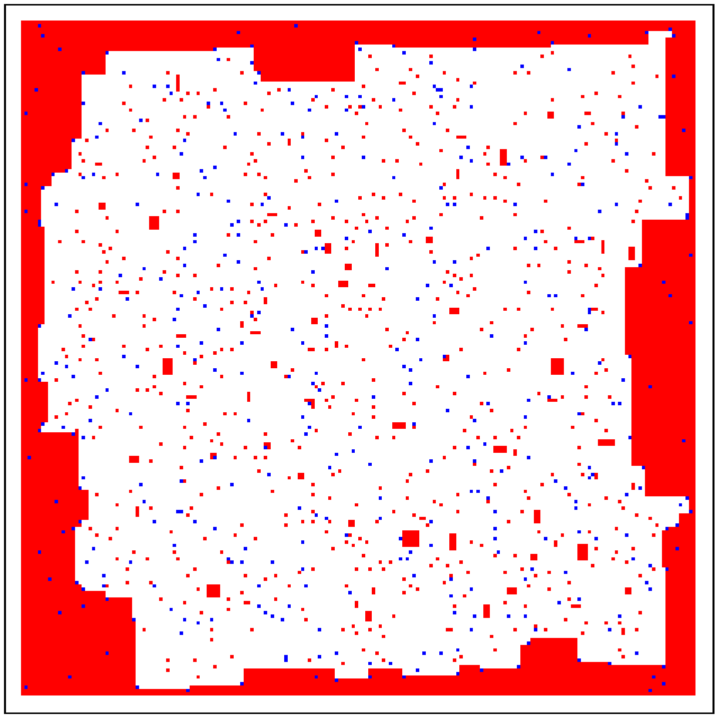
whereas without the 2 the rectangle would be filled.

**Theorem.** *There exists finite positive constants $c_1$ and $c_2$ so that*

*(1) If $q < c_1 p^2$, then $P(\xi_\infty(x) = 1) \to 1$ as $p \to 0$,*

*(2) If $q > c_2 p^2$, then $P(\xi_\infty(x) = 1) \to 0$ as $p \to 0$.*

The proof (G-McDonald, 1997) relies on the existence of *blocking loops*, which 1's cannot penetrate from outside. Such loops must have a 2 at every left turn.



Bootstrap percolation with 1 (red) boundary condition. For this picture, the density $q$ of $2's$ (blue) is 0.01, while the density of 1's is $p = 0.02$.

**Proof of a weaker statement than (1).**

(Due to R. Schonmann.)

Let $q < p^{2+\epsilon}$ for some $\epsilon > 0$. Let $N = 1/p^{1+\epsilon/3}$. Divide the plane into $N \times N$ squares. Call any such square *good* if it contains no 2, but each of its rows and columns contains a 1. Then

$$P(\text{a fixed square is not good}) \leq N^2 q + 2N(1-p)^N \to 0,$$

so that the probability that the square that contains $x$ is good, and is connected to infinitely many good squares, converges to 1. However, a.s. one of such infinitely many good squares is initially filled with 1's, and so eventually all sites connected to it are filled with 1's.

**Sketch of the proof of (2).**

Essentially, the problem reduces to showing that, when $q > c_2 p^2$, a infinite path with the following properties is likely to exist:

- The path starts at the origin, and it only moves up or to the right by one site,
- There are no 1's on the path,
- There is a 2 on the path at every right turn.

A blocking loop can then be created which cannot be penetrated from the outside. If the loop is short enough (of length $1/p^\gamma$), nothing much will happen inside either (by the "finite box" results). Therefore, the vast majority of sites will not turn into 1's.

Let $N = \alpha/p$ and divide the lattice into $N \times N$ squares, one of which has the lower left corner at the origin. Fix one such square $S$, and assume that the origin is connected to a site $x$ on its left edge by a path as specified above. If first $\alpha$ is chosen small, then $c_2$ is chosen very large, the probability that the path can be extended to the left side of the adjacent square to the right, *and* the diagonally adjacent Northeast square can be made arbitrarily close to 1.

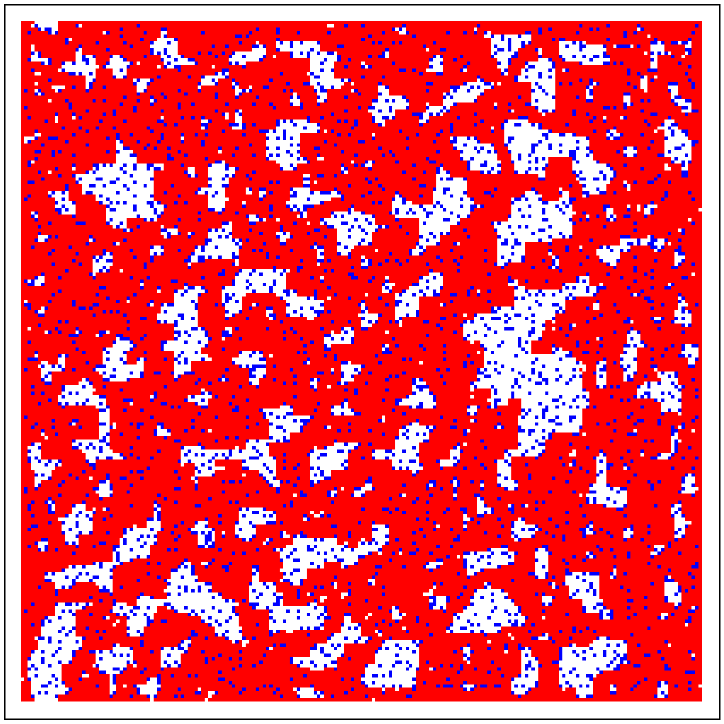
To see this, first note that a horizontal crossing of $S$ from $x$ without a 1 happens with probability $1 - \epsilon/4$. From each site of this crossing, make a vertical connection through $S$ and continued through its adjacent square $S'$ to the north. With probability $1 - \epsilon/4$, at least $1/2$ of these connections contain no 1. These combine for $\alpha^2/(2p^2)$ sites in $S'$. If $c_2$ is large enough, one of these sites is a 2 with probability $1-\epsilon/4$. Then there is a horizontal connection from the first such 2 to the square adjacent to the right of $S'$ with probability $1-\epsilon/4$.

## Open problems.

1. Are the two constants in the random environment theorem equal? Characterize their value(s).

2. Find the exact scaling for the modified bootstrap percolation in the random environment. In this CA, 0 turns into a 1 only in the presence of two *diagonally adjacent* nearest neighbor 1's.

3. Start bootstrap percolation with errors from all 1's, minus a density $q$ of 2's. The rule is the same as before, except that any 1 changes to 0 with probability $p$. In a rectangle without 1's and with 2's in its corners, no site can ever become a 1. Therefore every site eventually fixes in a 0 or a 2. Let $T$ be the last time the origin is 1. Conditioned on the origin starting at 1, compute the asymptotics of $T$ as $p, q \to 0$.



The dynamics described in problem 3. Here 1's are red, 2's are blue, $p = 0.2$ and $q = 0.1$. The (extremely slow) convergence to the final state on a $200 \times 200$ system is illustrated at time $t = 3212$.