Maximal spanning time for neighborhood growth on the Hamming plane

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Abstract

We consider a long-range growth dynamics on the two-dimensional integer lattice, initialized by a finite set of occupied points. Subsequently, a site x becomes occupied if the pair consisting of the counts of occupied sites along the entire horizontal and vertical lines through x lies outside a fixed Young diagram $\mathcal{Z}$. We study the extremal quantity $\mu(\mathcal{Z})$, the maximal finite time at which the lattice is fully occupied. We give an upper bound on $\mu(\mathcal{Z})$ that is linear in the area of the bounding rectangle of $\mathcal{Z}$, and a lower bound $\sqrt{s-1}$, where s is the side length of the largest square contained in $\mathcal{Z}$. We give more precise results for a restricted family of initial sets, and for a simplified version of the dynamics.

1 Introduction

The *Hamming plane* is the Cartesian product of two complete graphs on $\mathbb{Z}_+ = \{0, 1, 2 \ldots\}$, and so it has vertex set $\mathbb{Z}_+^2$ with an edge between any pair of sites that differ in a single coordinate. We refer to the points in $\mathbb{Z}_+^2$ as *sites*. Investigation of percolation and growth

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models on the Hamming plane and related highly connected graphs is a recent development [**Siv**, **GHPS**, **BBLN**, **Sli**], and this paper addresses an extremal quantity associated with a growth process introduced in [**GSS**].

We keep the terminology and notation from [**GSS**]. For $a, b \in \mathbb{N}$, we let $R_{a,b} = ([0, a-1] \times [0, b-1]) \cap \mathbb{Z}_+^2$ be the discrete $a \times b$ rectangle. A union $\mathcal{Z} = \bigcup_{(a,b)\in\mathcal{I}} R_{a,b}$ of rectangles over some finite set $\mathcal{I} \subseteq \mathbb{N}^2$ is called a *zero-set*. Note that it is possible that $\mathcal{Z} = \emptyset$. Also note that we restrict our consideration to finite zero-sets.

For a site $x \in \mathbb{Z}_+^2$, we denote by $L^h(x) \subseteq \mathbb{Z}_+^2$ and $L^v(x) \subseteq \mathbb{Z}_+^2$ the horizontal and vertical lines through x, respectively. The *neighborhood* of x then is $\mathcal{N}(x) = L^h(x) \cup L^v(x)$. The row and column counts of x in a set $A \subseteq \mathbb{Z}_+^2$ are given by

$$\texttt{row}(x, A) = |L^h(x) \cap A| \text{ and } \texttt{col}(x, A) = |L^v(x) \cap A|.$$

A zero-set $\mathcal{Z}$ determines a *neighborhood growth transformation* $\mathcal{T} : 2^{\mathbb{Z}_+^2} \to 2^{\mathbb{Z}_+^2}$ as follows. Fix $A \subseteq \mathbb{Z}_+^2$. If $x \in A$, then $x \in \mathcal{T}(A)$. If $x \notin A$, then $x \in \mathcal{T}(A)$ if and only if the pair of row and column counts of x lies outside the zero-set, i.e., $(\texttt{row}(x, A), \texttt{col}(x, A)) \notin \mathcal{Z}$. The *neighborhood growth dynamics* is given by the discrete time trajectory obtained by iteration of $\mathcal{T}$: $A_t = \mathcal{T}^t(A)$, for $t \geq 0$. We call sites in A_t *occupied* and sites in A_t^c *empty*. See Figure 1.1 for an example of neighborhood growth dynamics.

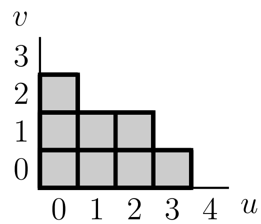
The simplest example, *line growth*, introduced as *line percolation* in [**BBLN**], is given by a rectangular zero-set $\mathcal{Z} = R_{a,b}$ for some $a, b \in \mathbb{N}$. Another special case, perhaps the most important one, is *threshold growth*, which is determined by an integer *threshold* $\theta \geq 1$. This natural growth rule is defined on an arbitrary graph as follows: a site x becomes occupied when the number of already occupied sites among its neighbors is at least θ. This dynamics was first introduced on trees in [**CLR**] and is typically called *bootstrap percolation*. The most common setting, with many deep and surprising results, is a graph of the form $[k]^\ell$, a Cartesian product of path graphs of k points, and thus with standard nearest neighbor lattice connectivity [**AL**, **GG**, **Hol**, **BB**, **GHM**, **BBDM**]. On the Hamming plane, threshold growth is given by the triangular zero-set $\mathcal{Z} = \{(u, v) : u + v \leq \theta - 1\}$; see [**GHPS**] and [**GSS**] for further background.

Define the set of eventually occupied sites by $A_\infty = \mathcal{T}^\infty(A) = \bigcup_{t\geq 0} A_t$. The set A *spans* if $A_\infty = \mathbb{Z}_+^2$. For a fixed zero-set $\mathcal{Z}$, we let $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ denote the collection of all finite spanning subsets of $\mathbb{Z}_+^2$. It follows from Theorem 2.8 in [**GSS**] that, for any $A \in \mathcal{A}$, the *spanning time*

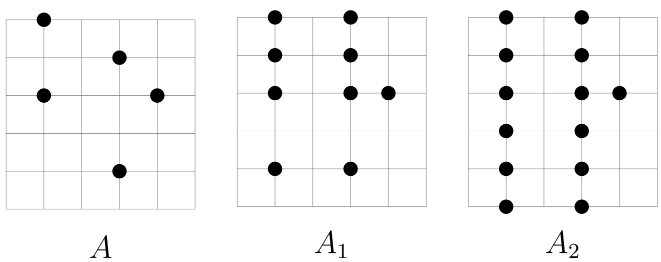
$$\tau(\mathcal{Z}, A) = \min\{t \in \mathbb{N} : \mathcal{T}^t(A) = \mathbb{Z}_+^2\}$$

is finite. Our main focus of attention is the *maximal spanning time*, the extremal quantity defined by

$$\mu(\mathcal{Z}) = \sup\{\tau(\mathcal{Z}, A) : A \in \mathcal{A}\}.$$



(a)



(b)

Figure 1.1: An example of neighborhood growth dynamics. The zero-set is given in (a), with the row count recorded on the u-axis and the column count on the v-axis (as in [**GSS**]). The initial set A is given in the leftmost panel of (b). The next two panels of (b) depict the subsequent two iterations. Observe that $\mathcal{T}^3(A) = \mathcal{T}^2(A)$, thus A does not span.

Theorem 2.8 in [**GSS**] shows that $\mu(\mathcal{Z}) < \infty$ by providing a very large upper bound (essentially the product of lengths of all rows of $\mathcal{Z}$, multiplied by their number). One of our results is a substantial improvement of that bound (see Theorem 1.1 below). Consequently, we provide an upper bound, linear in the area of the bounding rectangle of $\mathcal{Z}$, on the number of time steps needed to decide whether or not any given initial set spans. Before we give further definitions and state our results, we give a brief review of related results in the literature.

The earliest work related to the present topic may be the following classic problem on matrix powers. Let $\mathcal{M}$ be the set of primitive non-negative $n \times n$ matrices. Then define $h = h(n)$ to be the smallest power that makes all elements of the power M^h nonzero for every $M \in \mathcal{M}$. To put this in the context of long range growth dynamics similar to ours, take a binary $n \times n$ matrix M, let $M_0 = M$ and, given M_{t-1}, make the (i, j)th entry of M_t equal to 1 (resp., 0) when (say) the scalar product of the entire row through (i, j) in M_{t-1} with the entire column through (i, j) in M is nonzero (resp., zero). Thus h is akin to μ: if $H(M)$ is the minimal time t for which all entries of M_t are 1, then $h = \max\{H(M) : M \in \{0, 1\}^{n,n}, H(M) < \infty\}$. See [**HV**] for the solution of this problem, and related results for analogous extremal quantities obtained by replacing $\mathcal{M}$ by some natural subsets of $\mathcal{M}$.

Extremal quantities in growth models have been of substantial interest. Perhaps the most natural one is the smallest cardinality of a spanning set. It is a famous folk theorem that this quantity equals exactly n for the bootstrap percolation on $[n]^2$ with $\theta = 2$. For bootstrap percolation on $[n]^d$ with $\theta = 2$, the smallest spanning sets have size $\lceil d(n-1)/2 \rceil + 1$ [**BBM**]. Not much was known for thresholds $\theta \geq 3$ in such lattice setting until a recent breakthrough [**MN**]. Smallest spanning sets have also been studied for bootstrap percolation on trees [**Rie**] and on hypergraphs [**BBMR**]. For Hamming graphs, [**BBLN**] determines the size of the smallest spanning sets for line growth, while [**GSS**] gives bounds for general neighborhood growth.

Maximal spanning time results are comparatively scarce. In [**BP2**], it is shown that the maximal spanning time for bootstrap percolation on $[n]^2$ with $\theta = 2$ is $13n^2/18 + O(n)$ (see also [**BP1**]). For bootstrap percolation on the hypercube $[2]^n$, the maximal spanning time is $\lfloor n^2/3 \rfloor$ when $\theta = 2$ [**Prz**] and $n^{-1+o(1)}2^n$ when $\theta \geq 3$ [**Har**]. A related clique-completion process is studied from this perspective in [**BPRS**]. It appears that in general the maximal spanning time is considerably more complex than the smallest size of a spanning set. In our case, a major difficulty is non-monotonicity of μ: if $\mathcal{Z} \subseteq \mathcal{Z}'$, then $\mathcal{A}(\mathcal{Z}') \subseteq \mathcal{A}(\mathcal{Z})$ but $\tau(A, \mathcal{Z}) \leq \tau(A, \mathcal{Z}')$ for all $A \in \mathcal{A}(\mathcal{Z}')$, thus it is not clear how $\mu(\mathcal{Z})$ and $\mu(\mathcal{Z}')$ compare. As announced, our first result is an improved upper bound from [**GSS**] on the maximal spanning time.

Theorem 1.1. *For any $m, n \in \mathbb{N}$ and zero-set $\mathcal{Z} \subseteq R_{m,n}$,*

$$\mu(\mathcal{Z}) \leq 2mn + 5.$$

We are able to obtain a better upper bound for a special class of initial sets, which will also provide a lower bound for $\mu(\mathcal{Z})$.

A finite set $A \subseteq \mathbb{Z}_+^2$ is *thin* if, for every site $x \in A$, either $\texttt{row}(x, A) = 1$ or $\texttt{col}(x, A) = 1$. That is, any point $x \in A$ has no other points of A either on the horizontal line or on the vertical line through x. We define $\mathcal{A}_{\text{th}}$ to be the set of all spanning thin sets, and let

$$\mu_{\text{th}}(\mathcal{Z}) = \max\{\tau(A, \mathcal{Z}) : A \in \mathcal{A}_{\text{th}}\}.$$

Arguably, thin sets are the simplest general family of initial sets and are for this reason used in [**GHPS**, **GSS**] and in Section 3. In certain circumstances, thin set constructions are close to optimal [**GSS**]; in the present context, the extent to which $\mu(\mathcal{Z})$ and $\mu_{\text{th}}(\mathcal{Z})$ are comparable remains unclear (see open problem 3 in Section 7). The utility of thin sets in part comes from their connection to a simplified growth dynamics, which we now introduce.

The row and column *enhancements* $\vec{r} = (r_0, r_1, \ldots) \in \mathbb{Z}_+^\infty$ and $\vec{c} = (c_0, c_1, \ldots) \in \mathbb{Z}_+^\infty$ are two weakly decreasing sequences of non-negative integers which increase row and column counts by fixed values. To be precise, the *enhanced neighborhood growth dynamics* [**GSS**] is given by the triple $(\mathcal{Z}, \vec{r}, \vec{c})$, which defines a growth transformation $\mathcal{T}_{\text{en}} : 2^{\mathbb{Z}_+^2} \to 2^{\mathbb{Z}_+^2}$ as follows:

$$\mathcal{T}_{\text{en}}(A) = A \cup \{(i,j) \in \mathbb{Z}_+^2 : (\texttt{row}((i,j),A) + r_j, \texttt{col}((i,j),A) + c_i) \notin \mathcal{Z}\}.$$

By default, we initialize the enhanced growth by the empty set, and we say that the pair of enhancements $(\vec{r}, \vec{c})$ *spans* for $\mathcal{Z}$ if $\bigcup_{t\geq 0} \mathcal{T}_{\text{en}}^t(\emptyset) = \mathbb{Z}_+^2$. We let $\mathcal{A}_{\text{en}}$ be the set of all pairs of enhancements $(\vec{r}, \vec{c})$ that have finite support and span for $\mathcal{Z}$. Next, we introduce the enhanced spanning time

$$\tau_{\text{en}}(\mathcal{Z}, \vec{r}, \vec{c}) = \inf\{t \in \mathbb{N} : \mathcal{T}_{\text{en}}^t(\emptyset) = \mathbb{Z}_+^2\},$$

and finally define the corresponding maximal quantity

$$\mu_{\text{en}}(\mathcal{Z}) = \max\{\tau_{\text{en}}(\mathcal{Z}, \vec{r}, \vec{c}) : (\vec{r}, \vec{c}) \in \mathcal{A}_{en}\}.$$

We next state a comparison result between μ_{th} and μ_{en}.

Theorem 1.2. *For all zero-sets $\mathcal{Z}$,*

$$\mu_{\text{en}}(\mathcal{Z}) - 2 \leq \mu_{\text{th}}(\mathcal{Z}) \leq 2\mu_{\text{en}}(\mathcal{Z}).$$

For a zero-set $\mathcal{Z}$, we let $s(\mathcal{Z})$ be the side of the largest square included in $\mathcal{Z}$, that is, the integer $s \geq 0$ such that $R_{s,s} \subseteq \mathcal{Z}$ but $R_{s+1,s+1} \nsubseteq \mathcal{Z}$. The upper bound we obtain for μ_{en} and μ_{th} is linear in $s(\mathcal{Z})$, and is in this sense the best possible.

Theorem 1.3. *For any zero-set $\mathcal{Z}$,*

$$\mu_{\text{en}}(\mathcal{Z}) \leq 4s(\mathcal{Z}) + 1,$$
$$\mu_{\text{th}}(\mathcal{Z}) \leq 8s(\mathcal{Z}) + 2.$$

Moreover, there exists a sequence of zero-sets $\mathcal{Z}_n$ such that, as $n \to \infty$, $s(\mathcal{Z}_n) \to \infty$, $\liminf \mu_{\text{en}}(\mathcal{Z}_n)/s(\mathcal{Z}_n) > 0$, and $\liminf \mu_{\text{th}}(\mathcal{Z}_n)/s(\mathcal{Z}_n) > 0$.

The next theorem gives the lower bounds on the maximal spanning times. We do not know whether these are in any sense optimal (see open problem 2 in Section 7).

Theorem 1.4. *For any zero-set $\mathcal{Z} \neq \emptyset$,*

$$\mu_{\text{en}}(\mathcal{Z}) \geq s(\mathcal{Z})^{1/2}$$

and

$$\mu(\mathcal{Z}) \geq \mu_{\text{th}}(\mathcal{Z}) \geq (s(\mathcal{Z}) - 1)^{1/2}.$$

Finally, we state our results on two simplest special cases. When zero-set $\mathcal{Z}$ is a rectangle, that is, when $\mathcal{T}$ is a line growth, we can give an explicit formula for $\mu(Z)$.

Proposition 1.5. *If $\mathcal{Z} = R_{m,n}$, then*

$$\mu(\mathcal{Z}) = \mu_{\text{th}}(\mathcal{Z}) = \begin{cases} 2\min\{m,n\}, & \text{if } m \neq n \text{ or } m = n = 1, \\ 2n-1, & \text{if } m = n \neq 1. \end{cases}$$

When $\mathcal{Z}$ is the union of two rectangles, the precise value of $\mu(\mathcal{Z})$ is already unknown, although we are able to provide much better bounds than those that follow from our general theorems. The lower bound we give in this section can be improved by a constant factor in many cases (see Lemma 3.3), but we do not know how close either of the bounds is to $\mu(\mathcal{Z})$. (See also the first open problem in Section 7.)

Proposition 1.6. *Assume that $\mathcal{Z} = R_{a,b} \cup R_{c,d}$, with $a > c$ and $d > b$. Then*

$$s(\mathcal{Z}) \leq \mu_{\text{th}}(\mathcal{Z}) \leq \mu(\mathcal{Z}) \leq 2(b+c).$$

Outline of Sections 2–7

We start Section 2 with some additional notation and definitions. In the following two subsections, 2.2 and 2.3, we introduce a binary operation between Young diagrams that is used to characterize $\mathcal{A}_{\text{en}}$, and we also establish several further properties of the enhanced growth that make it much more manageable. Then, in subsection 2.4, we prove a few simple results on the change of our extremal quantities when the zero-set is perturbed by elimination of its longest row, column, or both. The final preliminary subsection explains how we can arrange the points in a thin set so that the resulting growth dynamics has convenient properties.

In Section 3 we give the arguments for the two special families of zero-sets that we consider in Propositions 1.5 and 1.6. Then we proceed to prove Theorem 1.2 in Section 4. While the basic principle on how to create a thin set from enhancements and vice versa is easy to spot, some care is needed to obtain the spanning properties and to compare spanning times.

The major part of Section 5 is the proof of Theorem 1.1. At the heart of our argument is the simple but powerful Lemma 5.1, which implies steady accumulation of points on horizontal or vertical lines as new points become occupied. This enables us to control the time when enough lines accumulate enough occupied points to become completely occupied and then induce complete occupation of $\mathbb{Z}_+^2$. The proof of the upper bounds in Theorem 1.3 is based on the fact that we can bound the spanning time in the enhanced dynamics by a linear expression in the number of different enhancement counts. The sequence that proves the last statement of Theorem 1.3 can be taken to be $\mathcal{Z}_n = R_{n,n}$, so that we may apply Proposition 1.5 on the line growth.

To prove lower bounds in Section 6, we need to exhibit enhancements that take a sufficiently long time to span. If $\mathcal{Z}$ has more than $\sqrt{s}$ of rows (or columns) of different lengths at least s, then we can set column (or row) enhancements to be zero. Otherwise, we use "staircase" enhancements, whereby both row and column enhancements decrease by one until they reach 0. We conclude with a selection of open questions in Section 7.

2 Preliminaries

2.1 Notation and Terminology

We define the partial order $\preceq$ on $\mathbb{Z}_+^2$ as follows. For two sites $z = (i, j)$ and $z' = (i', j')$, $z \preceq z'$ if and only if $i \le i'$ and $j \le j'$.

A *Young diagram* is then a set of sites $X \subseteq \mathbb{Z}_+^2$ such that $z' \in X$ and $z \preceq z'$ implies $z \in X$ for all sites $z, z' \in \mathbb{Z}_+^2$. Observe that any zero-set is a Young diagram.

For $A \subseteq \mathbb{Z}_+^2$, we denote the respective projections of A onto the x-axis and y-axis by $\pi_x(A)$ and $\pi_y(A)$.

Consider a vector $\vec{r} = (r_0, r_1, \ldots) \in \mathbb{Z}_+^\infty$ with weakly decreasing entries. The size of $\vec{r}$ is $|\vec{r}| = \sum_i r_i$. The support of $\vec{r}$ is the smallest interval $[0, N-1]$ such that $r_i = 0$ for $i \ge N$. We will often write $\vec{r}$ as a finite vector, omitting its zero coordinates.

The dynamics given by the growth transformation $\mathcal{T}$ is sometimes called the *regular dynamics* when it needs to be distinguished from the enhanced version.

We say a set $B \subseteq \mathbb{Z}_+^2$ is *covered* at time t by either regular or enhanced growth dynamics if every site of B is occupied at time t by the respective dynamics.

A *line* in $\mathbb{Z}_+^2$ is either $L^h(x)$ (also called a *row*) or $L^v(x)$ (also called a *column*) for some $x \in \mathbb{Z}_+^2$.

2.2 Operations with Young diagrams

Let X be a Young diagram and $k \in \mathbb{N}$. We define reductions of X obtained by removing the k leftmost columns or k bottommost rows of X,

$$X^{\downarrow k} = \{(u, v-k) : (u, v) \in X, v \ge k\},$$
$$X^{\leftarrow k} = \{(u-k, v) : (u, v) \in X, u \ge k\},$$

and the diagonal shift of X,

$$X^{\swarrow k} = X^{\leftarrow k \downarrow k}.$$

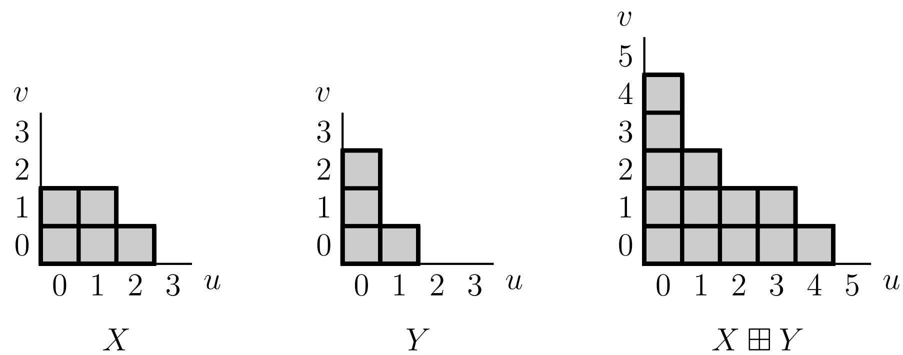


Figure 2.1: An example of the infimal sum of two Young diagrams.

Given two Young diagrams X, Y, we define the *infimal sum* of X and Y by

$$X \boxplus Y = (X^c + Y^c)^c$$

where $X^c = \mathbb{Z}_+^2 \setminus X$ and $X + Y = \{x + y : x \in X, y \in Y\}$. See Figure 2.1 for an example. For a Young diagram X, define its closure $\overline{X} = X \cup (\mathbb{Z}^2 \setminus \mathbb{Z}_+^2)$ and its height function $\phi_X : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ so that $\overline{X} + [0,1]^2 = \{(x,y) \in \mathbb{R}^2 : y \leq \phi_X(x)\}$. Then the terminology comes from the fact that $\phi_{X \boxplus Y} = \phi_X \Box \phi_Y$, where $\Box$ is the infimal convolution (see for example Section 5 of [**Roc**]).

The following lemma in particular establishes that the set of Young diagrams equipped with the operation $\boxplus$ is a commutative monoid. We omit the routine proof.

Lemma 2.1. *Let X, Y, Z, and X_λ, $\lambda \in \Lambda$, be Young diagrams, where Λ is an arbitrary index set. The infimal sum has the following properties:*

(1) *$X \boxplus Y$ is a Young diagram.*

(2) *$X \boxplus \emptyset = X$.*

(3) *$(X \boxplus Y) \boxplus Z = X \boxplus (Y \boxplus Z)$.*

(4) *$X \boxplus Y = Y \boxplus X$.*

(5) *$\cap_{\lambda \in \Lambda}(X_\lambda \boxplus Y) = (\cap_{\lambda \in \Lambda} X_\lambda) \boxplus Y$.*

(6) *If $X \subseteq Y$, then $X \boxplus Z \subseteq Y \boxplus Z$.*

Assume Y and Z are Young diagrams. Let $\Delta(Z, Y)$ consist of all Young diagrams X such that $Z \subseteq X \boxplus Y$. The *infimal difference* of Z and Y is defined as

$$Z \boxminus Y = \bigcap_{X \in \Delta(Z,Y)} X.$$

Lemma 2.2. *Let Y, Z be Young diagrams. The infimal difference has the following properties:*

(1) $Z \boxminus Y \in \Delta(Z, Y)$.

(2) $Z \boxminus Y \subseteq Z$.

Proof. Property (1) holds since an intersection of Young diagrams is a Young diagram and by Lemma 2.1 (5). Property (2) holds since $Z \in \Delta(Z, Y)$ implies $Z \boxminus Y \subseteq Z$. $\square$

2.3 Enhanced growth

Given a pair of enhancements $(\vec{r}, \vec{c})$, we form a pair of Young diagrams $(\mathcal{R}, \mathcal{C})$ such that the row counts of $\mathcal{R}$ are given by $\vec{r}$ and the column counts of $\mathcal{C}$ are given by $\vec{c}$. Therefore we use the two pairs interchangeably to describe the enhanced dynamics. The following lemma explains why enhanced growth is simpler than regular growth.

Lemma 2.3. *Let $(\mathcal{R}, \mathcal{C})$ be enhancements that span for a zero-set $\mathcal{Z}$. The set of occupied sites A_t satisfies the following for all $t \geq 0$:*

(1) *The set A_t is a Young diagram.*

(2) *The concave corners of A_t must grow: if $(i-1, j), (i, j-1) \in \overline{A_t}$, then $(i, j) \in \overline{A_{t+1}}$ for any $i, j \geq 0$.*

(3) *The sites with identical enhancements become occupied simultaneously: if $r_j = r_{j'}$ and $c_i = c_{i'}$, then $(i, j) \in A_t$ if and only if $(i', j') \in A_t$.*

Proof. We prove (1) by induction. When $t = 0$, $A_0 = \emptyset$. Now suppose A_t is a Young diagram for some $t \geq 0$. Assume $x = (i, j) \notin A_t$, $x' = (i', j') \notin A_t$, and $x \preceq x'$. As A_t is a Young diagram, $\texttt{row}(x, A_t) \geq \texttt{row}(x', A_t)$ and $\texttt{col}(x, A_t) \geq \texttt{col}(x', A_t)$. As $c_i \geq c_{i'}$ and $r_j \geq r_{j'}$, and $\mathcal{Z}$ is a Young diagram, $(\texttt{row}(x', A_t) + r_{j'}, \texttt{col}(x', A_t) + c_{i'}) \notin \mathcal{Z}$ implies $(\texttt{row}(x, A_t) + r_j, \texttt{col}(x, A_t) + c_i) \notin \mathcal{Z}$. Therefore, A_{t+1} is also a Young diagram.

To prove (2), assume that $(i, j) \in \mathbb{Z}_+^2 \setminus A_{t+1}$, and that $(i-1, j)$ and $(i, j-1)$ are both in $\overline{A_t}$. We claim that no point $x' = (i', j')$ with $i' \geq i$ and $j' \geq j$ ever gets occupied. Assume, to the contrary, that such a point is first occupied at some time t'. By (1), $t' > t$. Moreover, using the minimality of t' and the assumption, $(\texttt{row}(x', A_{t'-1}) + r_{j'}, \texttt{col}(x', A_{t'-1}) + c_{i'}) \preceq (i + r_j, j + c_i) \in \mathcal{Z}$. Hence x' does not get occupied at time t', a contradiction that proves the claim, which in turn contradicts the assumption that the enhancements $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$.

The statement (3) is also proved by induction. We first observe that it is valid for $t = 0$. Now assume its validity at time t, and let $x = (i, j)$ and $x' = (i', j')$ be such that

$r_j = r_{j'}$ and $c_i = c_{i'}$. By the induction hypothesis, for every $k \in \mathbb{Z}_+$, $(k, j) \in A_t$ if and only if $(k, j') \in A_t$. This implies that $\texttt{row}(x', A_t) = \texttt{row}(x, A_t)$, and similarly $\texttt{col}(x', A_t) = \texttt{col}(x, A_t)$. Then $(\texttt{row}(x', A_t) + r_{j'}, \texttt{col}(x', A_t) + c_{i'}) = (\texttt{row}(x, A_t) + r_j, \texttt{col}(x, A_t) + c_i)$, so that $x \in A_{t+1}$ if and only if $x' \in A_{t+1}$. $\square$

Suppose $(\mathcal{R}, \mathcal{C})$ are given enhancements. Partition $\mathbb{Z}_+$ into disjoint intervals $I_0, \dots, I_M$ so that, for any i, all columns of $\mathcal{C}$ with indices in I_i have equal height, and so that the common height of columns indexed by I_i is strictly decreasing with i. Analogously, let $J_0, \dots, J_N$ be the partition of $\mathbb{Z}_+$ into intervals J_j of indices of rows of $\mathcal{R}$, so that the rows indexed by J_j have common length that is strictly decreasing with j.

Lemma 2.4. *Fix a zero-set $\mathcal{Z}$ and let $(\mathcal{R}, \mathcal{C})$ be enhancements that span for $\mathcal{Z}$. Let the partitions I_i, $i = 0, \dots, M$ and J_j, $j = 0, \dots, N$, be defined as above. Then*

$$\bigcup_{i+j<t} I_i \times J_j \subseteq A_t$$

for all $t \geq 0$.

Proof. We use induction, beginning with the trivial base case $t = 0$. The inductive step follows from parts (2) and (3) of Lemma 2.3. $\square$

Corollary 2.5. *Let $\mathcal{R}$ have row counts with support $[0, M-1]$ and $\mathcal{C}$ have column counts with support $[0, N-1]$. If $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$, then $\tau_{\text{en}}(\mathcal{Z}, \mathcal{R}, \mathcal{C}) \leq M + N + 1$.*

The next lemma provides the key connection between the infimal sum and enhanced growth.

Lemma 2.6. *Fix a zero-set $\mathcal{Z}$. The enhancements $(\mathcal{R}, \mathcal{C})$ span for $\mathcal{Z}$ if and only if*

$$\mathcal{Z} \subseteq \mathcal{R} \boxplus \mathcal{C}.$$

Proof. The pair of enhancements $(\mathcal{R}, \mathcal{C})$ does not span if and only if there exists $(i, j) \notin \overline{\mathcal{T}_{\text{en}}^{\infty}(\emptyset)}$ so that $(i-1, j)$ and $(i, j-1)$ are both in $\overline{\mathcal{T}_{\text{en}}^{\infty}(\emptyset)}$. For this to happen, we must have $(i + r_j, j + c_i) \in \mathcal{Z}$. But $(i, c_i) \notin \mathcal{C}$ and $(r_j, j) \notin \mathcal{R}$, thus $\mathcal{R}^c + \mathcal{C}^c \not\subseteq \mathcal{Z}^c$, and so $\mathcal{Z} \not\subseteq \mathcal{R} \boxplus \mathcal{C}$.

Conversely, if $\mathcal{Z} \not\subseteq \mathcal{R} \boxplus \mathcal{C}$, there exist $(i, b) \notin \mathcal{R}$ and $(a, j) \notin \mathcal{C}$, so that $(i + a, j + b) \in \mathcal{Z}$. Then $b \geq c_i$, $a \geq r_j$, and so $(i + r_j, j + c_i) \in \mathcal{Z}$. Thus no point outside of $([0, i-1] \times [0, \infty)) \cup ([0, \infty) \times [0, j-1])$ becomes occupied, and consequently $(\mathcal{R}, \mathcal{C})$ does not span. $\square$

2.4 Perturbations of zero-sets

Let $\tau_{\text{line}}(\mathcal{Z}, A)$ be the first time that the regular dynamics given by $\mathcal{Z}$ covers a line in $\mathbb{Z}_+^2$ starting from A. Define

$$\mu_{\text{line}}(\mathcal{Z}) = \max_{A\in\mathcal{A}}\{\tau_{\text{line}}(\mathcal{Z}, A)\}.$$

We omit the simple proof of the following lemma.

Lemma 2.7. *For any zero-set $\mathcal{Z}$,*

$$\mu(\mathcal{Z}) \le \mu_{\text{line}}(\mathcal{Z}) + \max\left\{\mu\left(\mathcal{Z}^{\downarrow 1}\right), \mu\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}.$$

As already remarked, μ is not apparently monotone with respect to inclusion. We do however have a weaker form of monotonicity which is the subject of the next lemma.

Lemma 2.8. *For any zero-set $\mathcal{Z} \neq \emptyset$,*

$$\mu(\mathcal{Z}) \ge \max\left\{\mu\left(\mathcal{Z}^{\downarrow 1}\right), \mu\left(\mathcal{Z}^{\leftarrow 1}\right)\right\}.$$

The same inequlity holds when μ is replaced by μ_{th} or μ_{en}.

Proof. To prove the first inequality, by symmetry we only need to show $\mu(\mathcal{Z}) \ge \mu\left(\mathcal{Z}^{\leftarrow 1}\right)$. Assume $A \in \mathcal{A}(\mathcal{Z}^{\leftarrow 1})$. Let $A' = A \cup (\{M\} \times [M, M+N])$. If M and N are large enough, then $A' \in \mathcal{A}(\mathcal{Z})$ and $\tau(\mathcal{Z}^{\leftarrow 1}, A) \le \tau(\mathcal{Z}, A') \le \mu(\mathcal{Z})$. Therefore $\mu(\mathcal{Z}^{\leftarrow 1}) \le \mu(\mathcal{Z})$. Observe also that, for sufficiently large M and N, A' is thin provided A is thin. Therefore the preceding argument also proves that $\mu_{\text{th}}(\mathcal{Z}^{\leftarrow 1}) \le \mu_{\text{th}}(\mathcal{Z})$.

To prove the inequality for μ_{en}, we again only need to show $\mu_{\text{en}}(\mathcal{Z}) \ge \mu_{\text{en}}\left(\mathcal{Z}^{\leftarrow 1}\right)$. Assume $(\vec{r}, \vec{c}) \in \mathcal{A}_{\text{en}}(\mathcal{Z}^{\leftarrow 1})$, where $\vec{r} = (r_0, \dots, r_{N_0})$ and $\vec{c} = (c_0, \dots, c_{M_0})$. Let

$$\vec{r}\,' = (r_0 + 1, \dots, r_{N_0} + 1, 1, \dots, 1, 0, \dots)$$

be row enhancements with finite support $[0, N]$. For N large enough, $(\vec{r}\,', \vec{c}) \in \mathcal{A}_{\text{en}}(\mathcal{Z})$, and $\tau_{\text{en}}(\mathcal{Z}^{\leftarrow 1}, \vec{r}, \vec{c}) \le \tau_{\text{en}}(\mathcal{Z}, \vec{r}\,', \vec{c}) \le \mu_{\text{en}}(\mathcal{Z})$. Thus $\mu(\mathcal{Z}^{\leftarrow 1}) \le \mu_{\text{en}}(\mathcal{Z})$. □

The next lemma gives a converse inequality for enhanced growth.

Lemma 2.9. *For any zero-set $\mathcal{Z}$,*

$$\mu_{\text{en}}(\mathcal{Z}) \le \mu_{\text{en}}(\mathcal{Z}^{\swarrow 1}) + 2.$$

Proof. Assume that $\mathcal{R}$ and $\mathcal{C}$, with respective number of rows and columns N_0 and M_0, span for the zero-set $\mathcal{Z}$. Then $\mathcal{R}^{\leftarrow 1}$ and $\mathcal{C}^{\downarrow 1}$ span for $\mathcal{Z}^{\swarrow 1}$ and do so in at most $\mu_{\text{en}}(\mathcal{Z}^{\swarrow 1})$ steps. The two enhanced dynamics, given by $(\mathcal{Z}, \mathcal{R}, \mathcal{C})$ and $(\mathcal{Z}^{\swarrow 1}, \mathcal{R}^{\leftarrow 1}, \mathcal{C}^{\downarrow 1})$, agree on the rectangle $[0, M_0 - 1] \times [0, N_0 - 1]$ by Lemma 2.3 (1). (Note that, within this rectangle,

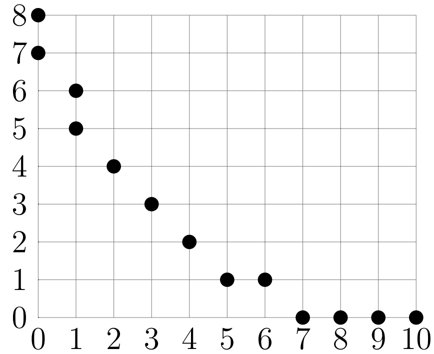


Figure 2.2: A thin set in the standard arrangement with $\vec{r} = (4, 2)$, $\vec{c} = (2, 2)$, and $w = 3$.

the row and column counts in the second dynamics equal the corresponding counts in the first dynamics, diminished by 1.) Therefore, the dynamics given by $(\mathcal{Z}, \mathcal{R}, \mathcal{C})$ cover this rectangle by time $\mu_{\text{en}}(\mathcal{Z}^{\swarrow 1})$, and then need at most two additional steps to fully occupy $\mathbb{Z}_+^2$. The inequality follows. $\square$

2.5 Thin sets

As we will see in the next lemma, it is advantageous to permute rows and columns to arrange sites in a thin set in a certain manner reminiscent of convexity (see Figure 2.2). We formalize this arrangement next.

Two sets $A_1, A_2 \subseteq \mathbb{Z}_+^2$ are *equivalent* if there are permutations of rows and columns of $\mathbb{Z}_+^2$ that map A_1 to A_2. It is clear that the spanning times of two equivalent sets are the same.

An equivalence class of thin sets is given by finite (possibly empty) weakly decreasing vectors $\vec{r}$ and $\vec{c}$ with integer entries of at least 2, which specify the number of occupied sites in the rows and columns that contain at least two sites, and a number $w \geq 0$ of isolated occupied sites. We now identify a specific representative of this equivalence class.

We say that a thin set A is in the *standard arrangement* (see Figure 2.2) if the following hold:

- The row counts $\texttt{row}((0, j), A)$, $j \geq 0$, and column counts $\texttt{col}((i, 0), A)$, $i \geq 0$, are weakly decreasing.
- If $x, x' \in A$ and $x \preceq x'$, then x and x' are on the same line.

To achieve the standard arrangement, let N_0 (resp., M_0) be the number of entries of $\vec{r}$ (resp., $\vec{c}$), and consider the rectangle $R_{M_0+w+|\vec{r}|, N_0+w+|\vec{c}|}$. The sites in A comprise, in order, the following diagonally adjacent intervals connecting the top left corner of this rectangle with its bottom right corner: vertical intervals of c_i sites, $i = 0, \dots, M_0 - 1$, followed by w single sites, and followed by horizontal intervals of r_i sites, $i = N_0 - 1, \dots, 0$ (see Figure 2.2). It is straightforward to see that the standard arrangement is unique. We record the following observation for later use.

Lemma 2.10. *Suppose A is a thin set in the standard arrangement, with M_0, N_0 and w as above. Then any $(i, j) \in A$ satisfies*

$$i + j \geq w + M_0 + N_0 - 1.$$

The next lemma shows that growth from thin sets is similar to enhanced growth (compare with Lemma 2.3).

Lemma 2.11. *Let A be a thin set in the standard arrangement. Assume $Y \subseteq \mathbb{Z}_+^2$ is a Young diagram. Let $A_0 = A \cup Y$ and $A_1 = \mathcal{T}(A_0)$.*

(1) *If $x \preceq x'$, then for all $t \geq 0$, $\mathtt{row}(x, A_0) \geq \mathtt{row}(x', A_0)$ and $\mathtt{col}(x, A_0) \geq \mathtt{col}(x', A_0)$.*

(2) *If $x \preceq x'$ and $x' \in A_1 \setminus A$, then $x \in A_1$.*

(3) *The set A_1 is the union of a Young diagram and A.*

Moreover, $A_t = \mathcal{T}^t(A_0)$ is the union of a Young diagram and A for all $t \geq 0$.

Proof. To prove (1), it is, by symmetry, enough to prove the inequality for the row counts. Let $x = (i, j)$ and $x' = (i', j')$, where $i \leq i'$ and $j \leq j'$. Let $I' = \{a \in \mathbb{Z}_+ : (a, j') \in Y\}$, $I = \{a \in \mathbb{Z}_+ : (a, j) \in Y\}$, $J' = \{a \in \mathbb{Z}_+ : (a, j') \in A\}$, and $J = \{a \in \mathbb{Z}_+ : (a, j) \in A\}$. As Y is a Young diagram, $I' \subseteq I$. As A is in the standard arrangement, $|J' \setminus I| \leq |J \setminus I|$. Therefore,

$$\begin{aligned}
\mathtt{row}(x', A_0) &= |I' \cup J'| \\
&= |I'| + |(J' \cap I) \setminus I'| + |J' \setminus I| \\
&\leq |I'| + |I \setminus I'| + |J \setminus I| \\
&= |I \cup J| \\
&= \mathtt{row}(x, A_0).
\end{aligned}$$

This establishes (1), which immediately implies (2). Then (3) follows, as $A_1 = A \cup Y_1$, where

$$Y_1 = \bigcup_{x' \in A_1 \setminus A} \{x : x \preceq x'\}$$

is a Young diagram. Finally, the last claim follows by induction. $\square$

3 Special cases

In this section we prove Propositions 1.5 and 1.6 for the two simplest families of zero-sets.

Lemma 3.1. *Let $\mathcal{Z} = R_{m,n}$ and let $A \subseteq \mathbb{Z}_+^2$ be such that $\mathcal{T}^2(A) \setminus \mathcal{T}(A) \neq \emptyset$. At least one column and at least one row not covered in A is covered in $\mathcal{T}^2(A)$.*

Proof. By symmetry, we only need to show the existence of a row with the claimed property. Suppose every row covered in $\mathcal{T}(A)$ is already covered in A. For every $y \in \mathcal{T}(A) \setminus A$, $L^v(y) \subseteq \mathcal{T}(A)$, as a site becomes occupied only if at least one of the lines through it is covered. Thus the column count of any $x \notin \mathcal{T}(A)$ remains unchanged, and so $\texttt{col}(x, \mathcal{T}(A)) = \texttt{col}(x, A) < n$. Now, pick an $x \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$. As $\texttt{col}(x, \mathcal{T}(A)) < n$, $\texttt{row}(x, \mathcal{T}(A)) \geq m$, and therefore $L^h(x)$ is covered in $\mathcal{T}^2(A)$. □

Proof of Proposition 1.5. We only address the case when $m > n$, as the case $m < n$ follows by symmetry, and the case $m = n$ is similar. Let A be a finite spanning set for $\mathcal{Z}$. By Lemma 3.1, in every two consecutive time steps until A spans, at least one row and at least one column must be covered, and consequently, there are at least $n-1$ covered rows and $n-1$ covered columns at time $t = 2n-2$. At $t = 2n-1$, every column containing at least one site that lies outside of the $n-1$ covered rows and $n-1$ covered columns is covered. As A spans, there must be at least m columns covered at $t = 2n-1$. Therefore $\tau(\mathcal{Z}, A) \leq 2n$.

For the lower bound let A be a thin set in the standard arrangement given by $\vec{r} = (m-1, m-2, \ldots, 2)$, $\vec{c} = (n, n-1, \ldots, 2)$, and $w = 2$. It is straightforward to check that A spans in $2n$ steps. □

Lemma 3.2. *Assume that $\mathcal{Z}$ is as in Proposition 1.6, and let $A \subseteq \mathbb{Z}_+^2$ such that $\mathcal{T}^2(A) \setminus \mathcal{T}(A) \neq \emptyset$. At least one line that is not covered in A is covered in $\mathcal{T}^2(A)$.*

Proof. Suppose $x \in \mathcal{T}^2(A) \setminus \mathcal{T}(A)$. If either $\texttt{row}(x, \mathcal{T}(A)) \geq a$ or $\texttt{col}(x, \mathcal{T}(A)) \geq d$, then $L^h(x)$ or $L^v(x)$ is covered in $\mathcal{T}^2(A)$. Otherwise, $c \leq \texttt{row}(x, \mathcal{T}(A)) < a$ and $b \leq \texttt{col}(x, \mathcal{T}(A)) < d$. Moreover, there exists a $y \in \mathcal{N}(x)$ such that $y \in \mathcal{T}(A) \setminus A$. Without loss of generality, assume $y \in L^h(x)$. Then either $\texttt{col}(y, A) \geq d$, or $b \leq \texttt{col}(y, A) < d$ and $c \leq \texttt{row}(y, A) < a$. If $\texttt{col}(y, A) \geq d$, then column $L^v(y)$ is covered in $\mathcal{T}(A)$. Otherwise, $b \leq \texttt{col}(y, A) < d$ and $c \leq \texttt{row}(y, A) = \texttt{row}(x, A) < a$. As $x \notin \mathcal{T}(A)$, $\texttt{col}(x, A) < b$; then $\texttt{col}(x, \mathcal{T}(A)) \geq b$ implies there exists a $z \in L^v(x)$ such that $z \in \mathcal{T}(A) \setminus A$. Then $\texttt{col}(x, A) = \texttt{col}(z, A) < b$ which implies that $\texttt{row}(z, A) \geq a$. Thus $L^h(z)$ is covered in $\mathcal{T}(A)$. □

Lemma 3.3. *Assume that $\mathcal{Z}$ is as in Proposition 1.6, and that $b \leq c$. Then*

$$\mu_{\text{th}}(\mathcal{Z}) \geq \min\{2c, \max\{2b, 2(d-b)-1\}\}.$$

Proof. If $d-b>c$, then by Lemma 2.8 and Proposition 1.5, $\mu_{\text{th}}(\mathcal{Z}) \geq \mu_{\text{th}}(\mathcal{Z}^{\downarrow b}) = \mu_{\text{th}}(R_{c,d-b}) = 2c$. Assume now that $d-b \leq c$. Then the comparison with $\mu_{\text{th}}(R_{c,d-b})$ gives $\mu_{\text{th}}(\mathcal{Z}) \geq 2(d-b)-1$. To also get the lower bound $2b$, let $\mathcal{Z}' = \mathcal{Z}^{\leftarrow(c-b)}$. Let A be a thin set in the standard arrangement given by $\vec{r} = (n-1, n-2, \ldots, 2)$, $\vec{c} = (b, b-1, \ldots, 2)$, and $w=2$, where $n = \max\{a-c+b, d\}$. One can check that $\tau(\mathcal{Z}', A) = 2b+1$, if $a-c+b \geq d$, and $\tau(\mathcal{Z}', A) = 2b$, otherwise. Therefore, by Lemma 2.8, $\mu_{\text{th}}(\mathcal{Z}) \geq \mu_{\text{th}}(\mathcal{Z}') \geq 2b$. □

Proof of Proposition 1.6. Without loss of generality assume $b \leq c$. By Lemma 3.2, at least one row or column is covered in every two consecutive time steps, and Lemma 2.7 gives

$$\mu(\mathcal{Z}) \leq 2 + \max\left\{\mu\left(\mathcal{Z}^{\leftarrow 1}\right), \mu\left(\mathcal{Z}^{\downarrow 1}\right)\right\},$$

and the upper bound follows by induction. To prove the lower bound, observe that $s(\mathcal{Z}) = \min\{c, d\}$ and that $\max\{2b, 2(d-b)-1\} \geq d$, and apply Lemma 3.3. □

4 Enhanced growth vs. growth from thin sets

In this section we prove Theorem 1.2.

Lemma 4.1. *For any zero-set $\mathcal{Z}$,*

$$\mu_{\text{en}}\left(\mathcal{Z}^{\swarrow 1}\right) \leq \mu_{\text{th}}(\mathcal{Z}).$$

Proof. Let $(\mathcal{R}, \mathcal{C})$ be some enhancements that span for $\mathcal{Z}^{\swarrow 1}$, given respectively by infinite vectors $\vec{r} = (r_0, r_1, \ldots)$ and $\vec{c} = (c_0, c_1, \ldots)$ of respective finite supports $[0, N_0-1]$ and $[0, M_0-1]$. Form infinite vectors $\vec{r}^{\,+} = (r_i^+ : i \geq 0) = (r_0+1, r_1+1, \ldots, r_{N_0-1}+1, 1, 1, \ldots)$ and $\vec{c}^{\,+} = (c_j^+ : j \geq 0) = (c_0+1, c_1+1, \ldots, c_{M_0-1}+1, 1, 1, \ldots)$.

Observe that the enhancements $\vec{r}^{\,+}, \vec{c}^{\,+}$ span for $\mathcal{Z}$. In fact, the enhanced dynamics with zero-set $\mathcal{Z}^{\swarrow 1}$ and enhancements $\vec{r}, \vec{c}$ and the enhanced dynamics with zero-set $\mathcal{Z}$ and enhancements $\vec{r}^{\,+}, \vec{c}^{\,+}$ have the same occupied set B_t^+ at any time $t \geq 0$. It is important to note that when B_t^+ covers the rectangle R_{M_0+1, N_0+1}, it in fact fully occupies $\mathbb{Z}_+^2$.

Choose N large enough so that the square $S = R_{N,N}$ satisfies $R_{M_0+1,N_0+1} \subseteq S$ and $\mathcal{Z} \subseteq S$. For $i < N$ let $r_i' = r_i^+$ and $c_i' = c_i^+$, and for $i \geq N$ let $r_i' = c_i' = 0$. Let B_t be the set of occupied sites at time t under the enhanced dynamics given by $(\mathcal{Z}, \vec{r}\,', \vec{c}\,')$. By Lemma 2.3 (1), $B_t \cap S = B_t^+ \cap S$ for all $t \geq 0$ and therefore the enhancements $\vec{r}\,', \vec{c}\,'$ span for $\mathcal{Z}$.

Define a thin set A in the standard arrangement given by the row vector $(r_0', \ldots, r_{N_0-1}')$, the column vector $(c_0', \ldots, c_{M_0-1}')$, $w = 2N - N_0 - M_0 \geq 2$ isolated points. By Lemma 2.10,

$A \subseteq \{(i,j) : i+j \geq 2N-1\}$, and therefore $S \cap A = \emptyset$. Let A_t be the set of occupied sites at time t starting from A under the regular dynamics given by $\mathcal{Z}$. We claim that

$$A_t \cap S = B_t \cap S, \tag{4.1}$$

for all $t \geq 0$. We use induction to prove (4.1), which clearly holds at $t = 0$. Assume that (4.1) holds for some $t \geq 0$. Let $x \in S \setminus A_t = S \setminus B_t$. By Lemma 2.11, $\mathcal{N}(x) \cap A_t \cap A^c = \mathcal{N}(x) \cap (A_t \cap S)$ and again by Lemma 2.3 (1), $\mathcal{N}(x) \cap B_t = \mathcal{N}(x) \cap (B_t \cap S)$. Therefore $x \in A_{t+1}$ if and only if $x \in B_{t+1}$ which proves the induction step.

When the enhanced dynamics given by $(\mathcal{Z}, \vec{r}', \vec{c}')$ covers S, the enhanced dynamics given by $(\mathcal{Z}, \vec{r}^+, \vec{c}^+)$ also covers S, and thus fully occupies $\mathbb{Z}_+^2$. Therefore, by (4.1), $\tau_{\text{en}}(\mathcal{Z}, \vec{r}^+, \vec{c}^+) \leq \tau(\mathcal{Z}, A) \leq \mu_{\text{th}}(\mathcal{Z})$, and then $\tau_{\text{en}}(\mathcal{Z}^{\swarrow 1}, \vec{r}, \vec{c}) = \tau_{\text{en}}(\mathcal{Z}, \vec{r}^+, \vec{c}^+) \leq \mu_{\text{th}}(\mathcal{Z})$. As $\vec{r}$ and $\vec{c}$ are arbitrary enhancements that span for $\mathcal{Z}^{\swarrow 1}$, the proof is concluded. $\square$

Lemma 4.2. *For any Young diagram $Y \in \mathcal{A}$, $\tau(\mathcal{Z}, Y) \leq \mu_{\text{en}}(\mathcal{Z})$.*

Proof. The enhanced spanning time for the pair of enhancements $(\mathcal{Z}, \emptyset)$ is given by the number of distinct row counts (including 0) in $\mathcal{Z}$. We also have $\tau(\mathcal{Z}, \mathcal{Z}) \leq \tau_{\text{en}}(\mathcal{Z}, \mathcal{Z}, \emptyset) \leq \mu_{\text{en}}(\mathcal{Z})$. By Lemma 3.1 in [**GSS**], the Young diagram $Y \in \mathcal{A}$ if and only if $\mathcal{Z} \subseteq Y$. Therefore by monotonicity $\tau(\mathcal{Z}, Y) \leq \tau(\mathcal{Z}, \mathcal{Z}) \leq \mu_{\text{en}}(\mathcal{Z})$ $\square$

Lemma 4.3. *For any $\mathcal{Z}$, $\mu_{\text{th}}(\mathcal{Z}) \leq 2\mu_{\text{en}}(\mathcal{Z})$.*

Proof. Let $A \in \mathcal{A}$ be a thin set in the standard arrangement. Let X be the set of points $x \in \mathbb{Z}_+^2 \setminus A$ such that $x \preceq x'$ for some $x' \in A$, that is, X is the set of points strictly below A (see Figure 4.1). Let $\ell = \inf\{t \geq 0 : X \subseteq A_t\}$ be the first time at which the regular dynamics starting from A covers X.

In order to obtain an upper bound on ℓ we consider enhanced growth with enhancements $\mathcal{R}$ and $\mathcal{C}$ that are given by the row and column counts of A. Thus, for any $x = (i,j) \in \mathbb{Z}_+^2$ the row and column enhancements for x are given by $r_j = \texttt{row}(x, A)$ and $c_i = \texttt{col}(x, A)$. The pair $(\mathcal{R}, \mathcal{C})$ spans for $\mathcal{Z}$ under the enhanced dynamics as the proof for the two-Y construction in Lemma 3.3 of [**GSS**] applies. (In fact, the induction argument from [**GSS**] is simpler in this case.)

For $t \geq 0$, let B_t denote the set of occupied sites under enhanced growth given by $(\mathcal{Z}, \mathcal{R}, \mathcal{C})$. First we claim that, for $t \geq 0$,

$$A_t \cap X = B_t \cap X. \tag{4.2}$$

We proceed to prove (4.2) by induction. For $t = 0$, (4.2) holds as $A_0 \cap X = \emptyset = B_0 \cap X$. Suppose (4.2) is true for some $t \geq 0$. By Lemma 2.3 (1) and Lemma 2.11 (3), for $x \in X \setminus A_t = X \setminus B_t$,

$$\mathcal{N}(x) \cap A_t \cap A^c = \mathcal{N}(x) \cap B_t.$$

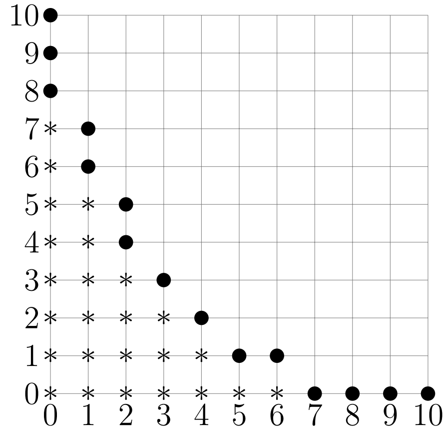


Figure 4.1: Illustration of the proof of Lemma 4.3. The thin set A is marked by solid circles, and the set X by asterisks.

Therefore, $\texttt{row}(x, A_t) = \texttt{row}(x, B_t) + \texttt{row}(x, A)$, and the analogous equality holds for the column counts. It follows that $A_{t+1} \cap X = B_{t+1} \cap X$, which thus establishes (4.2) for all t.

By (4.2), $\ell \leq \mu_{\text{en}}(\mathcal{Z})$. Now, $A \cup X$ is a Young diagram that includes A, and therefore spans, and is also covered by A_ℓ. By Lemma 4.2, $\tau(\mathcal{Z}, A_\ell) \leq \mu_{\text{en}}(\mathcal{Z})$ and therefore $\tau(\mathcal{Z}, A) \leq \ell + \mu_{\text{en}}(\mathcal{Z}) \leq 2\mu_{\text{en}}(\mathcal{Z})$. □

Proof of Theorem 1.2. The upper bound on μ_{th} follows from Lemma 4.3, and the lower bound follows from Lemmas 2.9 and 4.1. □

5 Upper bounds

In this section we prove Theorem 1.1 and Theorem 1.3.

Lemma 5.1. *Let A be an initial set of occupied points and A_t the resulting set of occupied points at time t in the regular dynamics with zero-set $\mathcal{Z}$. Assume that $x \in A_t \setminus A$ and $y \in A_t \setminus A$ are not neighbors. Let w be the sole element of $L^h(x) \cap L^v(y)$ and z the sole element of $L^v(x) \cap L^h(y)$. Then either $w \in A_t$ or $z \in A_t$.*

Proof. Let (u_x, v_x) and (u_y, v_y) denote the row and column counts of x and y, respectively, in A_{t-1}. Since x and y are both occupied by time t, but neither of them is initially occupied, $(u_x, v_x) \notin \mathcal{Z}$ and $(u_y, v_y) \notin \mathcal{Z}$. The pair of row and column counts of w (resp., z) in A_{t-1} is given by (u_x, v_y) (resp., (u_y, v_x)).

Assume $u_x \leq u_y$ and $v_x \leq v_y$. Then $(u_x, v_y) \notin \mathcal{Z}$ and $(u_y, v_x) \notin \mathcal{Z}$, and consequently w and z are both occupied at time t. The same conclusion holds if $u_y \leq u_x$ and $v_y \leq v_x$.

Otherwise suppose without loss of generality that $u_x \leq u_y$ and $v_y \leq v_x$. Then we claim $(u_y, v_x) \notin \mathcal{Z}$. If not, then $(u_x, v_x) \in \mathcal{Z}$ and $(u_y, v_y) \in \mathcal{Z}$, a contradiction. Thus $z \in A_t$. $\square$

Proof of Theorem 1.1. Suppose that $\mathcal{Z} \subseteq R_{m,n}$. Fix a set A of initially occupied sites that spans. Also fix a time $k > 0$ to be specified later. Suppose that $A_k \backslash A_{k-1} \neq \emptyset$ so that $\mu(\mathcal{Z}) \geq k$.

For $i > 0$, let $B_i = A_i \backslash A_{i-1}$ be the set of sites that become occupied at time i. A horizontal line with at least m sites in A_i or a vertical line with at least n sites in A_i is said to be *saturated* at time i. Let I denote the set of times $i \in [1, k]$ such that no line becomes saturated or covered at time i. We claim that

$$|[1, k] \setminus I| \leq 2m + 2n - 1. \tag{5.1}$$

To prove (5.1), we make a few observations. If a horizontal or a vertical line is saturated at time i, then it becomes covered by time $i + 1$. Once $n + m - 1$ lines become saturated (resp., covered), at least n horizontal lines or at least m vertical lines will be saturated (resp., covered). If at least $n + m - 1$ lines are saturated at time i, then at least $n + m - 1$ lines are covered by time $i + 1$ and all of $\mathbb{Z}_+^2$ is occupied by time $i + 2$. At time $k - 1$, $\mathbb{Z}_+^2$ is not covered, so there are at most $2(n + m - 2)$ times in $[1, k - 3]$ in which a line becomes either saturated or covered, which implies the claim.

Let $B = \cup_{i \in I} B_i$. If follows from (5.1) that

$$|B| \geq |I| \geq k - 2m - 2n + 1. \tag{5.2}$$

On any given horizontal (resp., vertical) line, there are less than m (resp., n) sites in B; otherwise, that line would become saturated by the last point added. Fix a $z \in B$ and suppose neither $L^h(z)$ nor $L^v(z)$ is saturated by time k. Define the sets

$$B_h = \big(L^h(z) \cap B\big) \bigcup \{z' \in B \backslash \mathcal{N}(z) : L^h(z) \cap L^v(z') \in A_k\}$$

and

$$B_v = B \backslash B_h.$$

The set B_v consists of sites z' that satisfy one of the following two properties: either $z' \in L^v(z) \backslash \{z\}$; or both $z' \in B \backslash \mathcal{N}(z)$ and $L^h(z) \cap L^v(z') \notin A_k$, in which case by Lemma 5.1, $L^v(z) \cap L^h(z') \in A_k$. See Figure 5.1 for an illustration of the division of B into the two sets. It follows that $|\pi_x(B_h)| \leq |L^h(z) \cap A_k|$ and $|\pi_y(B_v)| \leq |L^v(z) \cap A_k|$.

For any subset $B' \subseteq B$, we have

$$|B'| \leq (n-1)|\pi_x(B')|, \qquad |B'| \leq (m-1)|\pi_y(B')|.$$

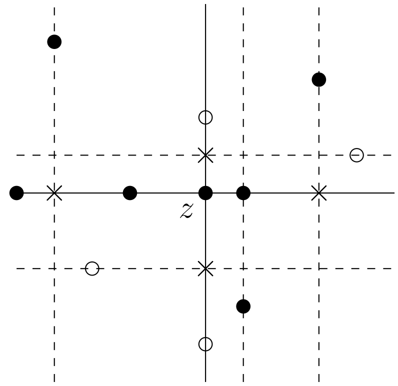


Figure 5.1: Illustration of the division of the set B into two sets B_h and B_v in the proof of Theorem 1.1. The points in B_h are marked by solid circles, and those in B_v by open circles. Sites outside B that are guaranteed to belong to A_k are marked by $\times$.

Therefore, the inequalities

$$|\pi_x(B_h)| \le \frac{k-2m-2n}{2(n-1)}, \qquad |\pi_y(B_v)| \le \frac{k-2m-2n}{2(m-1)} \tag{5.3}$$

cannot both be satisfied, as otherwise

$$|B| = |B_h| + |B_v| \le (n-1)|\pi_x(B_h)| + (m-1)|\pi_y(B_v)| \le k - 2m - 2n.$$

Assume $k \ge 2(m-1)(n-1) + 2m + 2n$. For any $z \in B$, we claim a line through z is covered at time $k+1$. If either $L^h(z)$ or $L^v(z)$ is saturated the claim is true. Otherwise one of the inequalities in (5.3) is not satisfied, and then either $|\pi_x(B_h)| \ge m$ or $|\pi_y(B_v)| \ge n$. In the former case $|L^h(z) \cap A_k| \ge |\pi_x(B_h)| \ge m$ and $L^h(z)$ is covered by time $k+1$. In the latter case $|L^v(z) \cap A_k| \ge |\pi_y(B_v)| \ge n$ and $L^v(z)$ is covered by time $k+1$.

Let H (resp., V) be the set of all $z \in B$ such that $L^h(z)$ (resp., $L^v(z)$) is covered at time $k+1$. By the previous paragraph, $B = H \cup V$. Let $k = 2(m-1)(n-1)+2m+2n+1 = 2mn+3$. For this choice of k, $|B| \ge 2(m-1)(n-1)+2$ by (5.2), and therefore either $|H| \ge (m-1)(n-1)+1$ or $|V| \ge (m-1)(n-1)+1$. As a horizontal line can contain at most $m-1$ sites in H, $|H| \ge (m-1)(n-1)+1$ implies that there are at least n horizontal lines with sites in H which are covered by time $k+1$. Similarly, as a vertical line can contain at most $n-1$ sites in V, $|V| \ge (m-1)(n-1)+1$ implies that there are at least m vertical lines with sites in V which are covered by time $k+1$. In either case, $\mathbb{Z}_+^2$ is fully occupied by time $k+2 = 2mn+5$. $\square$

We now proceed to prove the better upper bound for enhanced growth. Let $\widetilde{\mathcal{A}}_{\text{en}}$ consist of pairs $(\mathcal{R}, \mathcal{C}) \in \mathcal{A}_{\text{en}}$ such that $\mathcal{R} \subseteq \mathcal{Z}$ and $\mathcal{C} \subseteq \mathcal{Z}$.

Lemma 5.2. *For any zero-set $\mathcal{Z}$,*

$$\mu_{\mathrm{en}}(\mathcal{Z}) = \sup\{\tau_{\mathrm{en}}(\mathcal{Z},\mathcal{R},\mathcal{C}) : (\mathcal{R},\mathcal{C}) \in \widetilde{\mathcal{A}}_{\mathrm{en}}\}.$$

Proof. Fix $(\mathcal{R},\mathcal{C}) \in \mathcal{A}_{\mathrm{en}}$ and let $\mathcal{R}' = \mathcal{Z} \boxminus \mathcal{C}$ and $\mathcal{C}' = \mathcal{Z} \boxminus \mathcal{R}'$. By Lemmas 2.2 and 2.6, $(\mathcal{R}',\mathcal{C}')$ span for $\mathcal{Z}$, and $(\mathcal{R}',\mathcal{C}') \in \widetilde{\mathcal{A}}_{\mathrm{en}}$. Moreover, $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{C}' \subseteq \mathcal{C}$, and so $\tau_{\mathrm{en}}(\mathcal{Z},\mathcal{R}',\mathcal{C}') \geq \tau_{\mathrm{en}}(\mathcal{Z},\mathcal{R},\mathcal{C})$. This shows that $\mu_{\mathrm{en}}(\mathcal{Z}) \leq \sup\{\tau_{\mathrm{en}}(\mathcal{Z},\mathcal{R},\mathcal{C}) : (\mathcal{R},\mathcal{C}) \in \widetilde{\mathcal{A}}_{\mathrm{en}}\}$, and the other inequality is trivial. □

Proof of Theorem 1.3. Let $s = s(\mathcal{Z})$. The zero-set $\mathcal{Z}$ contains at most s rows and s columns that are longer than s, so the number of distinct non-zero row counts of $\mathcal{Z}$ is at most $2s$, and the same upper bound holds for the column counts of $\mathcal{Z}$. For every $(\mathcal{R},\mathcal{C}) \in \widetilde{\mathcal{A}}_{\mathrm{en}}$, $\mathcal{R} \subseteq \mathcal{Z}$ and $\mathcal{C} \subseteq \mathcal{Z}$ and so the number of distinct row counts of $\mathcal{R}$, including zero, is at most $2s+1$, and again the same upper bound holds for the number of distinct nonzero column counts of $\mathcal{C}$. By Corollary 2.5, $A_{2s+2s+1} = \mathbb{Z}_+^2$. Thus, for every $(\mathcal{R},\mathcal{C}) \in \widetilde{\mathcal{A}}_{\mathrm{en}}$, $\tau_{\mathrm{en}}(\mathcal{Z},\mathcal{R},\mathcal{C}) \leq 2s+2s+1 = 4s+1$ and therefore by Lemma 5.2, $\mu_{\mathrm{en}}(\mathcal{Z}) \leq 4s+1$. The upper bound on μ_{th} follows from Theorem 1.2. To prove the last statement, we may take $\mathcal{Z}_n = R_{n.n}$ and apply Proposition 1.5 and Theorem 1.2. □

6 Lower bounds

In this section we prove Theorem 1.4.

For $a,b,k \in \mathbb{N}$, define the (a,b)-staircase of size k as the following union of rectangles:

$$S_{a,b,k} = \bigcup_{h=1}^{k} R_{bh,a(k+1-h)} = \left\{(i,j) \in \mathbb{Z}_+^2 : \left\lfloor \frac{i}{b} \right\rfloor + \left\lfloor \frac{j}{a} \right\rfloor \leq k-1 \right\}.$$

This Young diagram has row counts of length $b, 2b, \cdots, kb$, each with multiplicity a, and column counts $a, 2a, \cdots, ka$, each with multiplicity b.

Lemma 6.1. *Fix $a,b \in \mathbb{N}$. For any $k \geq 1$, $S_{a,b,k}$ lies strictly below the line $ax+by = (k+1)ab$ and contains the set of sites $\{(i,j) : ai+bj < kab\}$ Furthermore, if $k_1,k_2 \in \mathbb{N}$,*

$$S_{a,b,k_1} \boxplus S_{a,b,k_2} = S_{a,b,k_1+k_2}. \tag{6.1}$$

Proof. This is a straightforward verification. □

The next lemma provides a general method for establishing lower bounds; see Figure 6.1 for an illustration.

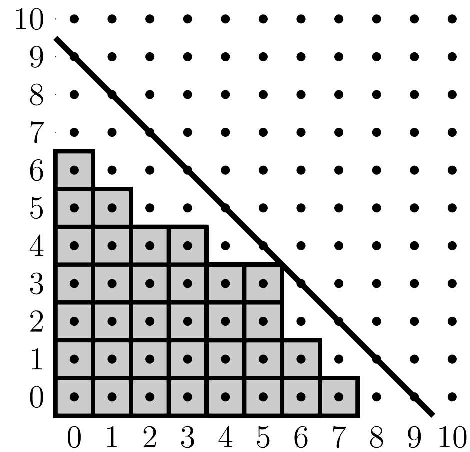


Figure 6.1: Illustration of Lemma 6.2, with $a = b = 1$, $k = 9$, and $(i_0, j_0) = (5, 3)$.

Lemma 6.2. *Fix a zero-set $\mathcal{Z}$ and two positive integers a and b. Let k be the smallest positive integer such that $ai + bj < kab$ for all $(i, j) \in \mathcal{Z}$. Let $(i_0, j_0) \in \mathcal{Z}$ be a point that satisfies $(k-1)ab \le ai_0 + bj_0$. Then*

$$\mu_{\text{en}}(\mathcal{Z}) \ge \min\left(\left\lceil \frac{i_0+1}{b} \right\rceil, \left\lceil \frac{j_0+1}{a} \right\rceil\right).$$

Proof. By Lemma 6.1, $\mathcal{Z} \subseteq S_{a,b,k}$. Assume $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 \ge k$. By (6.1) and Lemma 2.6, the row and column enhancements $\mathcal{R} = S_{a,b,k_1}$ and $\mathcal{C} = S_{a,b,k_2}$ span for $\mathcal{Z}$.

Let $m = i_0 + 1$ and $n = j_0 + 1$. Then $R_{m,n} \subseteq \mathcal{Z}$. Moreover, $k_1 = \lceil \frac{m}{b} \rceil$ and $k_2 = \lceil \frac{n}{a} \rceil$ satisfy $k_1 + k_2 \ge k$.

In this paragraph we consider the dynamics with zero-set $R_{m,n}$. As $R_{m,n} \subseteq \mathcal{Z}$, $\mathcal{R}$ and $\mathcal{C}$ span for $R_{m,n}$. Let $\ell = \min(k_1, k_2)$. At any time $t \le \ell$, exactly a new rows and b new columns become covered. Therefore, $\mathbb{Z}_+^2$ does not become occupied by time ℓ.

By monotonicity, $\tau_{\text{en}}(\mathcal{Z}, \mathcal{R}, \mathcal{C}) \ge \tau_{\text{en}}(R_{m,n}, \mathcal{R}, \mathcal{C})$, and therefore $\mu_{\text{en}}(\mathcal{Z}) \ge \ell$. □

Proof of Theorem 1.4. Let $s = s(\mathcal{Z})$. Let k_r (resp., k_c) denote the largest number of rows (resp., columns) in $\mathcal{Z}$ that are all of the same length and are of length at least s. If $k_r < s^{1/2}$, then there are at least $s^{1/2}$ rows of different lengths in $\mathcal{Z}$. Let $(\mathcal{R}, \mathcal{C})$ be enhancements such that $\mathcal{R} = \mathcal{Z}$ and $\mathcal{C} = \emptyset$. Then rows with different lengths are covered at different times, which gives a spanning time of at least $s^{1/2}$. Similarly, if $k_c < s^{1/2}$, $\mathcal{R} = \emptyset$ and $\mathcal{C} = \mathcal{Z}$ induce a spanning time of at least $s^{1/2}$.

Otherwise, $k_r \ge s^{1/2}$ and $k_c \ge s^{1/2}$. Let d_c be the number of columns with length strictly larger than the common length of the k_c columns. Analogously, let d_r be the number of rows with length strictly larger than that of the k_r rows. Define $\mathcal{Z}' = \mathcal{Z}^{\leftarrow d_c \downarrow d_r}$. We next use Lemma 6.2 with $a = b = 1$ and zero-set $\mathcal{Z}'$. There exists $(i_0, j_0) \in \mathcal{Z}'$ such that $i_0 + 1 \geq k_c$, $j_0 + 1 \geq k_r$ and the line with slope -1 through $(i_0 + 1, j_0 + 1)$ does not intersect $\mathcal{Z}'$. By Lemma 6.2 $\mu_{\text{en}}(\mathcal{Z}') \geq \min\{i_0 + 1, j_0 + 1\} \geq \min(k_c, k_r) \geq s^{1/2}$. By Lemma 2.8, $\mu_{\text{en}}(\mathcal{Z}') \leq \mu_{\text{en}}(\mathcal{Z})$, so $\mu_{\text{en}}(\mathcal{Z}) \geq s^{1/2}$. This proves the inequality for μ_{en}.

The remainder of the theorem follows from Lemma 4.1 and the observation that if $R_{s,s} \subseteq \mathcal{Z}$, then $R_{s-1,s-1} \subseteq \mathcal{Z}^{\swarrow 1}$. $\square$

7 Open questions

1. Apart from the line growth (addressed in Proposition 1.5), is there any other family of zero-sets with easily computable maximal spanning time? For example, is there an explicit formula for $\mu(\mathcal{Z})$ or for $\mu_{\text{en}}(\mathcal{Z})$ for threshold growth $\mathcal{Z} = \{(u, v) : u+v \leq \theta - 1\}$? We know that $\theta + 1 \leq \mu(\mathcal{Z}) \leq 2\theta^2 + 5$. Can $\mu(\mathcal{Z})$ be determined explicitly for the L-shaped $\mathcal{Z}$ considered in Proposition 1.6?

2. Let $\alpha_c^{\min}$ be the infimum of the the set of exponents $\alpha > 0$ for which
$$\inf_{\mathcal{Z}} \frac{\mu_{\text{en}}(\mathcal{Z})}{s(\mathcal{Z})^\alpha} = 0.$$
What is the value of $\alpha_c^{\min}$? We suspect that the answer is $1/2$ and know that $1/2 \leq \alpha_c^{\min} \leq \alpha_c^{\max} = 1$ (from Theorem 1.3 and 1.4). Here, $\alpha_c^{\max}$ is defined to be the supremum of the exponents $\alpha > 0$ for which
$$\sup_{\mathcal{Z}} \frac{\mu_{\text{en}}(\mathcal{Z})}{s(\mathcal{Z})^\alpha} = \infty.$$

3. Is there a finite exponent $\alpha > 0$ so that
$$\sup_{\mathcal{Z}} \frac{\mu(\mathcal{Z})}{s(\mathcal{Z})^\alpha} < \infty?$$

4. Call a set $A \subseteq \mathbb{Z}_+^2$ *k-thin* if, for every $x \in A$, either $\texttt{row}(x, A) \leq k$ or $\texttt{col}(x, A) \leq k$. Observe that 1-thin sets are exactly the thin ones. Let $\mu_{\text{th}}^k(\mathcal{Z})$ be the maximal finite spanning time of a k-thin set. Do there exist finite constants $c_k, C_k > 0$ such that
$$c_k\, \mu_{\text{th}}(\mathcal{Z}) \leq \mu_{\text{th}}^k(\mathcal{Z}) \leq C_k\, \mu_{\text{th}}(\mathcal{Z})?$$

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References

[**AL**] M. Aizenman, J. L. Lebowitz, *Metastability effects in bootstrap percolation*, Journal of Physics A: Mathematical and General 21 (1988), 3801–3813.

[**BB**] J. Balogh, B. Bollobás, *Bootstrap percolation on the hypercube*, Probability Theory and Related Fields 134 (2006), 624–648.

[**BBDM**] J. Balogh, B. Bollobás, H. Duminil-Copin, R. Morris, *The sharp threshold for bootstrap percolation in all dimensions*, Transactions of the American Mathematical Society 364 (2012), 2667–2701.

[**BBLN**] P. N. Balister, B. Bollobás, J. D. Lee, B. P. Narayanan, *Line percolation*, Random Structures & Algorithms 52 (2018), 597–616.

[**BBM**] J. Balogh, B. Bollobás, R. Morris, *Bootstrap percolation in high dimensions*, Combinatorics, Probability and Computing 19 (2010), 643–692.

[**BBMR**] J. Balogh, B. Bollobás, R. Morris, O. Riordan, *Linear algebra and bootstrap percolation*, Journal of Combinatorial Theory, Series A, 119 (2012), 1328–1335.

[**BP1**] F. Benevides. M. Przykucki, *On slowly percolating sets of minimal size in bootstrap percolation*, Electronic Journal of Combinatorics 20 (2013), 1–20.

[**BP2**] F. Benevides, M. Przykucki, *Maximum percolation time in two-dimensional bootstrap percolation*, SIAM Journal on Discrete Mathematics 29 (2015), 224–251.

[**BPRS**] B. Bollobás, M. Przykucki, O. Riordan, J. Sahasrabudhe, *On the maximum running time in graph bootstrap percolation*, Electronic Journal of Combinatorics 24 (2017), Paper #P2.16 1–20.

[**CLR**] J. Chalupa, P. L. Leath, G. R. Reich, *Bootstrap percolation on a Bethe lattice*, Journal of Physics C 12 (1979), L31–L35.

[**GG**] J. Gravner, D. Griffeath, *First passage times for the threshold growth dynamics on* $\mathbb{Z}^2$, Annals of Probability 24 (1996), 1752–1778.

[**GHM**] J. Gravner, A. E. Holroyd, R. Morris, *A sharper threshold for bootstrap percolation in two dimensions*, Probability Theory and Related Fields 18 (2010), 1–23.

[**GHPS**] J. Gravner, C. Hoffman, J. Pfeiffer, D. Sivakoff, *Bootstrap percolation on the Hamming torus*, Annals of Applied Probability 25 (2015), 287–323.

[**GSS**] J. Gravner, D. Sivakoff, E. Slivken, *Neighborhood growth dynamics on the Hamming plane*, Electronic Journal of Combinatorics 24 (2017), Paper #P4.29, 1–55.

[**Har**] I. Hartarsky, *Maximal bootstrap percolation time on the hypercube via generalised snake-in-the-box*. Electronic Journal of Combinatorics 25 (2018), Paper #P3.10, 1–12.

[**Hol**] A. E. Holroyd, *Sharp metastability threshold for two-dimensional bootstrap percolation*, Probability Theory and Related Fields 125 (2003), 195–224.

[**HV**] J. C. Holladay, R. S. Varga, *On powers of non-negative matrices*, Proceedings of the American Mathematical Society 9 (1958), 631–634.

[**MN**] N. Morrison, J. A. Noel, *Extremal bounds for bootstrap percolation in the hypercube*, Journal of Combinatorial Theory, Series A 156 (2018), 61–84.

[**Prz**] M. Przykucki, *Maximal percolation time in hypercubes under* 2*-bootstrap percolation*, Electronic Journal of Combinatorics 19 (2012), 1–13.

[**Rie**] E. Riedl, *Largest and smallest minimal percolating sets in trees*, Electronic Journal of Combinatorics 19 (2012), Paper #P64, 1–18.

[**Roc**] R. T. Rockafellar, *Convex Analysis*, Princeton University Press (1996).

[**Siv**] D. Sivakoff, *Random site subgraphs of the d-dimensional Hamming torus*, Combinatorics, Probability and Computing 23 (2014), 290–315.

[**Sli**] E. Slivken, *Bootstrap percolation on the Hamming torus with threshold* 2. arXiv:1407.2317