Random Threshold Growth Dynamics

Tom Bohman*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
bohman@math.mit.edu

Janko Gravner†
Department of Mathematics
University of California, Davis
Davis, CA 95616
gravner@math.ucdavis.edu

Revised, January 1999

Abstract

A site in $\mathbf{Z}^2$ becomes occupied with a certain probability as soon as it sees at least a threshold number of already occupied sites in its neighborhood. Such randomly growing sets have the following regularity property: a large fully occupied set exists within a fixed distance (which does not increase with time) of every occupied point. This property suffices to prove convergence to an asymptotic shape.

1 Introduction

Deterministic threshold growth dynamics on $\mathbf{Z}^2$ are generated by one of simplest models for a growing droplet in the plane. This model has two parameters: a finite *neighborhood* of the origin $\mathcal{N} \subset \mathbf{Z}^2$, which defines the neighborhood of an arbitrary point z as $z + \mathcal{N}$, and a positive integer *threshold* θ. Given the initial droplet A, the extent of growth at time i is given by the i^{th} iterate, $\mathcal{D}^i(A)$, of the following transformation on subsets of $\mathbf{Z}^2$:

$$\mathcal{D}(A) = A \cup \{z \in \mathbf{Z}^2 : |(z + \mathcal{N}) \cap A| \geq \theta\}.$$

In words, a site z joins the occupied set at time i if it sees at least a threshold number of occupied points in its neighborhood at time $i - 1$.

We stress immediately that our considerations are restricted to two dimensions, although some of our methods can be extended beyond the planar case. In fact, an analogue of Theorem 6 holds in any dimension. However, combinatorial methods that lead to our main result, Theorem 2, become much trickier even for dynamics in $\mathbf{Z}^3$.

*research supported by NSF grant DMS-9627408
†supported in part by research grant J1-6157-0101-94 from the Slovenian Ministry of Science and Technology

If $\theta = 1$, then we have an instance of *additive* dynamics, as $\mathcal{D}(A \cup B) = \mathcal{D}(A) \cup \mathcal{D}(B)$. If $\theta > 1$, this fails to be the case, and thus nucleation and interaction questions become interesting ([GG2], [GG3]). Of even more basic importance are issues of regularity of persistent growth, which we now introduce. Let $\mathcal{D}^\infty(A) = \cup_{i=0}^\infty \mathcal{D}^i(A)$ be the final droplet. A set $A \subset \mathbf{Z}^2$ is said to *generate persistent growth* if $\mathcal{D}^{i+1}(A) \neq \mathcal{D}^i(A)$ for all i. Moreover, A *generates the plane* if $\mathcal{D}^\infty(A) = \mathbf{Z}^2$. We call the dynamics *supercritical* if there exists a finite set A that generates the plane. The first result states that a set that generates the plane must do so in a very orderly fashion, namely, the iterates have a unique asymptotic shape. For a proof, see [W], or [GG1] and [GG2].

Theorem 1. *Let $\mathcal{N}$ and θ be fixed and let $I = B_\infty(0, 1/2)$ be the 1×1 square centered at the origin. There exists a set $L \subset \mathbf{R}^2$ such that if A is finite and generates the plane then $(\mathcal{D}^n(A) + I)/n \to L$. Furthermore, the asymptotic shape L is a bounded convex polygon that can be explicitly computed given $\mathcal{N}$ and θ.*

The easiest way to define convergence in Theorem 1 is via Hausdorff metric. However, a slightly stronger notion is preferable here: if $S_n \subset \mathbf{R}^2$ are closed and $S \subset \mathbf{R}^2$ is closed and convex, then $S_n \to S$ will mean that for every $\epsilon > 0$, $(1-\epsilon)S \subset S_n \subset (1+\epsilon)S$ for a large enough n.

Theorem 1 holds for a very general class of neighborhoods and the limiting shape is independent of the initial droplet so long as the initial droplet generates the plane. However, for a fixed neighborhood and threshold there could be different nontrivial limiting shapes resulting from initial droplets that generate persistent growth but do not generate the plane. Certain neighborhoods allow this possibility, as our next example shows.

Example 1. Suppose

$$\mathcal{N} = \{(x, y) \in \mathbf{Z}^2 : |x|, |y| \leq 3 \text{ and either } x = 0 \text{ or } y = 0\}$$

and $\theta = 3$. If $A = \{z \in \mathbf{Z}^2 : \|z\|_1 \leq 1\}$ then

$$\mathcal{D}^n(A) = \{(x, y) \in \mathbf{Z}^2 : |x|, |y| \leq n + 1 \text{ and either } x = 0 \text{ or } y = 0\}$$

and therefore (in Hausdorff metric only, since the limit set is not convex)

$$\mathcal{D}^n(A)/n \to \{(x, y) \in \mathbf{R}^2 : |x|, |y| \leq 1 \text{ and either } x = 0 \text{ or } y = 0\}.$$

On the other hand, any finite set that generates the plane has octagonal asymptotic shape with vertices $(\pm 1, 0)$, $(0, \pm 1)$ and $(\pm 3/4, \pm 3/4)$ (to see how these are computed, consult [GG2]).

Deterministic threshold growth dynamics for which every set that generates persistent growth also generates the plane are called *omnivorous* dynamics. By Theorem 1, deterministic omnivorous dynamics have a unique nontrivial limiting shape (by "nontrivial" we mean not equal to $\{0\}$).

The aim of this paper is to prove a shape theorem for a version of *random* threshold growth dynamics. To describe the discrete time version, on which we will focus, let $N = |\mathcal{N} \setminus \{0\}|$ be the number of true neighbors and fix $N+1$ probabilities $p_0, \dots, p_N$ satisfying

$$0 = p_0 = \cdots = p_{\theta-1} < p_\theta \leq p_{\theta+1} \leq \cdots \leq p_N.$$

The state of the dynamics at time t is a random set $A_t \subset \mathbf{Z}^2$. Once A_t is known, $A_{t+1} \supset A_t$ is determined as follows: if $x \notin A_t$, then, independently of other sites and other times, $x \in A_{t+1}$ with probability $p_{|A_t \cap (x+\mathcal{N})|}$. The starting point will be some deterministic initial droplet $A_0 \subset \mathbf{Z}^2$.

We should mention that all of the methods in this paper work for the continuous time analogue (in which the non-zero rates $\lambda_\theta \leq \lambda_{\theta+1} \leq \cdots \leq \lambda_N$ are given instead of the probabilities), with only trivial modifications. Other generalizations are also possible, with more general monotone rules and waiting times (see [CD], [K]), but these would lead us astray from the combinatorial issues that are the central point of this paper.

At first glance, the dynamics described above do not differ appreciably from those in [S] and [CD]. Indeed, if $\theta = 1$, then the standard subadditive arguments work: the main observation is that, for any time t_0, one can get a lower bound on the original occupied set at any time $t \geq t_0$ by picking an occupied point at time t_0, throwing the other occupied points away, and re–starting the dynamics. If $\theta > 1$ this argument needs a substantial extra step, as the dynamics can only be successfully re–started from a fairly large set. As is the case in Example 1, it is possible that an $x \in A_t$ is nowhere near, say, a large fully occupied square; this renders the usual shape theorem (of [S] and [CD]) false.

On the other hand, whatever A_0 is, the final droplet in the random growth is the same as in the deterministic one: $A_\infty = \cup_{t \geq 0} A_t = \mathcal{D}^\infty(A_0)$. If A_0 generates the plane, then A_t eventually occupies every point as well, eliminating cases like the troublesome Example 1. Must a unique asymptotic shape exist under this assumption? We suspect that the answer is yes, although we are unable to prove this in full generality. Our main result is valid only for *box neighborhoods* $\mathcal{N}_\rho = \{z \in \mathbf{Z}^2 : ||z||_\infty \leq \rho\}$. These dynamics are supercritical iff $\theta \in [1, \rho(2\rho+1)]$ ([GG2]).

Theorem 2. *Assume that $\mathcal{N} = \mathcal{N}_\rho$ for some $\rho \geq 1$ and that the dynamics are supercritical. Then there exists a convex compact neighborhood of the origin S (that* *depends on $\mathcal{N}$, θ and the probabilities p_i) such that*

$$(A_t + I)/t \to S \qquad \text{as } t \to \infty$$

almost surely for any A_0 that generates the plane.

Theorem 2 only provides existence of the limiting shape; it has been long known that properties of this set are not easy to discern. Here are some questions one might ask. Does L have a differentiable boundary? Is L strictly convex? (This question was studied for the additive case in [DL].) Could L ever be an isotropic disc? Some of these issues will be addressed in a forthcoming paper ([BGG]). Another interesting class of questions concerns the speed of convergence to the limiting shape and the related roughness of the boundary of A_t. This issue has generated much interest in the case of first–passage percolation, a related additive dynamics. Kesten ([K]) and Alexander ([A]) have shown that, with probability 1, $(t - C\sqrt{t}\log t)S \subset A_t + I \subset (t + C\sqrt{t}\log t)S$ for a large enough t. (Here, and throughout the paper, C is a "generic constant," whose value is of no importance.) It is in fact not clear if $\sqrt{t}$ could be replaced by a lower power, for experimental studies suggest that $A_t + I$ differs by $t^{1/3}$ from a deterministic set. To get a flavor of physical arguments in favor of such power law, the reader can consult [KS], while [NP] has a rigorous lower bound on described fluctuations. It seems to be a challenge to adapt these methods from [K, A, NP] to the case when $\theta > 1$, although it is intuitively clear that increased θ should diminish boundary fluctuations, as tentacles are less likely to form. Further evidence for this is provided in Figure 1, which shows simulations of discrete–time threshold growth dynamics with range 1 box neighborhood, $\theta = 1, 2, 3$ and $p_\theta = \cdots = p_8 = 0.1$ at the time t when the radius of A_t reaches about 100. In all cases, A_0 is a 3×3 square.

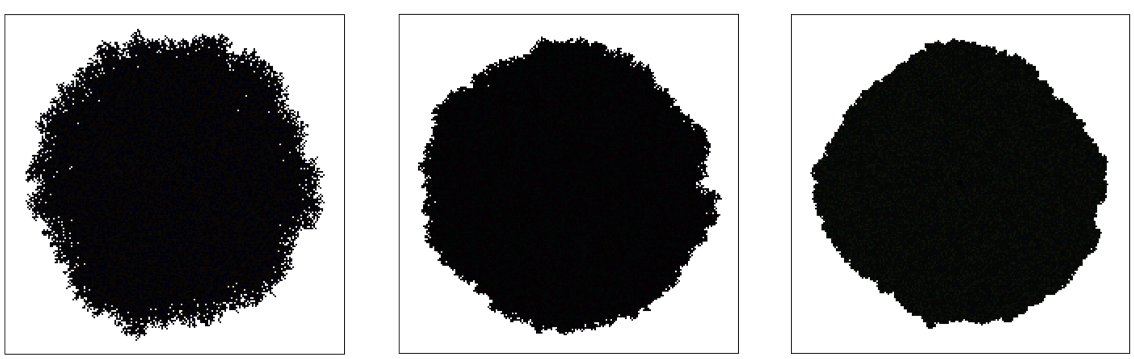

Figure 1. Range 1 dynamics with thresholds (from left to right) 1, 2, and 3.

We now proceed to discuss the main ideas in the proof of Theorem 2. As already pointed out, it is necessary to show that, with high probability, every site in A_t has a large fully occupied set nearby. It turns out that this property holds deterministically in box neighborhood cases.

Theorem 3. *If $\mathcal{N} = \mathcal{N}_\rho$, $x \in A_t$ and $B_\infty(x, 1000\rho^3(\log\rho + 1)) \cap A_0 = \emptyset$ then there exists a set $G \subset A_t \cap B_\infty(x, 1000\rho^3(\log\rho + 1))$ that generates the plane.*

Of course, the $1000\rho^3(\log\rho + 1)$ term should be viewed as constant in t, so a set that generates the plane is indeed very near x. It remains an open problem to estimate the smallest necessary size $R(\rho, \theta)$ of this term, but it is illustrative to check that $R(1, 2) = 1$ and $R(1, 3) = 2$. The methods used to prove Theorem 3 were first used in [B] to prove

Theorem 4. *Deterministic growth dynamics are omnivorous for box neighborhoods.*

This connection between Theorem 3 and Theorem 4 exemplifies how understanding deterministic dynamics can lead to progress in studying their random perturbations. The difficulty in generalizing either of these Theorems (and therefore Theorem 2) to more general neighborhoods lies in understanding the sets that generate the plane for non–box neighborhoods. A grasp on such sets thorough enough to prove either Theorem 3 or Theorem 4 would likely suffice to prove both. This impression is supported by our next result. We say that the threshold growth dynamics satisfy the *double threshold condition* if all sets $G \subset \mathcal{N}$ with $|G| \geq 2\theta$ generate the plane.

Theorem 5. *Assume that $-\mathcal{N} = \mathcal{N}$ and $\mathcal{N} \subset \mathcal{N}_\rho$. If the dynamics satisfy the double threshold condition, then the following hold:*

1. *Deterministic threshold growth dynamics are omnivorous.*
2. *If $x \in A_t$ and $B_\infty(x, 150\rho^3(\log\rho + 1)) \cap A_0 = \emptyset$ then there exists a set $G \subset A_t \cap B_\infty(x, 150\rho^3(\log\rho + 1))$ that generates the plane.*

Part 1 of Theorem 5 is given in Section 2 of [B], while Part 2 will be proven below. As a consequence of Theorem 5, we have the following random shape result.

Theorem 6. *Assume that $-\mathcal{N} = \mathcal{N}$ and the double threshold condition holds. Then there exists a convex compact neighborhood of the origin S such that*

$$(A_t + I)/t \to S \qquad \text{as } t \to \infty$$

almost surely for any A_0 that generates the plane.

The rest of this paper is organized as follows. Section 2 addresses the validity of the double threshold condition for a general class of neighborhoods. The proof of Theorem 3 is divided into Sections 3 and 4. This division is necessary because properties of sets that generate the plane differ for small and large θ. For $\mathcal{N} = \mathcal{N}_\rho$ and $\theta \leq \rho^2$ the double threshold condition is satisfied ([B]) and Theorem 3 follows from Part 2 of Theorem 5, the proof of which is given in Section 3. On the other hand, for $\rho^2 < \theta \leq \rho(2\rho+1)$ the double threshold condition does not necessarily hold. In this case the proof of Theorem 3 uses much more machinery from [B] and is given in Section 4. We note that it is not at all clear how to generalize this 'large threshold' proof to other neighborhoods; the special machinery from [B] is specific to box neighborhoods and it is not clear what the corresponding general machinery would be (or even if such machinery exists). Finally, in Section 5 we show how Theorems 2 and 6 follow from Theorems 3 and 5, respectively.

2 The Double Threshold Condition

Theorems 5 and 6 lead to the obvious question: when does the double threshold condition hold? For example, one can check by hand that the double threshold condition is always satisfied for range 1 box neighborhoods (that is, for $\theta = 1, 2, 3$), and is satisfied for range 2 box neighborhoods when $\theta \leq 8$ (for $\theta = 9$ the neighborhood without the four corner points does not grow, while for $\theta = 10$ the full neighborhood does not grow after the first time step). We now give a simple sufficient condition, which is far from optimal, but says that the double threshold condition holds when $\mathcal{N}$ is well behaved and θ is less than a constant multiple of $|\mathcal{N}|$. We assume that $\mathcal{N} = conv(\mathcal{N}) \cap \mathbf{Z}^2$, where $conv(\mathcal{N})$ is the Euclidean convex hull of $\mathcal{N}$ and that $\mathcal{N}$ is invariant with respect to transposition of the two coordinates and switching sign of either of them. Let ρ be the largest integer such that $(\rho, 0) \in \mathcal{N}$ and σ be the largest integer such that $(\sigma, \sigma) \in \mathcal{N}$. In other words, ρ and σ are chosen so that $\mathcal{N}_\rho$ is the smallest box neighborhood containing $\mathcal{N}$ and $\mathcal{N}_\sigma$ is the largest box neighborhood contained in $\mathcal{N}$.

Theorem 7. *If $\theta \leq \sigma^2/2$ then the double threshold condition holds.*

We conjecture that if $\theta \leq (|\mathcal{N}| - 4\rho - 1)/4$ (this is the number of elements of $\mathcal{N}$ in the first quadrant that do not lie on either axis) then the double threshold condition holds. This conjecture holds in the box neighborhood case: a version of the argument below tailored specifically to the box neighborhood $\mathcal{N}_\rho$ ([B]) shows that in this case the double threshold condition holds when $\theta \leq \rho^2$. On the other hand, a computer computation (along the lines of [GG3]) shows that in the box neighborhood case $\mathcal{N}_\rho$ does not grow if $\theta \geq 1.7\rho^2$ for ρ sufficiently large, demonstrating that the double threshold condition fails to hold for a wide range of supercritical dynamics, that is, at least when ρ is large and $1.7\rho^2 \leq \theta \leq \rho(2\rho+1)$.

An alternative to the argument given below shows that the double threshold condition holds when ρ is sufficiently large and $\theta < \pi\rho^2/16$. In some cases this bound is better than the one given by Theorem 7. The alternative argument is continuous in nature (i.e. cardinalities of subsets of the lattice are given as areas with appropriate error terms), and is omitted for the sake of brevity.

Finally we note that for general neighborhoods the double threshold condition is not monotone in θ (although we suspect that it is for the neighborhoods considered above). Consider the neighborhood $\mathcal{N} = \mathcal{N}_\rho \cup \{(\pm(2\rho+1), \pm(2\rho+1))\}$. The double threshold condition does not hold when $\theta = 2$ as $\mathcal{D}^\infty(A) = \mathcal{D}(A)$ for $A = \{(\pm(2\rho+1), \pm(2\rho+1))\}$. On the other hand, the arguments given below can be used to show that the double threshold condition holds for $4(\rho+1) \le \theta \le \rho^2/2$.

Proof of Theorem 7. Let $G \subset \mathcal{N}$ with $|G| \ge 2\theta$. Clearly, if $\mathcal{D}^n(G)$ contains a $\sigma \times \sigma$ square for some n then G generates the plane. We show that such a square emerges by showing that, as n increases, a concentration of $\mathcal{D}^n(G)$ develops around the origin; in particular, after noting that the origin is in $\mathcal{D}(G)$, we show that, for $z = (x,y) \in B_\infty(0, \lfloor\sigma/2\rfloor)\backslash\{0\}$,

$$|x| \ne |y| \text{ and } x \ne 0 \text{ and } y \ne 0 \Rightarrow z \text{ or } -z \in \mathcal{D}^{2\|z\|_\infty - 1}(G) \tag{1}$$

$$|x| = |y| \text{ or } x = 0, \text{ or } y = 0 \Rightarrow z \text{ or } -z \in \mathcal{D}^{2\|z\|_\infty}(G). \tag{2}$$

Since $B_\infty(0, \lfloor\sigma/2\rfloor) \subset z + \mathcal{N}$ for all $z \in B_\infty(0, \lfloor\sigma/2\rfloor)$, it then follows that

$$B_\infty(0, \lfloor\sigma/2\rfloor) \subset \mathcal{D}^{2\lfloor\sigma/2\rfloor+1}(G)$$

and G generates the plane.

Consider $z = (x,y) \in B_\infty(0, \lfloor\sigma/2\rfloor)$. Our main observation is that

$$b := |\mathcal{N}\backslash\left((z+\mathcal{N}) \cup (-z+\mathcal{N})\right)| \le 2(x^2 - |x|\cdot|y| + y^2). \tag{3}$$

Suppose (3) holds, and assume without loss of generality that $0 \le y \le x$. We prove (1) and (2) by induction on $\|z\|_\infty$; for fixed $\|z\|_\infty$ we first prove (1) and then use (1) to prove (2) (note that (1) is vacuous for $\|z\|_\infty = 1$). If $0 < y < x$ then it follows from our inductive assumption that

$$\begin{aligned} a :&= \left|\mathcal{D}^{2x-2}(G) \cap \left((z+\mathcal{N}) \cap (-z+\mathcal{N})\right)\right| \\ &\ge |\mathcal{D}^{2x-2}(G) \cap B_\infty(0, x-1)| \\ &\ge \lceil |B_\infty(0,x-1)|/2 \rceil \\ &= 2x^2 - 2x + 1. \end{aligned}$$

It then follows that the larger of $|(z+\mathcal{N}) \cap \mathcal{D}^{2x-2}(G)|$ and $|(-z+\mathcal{N}) \cap \mathcal{D}^{2x-2}(G)|$ is at least

$$\frac{1}{2}\left(|\mathcal{D}^{2x-2}(G) \cap \mathcal{N}| - a - b\right) + a \ge \frac{1}{2}|G| + \frac{a}{2} - \frac{b}{2} \ge \theta - \frac{1}{2}.$$

Thus, either z or $-z$ is in $\mathcal{D}^{2x-1}(G)$. If $y = x$ or $y = 0$ then the inductive assumption gives

$$a := \left|\mathcal{D}^{2x-1}(G) \cap ((z+\mathcal{N}) \cap (-z+\mathcal{N}))\right| \geq \left\lceil \frac{1}{2}\left((2x+1)^2 - 8\right)\right\rceil = 2x^2 + 2x - 3,$$

which implies that the larger of $|(z+\mathcal{N}) \cap \mathcal{D}^{2x-1}(G)|$ and $|(-z+\mathcal{N}) \cap \mathcal{D}^{2x-1}(G)|$ is at least $\theta - 1/2$, and it follows that either z or $-z$ is in $\mathcal{D}^{2x}(G)$.

It remains to prove (3). Assume without loss of generality that $0 \leq y \leq x$. For $v = (r, s) \in \mathbf{Z}^2$, let $\mathcal{O}_v$ be the octagon (or diamond or square) in $\mathbf{Z}^2$ with corners $\{(\pm r, \pm s), (\pm s, \pm r)\}$. It follows from the symmetry and convexity of $\mathcal{N}$ that $v \in \mathcal{N}$ implies $\mathcal{O}_v \subset \mathcal{N}$.

Let B be the part of $\mathcal{N} \backslash ((z+\mathcal{N}) \cup (-z+\mathcal{N}))$ on or above the x–axis.

Claim 8. $(r, s) \in B \Rightarrow s > \lfloor \sigma/2 \rfloor$.

Proof. Consider $v := (r, s) \in \mathcal{N}$. If $-\sigma \leq r \leq \sigma$ and $0 \leq s \leq \lfloor \sigma/2 \rfloor$ then either $v + z \in B_\infty(0, \sigma) \subset \mathcal{N}$ or $v - z \in B_\infty(0, \sigma) \subset \mathcal{N}$. If $|r| > \sigma$ and $0 \leq s \leq \lfloor \sigma/2 \rfloor$ then either $v + z \in \mathcal{O}_v \subset \mathcal{N}$ or $v - z \in \mathcal{O}_v \subset \mathcal{N}$. □

Claim 9. $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in B \Rightarrow |x_1 - x_2| < x$.

Proof. Assume for the sake of contradiction that $x_1 + x \leq x_2$. We consider two cases. If $z_1 + z$ lies below the line through z_1 and z_2 then consider the triangle T with vertices z_1, z_2 and $(x_1 + x, 0)$. By the convexity of $\mathcal{N}$, $T \cap \mathbf{Z}^2 \subset \mathcal{N}$. Thus $z_1 + z \in \mathcal{N}$ and $z_1 \in -z + \mathcal{N}$, which is a contradiction. On the other hand, if $z_1 + z$ lies above the line through z_1 and z_2 then $z_2 - z$ lies below this line. Furthermore, the second coordinate of $z_2 - z$ is positive by Claim 8. If T is now the triangle with vertices z_1, z_2 and $(x_2 - x, 0)$, then $z_2 - z \in T \cap \mathbf{Z}^2 \subset \mathcal{N}$ and so $z_2 \in z + \mathcal{N}$, a contradiction. □

If B is nonempty then let $u = (-p, q)$ be an element of B having the largest second component (i.e. $q = \max\{s : \exists (r, s) \in B\}$). We can assume $p \geq 0$ as $(0, \rho) \notin -z + \mathcal{N}$ (this follows from the assumption $B \neq \emptyset$) and $(0, \rho) \in z + \mathcal{N}$ implies that $\mathcal{N}$ intersected with the first quadrant is contained in $z + \mathcal{N}$ (this follows from the symmetry and convexity of $\mathcal{N}$).

Claim 10. *If $q < \sigma$ then $|B| \leq xy$.*

Proof. Suppose $q < \sigma$. If $v \in B$ and $v + e_2 \in \mathcal{N}$ then, by Claim 8 and the convexity and symmetry of $\mathcal{N}$, $v + e_2 \in B$. Therefore, $(r, s) \in B$ implies $r < -\sigma$.

Now, suppose there exists $v := (r, s) \in B$ with $s \leq q - y$. It follows from Claim 9 that $-p < r + x < -\sigma + \lfloor \sigma/2 \rfloor < 0$. Thus, $v + z$ is contained in the rectangle with corners at u and the origin, which is contained in $\mathcal{N}$. This is a contradiction; a point $(r, s) \in B$ must have $r > q - y$. By Claim 9, then, $|B| \leq xy$. □

We can therefore assume $q \geq \sigma$. We now divide B into two parts. Let $B' = \{(r,s) \in B : s > q - y\}$ and $B'' = B \backslash B'$. It follows from Claim 9 that

$$|B'| \leq xy.$$

To bound $|B''|$ we note that it follows from Claim 9 that B'' lies between $-z+\mathcal{N}$ and $z+\mathcal{N}$. Thus, $(r,s) \in B''$ is contained in the region bounded by the line $s = q - y$, $z + \mathcal{O}_u$ and $-z + \mathcal{O}_u$. We now show that neither the 'vertical part' of the octagon $z + \mathcal{O}_u$ nor the 'vertical part' of $-z + \mathcal{O}_u$ intersects the boundary of this region.

Claim 11. $(r,s) \in B'' \Rightarrow s > p + y$

Proof. We divide the argument into two cases. If $s < p - y$ then $r + x > q$ while $r - x < -q$. This implies $q - x < x - q$ and $x > q \geq \sigma$, which is a contradiction. If $p - y \leq s \leq p + y$ then $r - x < -q$ and $r + x > p + q - s - y$. This then gives $x - q > q - x + p - s - y$, which in turn gives $x + y > q \geq \sigma$, again a contradiction. □

We can thus conclude that all sites $(r,s) \in B''$ are contained in the triangular region given by

$$s \leq q - y,$$
$$r + s > (p + q) - (x + y)$$

and

$$r - s < (-p - q) + (x - y).$$

Therefore, $B'' = \emptyset$ if $x \leq y + p$ and

$$|B''| \leq (x - y - p)^2 \leq (x - y)^2$$

otherwise. This establishes (3) and thus concludes the proof of Theorem 7.

□

3 A Weight Function Argument

We prove both Theorem 3 and Part 2 of Theorem 5 by applying a weight function argument to the growth of the droplet around x (for other examples of weight function arguments in combinatorics see [BCG, pages 715-717],[ES]). The weight function we now define was introduced in [B] for the proof of Theorem 4. Let $Z_0 \subset Z_1 \subset \ldots \subset \mathbf{Z}^2$ be an arbitrary growing droplet (not necessarily given by any version of threshold growth). Our weight function considers a point z in the lattice 'heavy' or in a 'high energy state' if the neighborhood of z has a large intersection with the droplet but z is not yet in the droplet. To be precise, we set

$$w_t(z) = \begin{cases} |(z+\mathcal{N}) \cap Z_t| & z \notin Z_t \\ 0 & z \in Z_t. \end{cases} \tag{4}$$

We will observe the evolution over time of the total weight of some carefully chosen set X; that is, we will consider the function

$$W_t = W_t(X) := \sum_{z \in X} w_t(z) \tag{5}$$

where $X \subset \mathbf{Z}^2$ will be determined.

Now, for each $z \in Z_t \backslash Z_{t-1}$ there are two possible contributions to the change in the total wieght (i.e. $W_t - W_{t-1}$) when z joins the droplet: the weight of z itself goes to zero while the weight of each element of $(z+\mathcal{N})\backslash Z_t$ increases by one. Thus, if

$$\Delta_z := -\mathbf{1}_X(z)|(z+\mathcal{N}) \cap Z_{t-1}| + |(z+\mathcal{N}) \cap (X \backslash Z_t)|, \tag{6}$$

where $\mathbf{1}_X$ is the characteristic function of X, then we have

$$W_t = W_{t-1} + \sum_{z \in Z_t \backslash Z_{t-1}} \Delta_z. \tag{7}$$

In words Δ_z is the change in the total weight, or energy, that is caused by z joining the droplet. The central point in the proofs that follow is that when a droplet generated by random threshold growth exhibits continuing growth Δ_z is often (or on average) nonnegative. In other words, if every point (or the average point) causes a loss of energy then growth will eventually stop. The combinatorial consequence of this fact is that the droplet eventually becomes dense around most elements of the droplet. This density property can then be used to show that a subset of the droplet that generates the plane lies near all elements of the droplet.

Proof of Part 2 of Theorem 5. Let $R = R(\rho) = 150\rho^3(\log \rho + 1)$. We are assuming that $\mathcal{N} = -\mathcal{N}$, $\mathcal{N} \subset \mathcal{N}_\rho$ and the double threshold condition holds. We further assume $x \in A_n$ and $B_\infty(x, R) \cap A_0 = \emptyset$; in words, $x \in \mathbf{Z}^2$ joins the droplet at time n (i.e. $n = n(x, A_0)$ is the hitting time of x) and is far from the initial seed. Given these assumptions, we need only find

$$z \in B_\infty(x, R-\rho) \text{ such that } |(z+\mathcal{N}) \cap A_n| \geq 2\theta. \tag{8}$$

We use the weight function given in (4) and (5) where $Z_0 \subset Z_1 \subset \ldots$ is simply taken to be the droplet generated by random threshold growth dynamics, $A_0 \subset A_1 \subset \ldots$. The set over which we consider the total weight (the set X in equation (5)) is

$$X_r := \{z \in A_n : \|z - x\|_\infty \leq r\}$$

where the radius r is to be determined but will satisfy $r \leq R - \rho$. Since we are assuming $A_0 \cap B_\infty(x, R) = \emptyset$ we have

$$W_0 = 0. \tag{9}$$

Furthermore,

$$W_n = 0 \tag{10}$$

because $X_r \subset A_n$ (i.e. $w_n(z) = 0$ for all $z \in X_r$).

We begin by analyzing Δ_z for $z \in A_t \backslash A_{t-1}$ where $1 \leq t \leq n$. First note that only those elements of $A_t \backslash A_{t-1}$ in or near X_r influence the total weight when they join the droplet. To be precise, letting

$$B_r := \{z \in A_n : r < \|z - x\|_\infty \leq r + \rho\}$$

it follows from (6) that $z \notin X_r \cup B_r$ implies $\Delta_z = 0$. It also follows from (6) that for $z \in B_r$

$$\Delta_z \leq |(z + \mathcal{N}) \cap X_r| \leq \rho(2\rho + 1) \tag{11}$$

and for $z \in X_r$

$$\begin{aligned} \Delta_z &\leq -|(z + \mathcal{N}) \cap A_{t-1}| + |(z + \mathcal{N}) \cap (A_n \backslash A_t)| \\ &\leq -\theta + |(z + \mathcal{N}) \cap (A_n \backslash A_t)|. \end{aligned}$$

So, if $\Delta_z \geq 0$ for $z \in X_r$ then

$$|(z + \mathcal{N}) \cap (A_n \backslash A_t)| \geq \theta$$

and

$$\begin{aligned} |(z + \mathcal{N}) \cap A_n| &= |(z + \mathcal{N}) \cap A_{t-1}| + |(z + \mathcal{N}) \cap (A_t \backslash A_{t-1})| \\ &\quad + |(z + \mathcal{N}) \cap (A_n \backslash A_t)| \\ &\geq |(z + \mathcal{N}) \cap A_{t-1}| + |(z + \mathcal{N}) \cap (A_n \backslash A_t)| \\ &\geq 2\theta. \end{aligned}$$

Thus, if there exists $z \in X_r$ such that $\Delta_z \geq 0$ then we have (8) and we are done.

Now, we consider the evolution of W_t as t goes from 0 to n. It follows from (7) that

$$W_0 + \sum_{z \in B_r} \Delta_z + \sum_{z \in X_r} \Delta_z = W_n.$$

By (9) and (10) this implies

$$\sum_{z \in B_r} \Delta_z = -\sum_{z \in X_r} \Delta_z.$$

Now, if $\Delta_z \le -1$ for all $z \in X_r$, then by (11) we have

$$\rho(2\rho+1)|B_r| \ge |X_r|. \tag{12}$$

In other words, if (12) does not hold then there exists $z \in X_r$ such that $\Delta_z \ge 0$ which implies (8) and the proof is complete.

Assume for the sake of contradiction that (12) holds for $0 < r < R - \rho$. We consider the sequence of radii $r_0, r_1, \ldots$ where $r_i = \rho i$. Clearly,

$$|X_{r_i}| = |X_{r_{i-1}}| + |B_{r_{i-1}}|$$

and

$$|X_{r_0}| = 1.$$

Since (12) holds for all r_i we have

$$\begin{aligned} |X_{r_i}| &\ge \left(1 + \frac{1}{\rho(2\rho+1)}\right) |X_{r_{i-1}}| \\ &\ge \left(1 + \frac{1}{\rho(2\rho+1)}\right)^i |X_{r_0}| \\ &\ge \exp\left(\frac{i}{2\rho(2\rho+1)}\right). \end{aligned}$$

But, we also have the obvious bound $|X_{r_i}| \le (2r_i+1)^2 = (2\rho i+1)^2$. This leaves us with

$$(2\rho i+1)^2 \ge \exp\left(\frac{i}{2\rho(2\rho+1)}\right)$$

which is clearly a contradiction for $i \ge 150\rho^2(\log\rho + 1) - 1$. $\square$

4 Proof of Theorem 3

Throughout this section we assume our neighborhood is $\mathcal{N}_\rho$, our threshold satisfies $\rho^2 < \theta \le \rho(2\rho+1)$, $x \in A_n$ and $B_\infty(x,R) \cap A_0 = \emptyset$ where $R = R(\rho) = 1000\rho^3(\log\rho+1)$. We are looking for $G \subset B_\infty(x,R) \cap A_n$ that generates the plane. We argue indirectly; we assume no such G exists (i.e. we assume $B_\infty(x,R) \cap A_n$ does not generate the plane). This assumption implies a strong condition on the structure of A_n near x. We use this structural condition and the weight function given in (4) and (5) to arrive at a contradiction.

We begin with some preliminary definitions and results taken from [B]. For $z \in \mathbf{Z}^2$ let S_z be the $\rho \times \rho$ square whose lower left hand corner is z: $S_z = \{z + ie_1 + je_2 : 0 \le i,j < \rho\}$ (here $e_1 = (1,0)$ and $e_2 = (0,1)$). For an arbitrary $\mathcal{A} \subset \mathbf{Z}^2$ let

$V = \{S_z : S_z \subset \mathcal{A}\}$. We form a graph G on vertex set V by joining vertices S_z and S_y with an edge if and only if $|S_z \cap S_y| = \rho(\rho - 1)$. In other words, S_z and S_y are joined by an edge if $z \in \{y \pm e_1, y \pm e_2\}$. For each $U \subset V$ that defines a maximal connected component of G, the set

$$\mathcal{B} := \bigcup_{S_z \in U} S_z$$

is called a *jagged block* of $\mathcal{A}$.

Note that the definition of jagged blocks given here differs from the definition given in [B]. In [B] jagged blocks are defined only for $\mathcal{A} = \mathcal{D}^\infty(A)$ for some finite set A – a notion that is of no use in the present context. This difference in definitions yields an important distinction: the jagged blocks defined in [B] always satisfy

$$d_{xy} = \|x - y\|_\infty, \tag{13}$$

where, for vertices S_x and S_y in the graph defined above, d_{xy} is the length of a shortest path from S_x to S_y in G. A jagged block $\mathcal{B}$ as defined here may not satisfy (13), but if it does not satisfy (13), then $\mathcal{D}(\mathcal{B}) \neq \mathcal{B}$ (the proof of this fact is the same as the proof of equation (4) in [B]). In applying results about jagged blocks from [B], we can therefore assume that the jagged blocks in question satisfy (13) ; otherwise, $\mathcal{D}(\mathcal{B}) \neq \mathcal{B}$ and the desired property is either established immediately or upon repeated iteration of the growth rule.

A large part of the proof of Theorem 4 given in [B] is in showing that deterministic threshold growth dynamics cannot build a 'large' jagged block without simultaneously building something that generates the plane. The same is true for random threshold growth dynamics, and this is the central point of the argument in this section. This is not to say that the jagged blocks we consider generate the plane; in fact, they usually do not.

We now make precise the notion of a 'large' jagged block. Let

$$\kappa = \max\left\{ \left\lfloor \frac{\rho}{2} + 1 \right\rfloor, \left\lfloor \frac{\theta}{2\rho + 1} \right\rfloor \right\}.$$

For an arbitrary $z \in \mathbf{Z}^2$ let

$$\mathcal{C}_z = B_\infty(z, \kappa) \backslash \{z \pm \kappa e_1 \pm \kappa e_2\}.$$

This is a $(2\kappa + 1) \times (2\kappa + 1)$ square with its corners removed. We say that a jagged block $\mathcal{B}$ has a *center* $\mathcal{C}_\mathcal{B}$ if there exists z such that $\mathcal{C}_\mathcal{B} := \mathcal{C}_z \subset \mathcal{B}$. Think of a jagged block as being 'large' if it has a center.

Now we state some facts concerning jagged blocks. For finite $\mathcal{A} \subset \mathbf{Z}^2$ the *diameter* of $\mathcal{A}$, denoted $diam(\mathcal{A})$, is given by $diam(\mathcal{A}) = \max\{\|z - y\|_\infty : z, y \in \mathcal{A}\}$. For nonempty $\mathcal{A}, \mathcal{A}' \subset \mathbf{Z}^2$ the distance between $\mathcal{A}$ and $\mathcal{A}'$ is defined to be $d(\mathcal{A}, \mathcal{A}') = \min\{\|z - y\|_\infty : z \in \mathcal{A}, y \in \mathcal{A}'\}$.

Lemma 12. *If $\mathcal{B}$ is a jagged block with diam$(\mathcal{B}) > 2\rho^3$ then $\mathcal{B}$ generates the plane.*

The authors suspect that the bound given in Lemma 12 (which follows from Lemma 4 in [B]) is not the best possible, as results in [GG3] suggest that any jagged block with diameter larger than $C\rho^2$ generates the plane. However, the bound given in Lemma 12 suffices for the argument we give here.

Lemma 13. *Suppose $A_0 \subset A_1 \subset \cdots \subset A_n \subset \mathbf{Z}^2$ is generated via the random threshold growth rule. If $\mathcal{B}$ is a jagged block of A_n that both has a center and satisfies $d(\mathcal{B}, A_0) > \rho$ then there exists $z \in \mathcal{D}(A_n)\backslash A_n$ such that $d(\{z\}, \mathcal{B}) \leq \rho$.*

Lemma 13 follows from the Low Density Lemma in [B] and is the central fact concerning jagged blocks. Intuitively, the site z in Lemma 13 is found near the first element of $\mathcal{B}$ to join the droplet. It follows from Lemmas 12 and 13 that if $\mathcal{B}$ is a jagged block of A_n with center such that $d(\mathcal{B}, A_0) > 2\rho^3$ then $A_\infty = \mathbf{Z}^2$ (by Lemma 13, $\mathcal{B}$ grows until it is a distance less than ρ from A_0. Lemma 12 then implies that this large jagged block generates the plane).

In the course of the argument we make use of the following function. Let A be an arbitrary subset of $\mathbf{Z}^2$ and $H = \mathbf{Z}^2 \backslash A$ (the set of *holes* relative to A). Let $H = \{h_1, h_2, \dots\}$ be an arbitrary ordering of the holes. We define $\varphi_A : A \to H$ to be the map by which $z \in A$ is mapped to the nearest hole in ℓ_∞ distance where the first tie breaker is ℓ_1 distance and the second tie breaker is the ordering on H.

Lemma 14. *Let $h \in H$ where H is the set of holes relative to some $A \subset \mathbf{Z}^2$. If $|(h + \mathcal{N}) \cap A| < \theta$ and $\varphi_A^{-1}(h) \subset \mathcal{C}_h$ then either $A \cap B_\infty(h, 2\rho)$ generates the plane or*

$$\sum_{z \in \varphi_A^{-1}(h)} |(z + \mathcal{N}) \cap A| \leq |\varphi_A^{-1}(h)| 2\theta.$$

Lemma 14, which follows from the Local Averaging Claim in [B], provides a link between our knowledge concerning jagged blocks and the weight function we use in the proof.

With these preliminary definitions and results in hand we can discuss the 'structural implications' of our initial assumptions. We construct a sequence that contains information about both the random droplet's approach to x and the properties this approach to x deterministically has. Let

$$C_i = \begin{cases} A_i \cap B_\infty(x, R) & 0 \leq i \leq n \\ \mathcal{D}(C_{i-1}) \cap B_\infty(x, R) & n < i \leq m \end{cases}$$

where m is the 'final time' in the sense that $\mathcal{D}(C_m) \cap B_\infty(x, R) = C_m$. Since *any* set of eligible points (i.e. points whose neighborhoods contain at least a threshold number of already occupied points) can join the droplet at each time step in random threshold growth, we can think of $C_0, C_1, \ldots, C_m$ as being generated by the random threshold growth rule if we think of adjoining the set $B_\infty(x, R+\rho)\backslash B_\infty(x, R)$ to each C_i.

Note that if $C_m = B_\infty(x, R)$ then, by Lemma 12, $G := A_n \cap B_\infty(x, R)$ generates the plane. Since this contradicts our initial assumption, $C_m \neq B_\infty(x, R)$.

Consider an arbitrary jagged block $\mathcal{B}$ of C_m with center. Suppose $\mathcal{B} \subset B_\infty(x, R-\rho)$. Since the ℓ_∞ distance between $\mathcal{B}$ and our 'adjoined initial droplet' is greater than ρ we can apply Lemma 13. Thus, there exists $z \in \mathbf{Z}^2$ such that $z \in \mathcal{D}(C_m)\backslash C_m$ and $z \in B_\infty(x, R)$. This contradicts $C_m = \mathcal{D}(C_m) \cap B_\infty(x, R)$. We have shown

$$\mathcal{B} \text{ is a jagged block of } C_m \text{with center} \Rightarrow \mathcal{B} \not\subset B_\infty(x, R-\rho).$$

Furthermore, Lemma 12 implies that the diameter of $\mathcal{B}$ is less than $2\rho^3$. Thus,

$$\mathcal{B} \text{ is a jagged block of } C_m \text{ with center} \Rightarrow \mathcal{B} \cap B_\infty(x, R-3\rho^3) = \emptyset.$$

Since, for any $z \in \mathbf{Z}^2$, $\mathcal{C}_z$ itself is a jagged block with center, we now have

$$z \in B_\infty(x, R-3\rho^3) \Rightarrow \mathcal{C}_z \not\subset C_m. \tag{14}$$

This fact is the 'structural implication' of the assumption that no $G \subset C_m \cap B_\infty(x, R)$ generates the plane mentioned in the first paragraph of this section. We now forget about jagged blocks and use (14) and the weight function given in (4) and (5) to achieve a contradiction. In this application of (4) and (5) the sequence $Z_1, Z_2, \ldots$ is taken to be $C_0 \subset C_1 \subset \ldots \subset C_m \subset C_m \subset \ldots$, and the set over which we consider the total weight (the set X in (5)) is

$$X_r := B_\infty(x, r) \cap C_m$$

where $r < R - 3\rho^3$ is to be determined.

What happens to W_t as t goes from 0 to m? We clearly have

$$W_0 = W_m = 0 \tag{15}$$

For z in

$$B_r := (B_\infty(x, r+\rho)\backslash B_\infty(x, r)) \cap C_m,$$

$\Delta_z \leq \rho(2\rho+1)$. For $z \notin X_r \cup B_r$, $\Delta_z = 0$. These facts together with (15) give

$$\sum_{z \in X_r} \Delta_z \geq -(2\rho+1)\rho|B_r|. \tag{16}$$

We now analyze Δ_z for $z \in X_r$. This analysis differs from that of Section 3. Instead of considering Δ_z for each $z \in X_r$ individually, we consider sums of Δ_z's over subsets of X_r. Before doing so we note that for $z \in X_r \cap (C_t\backslash C_{t-1})$

$$\Delta_z \leq -|(z+\mathcal{N}) \cap C_{t-1}| + |(z+\mathcal{N}) \cap (C_m\backslash C_t)|$$

Thus

$$\begin{aligned}|(z+\mathcal{N})\cap C_m| &= |(z+\mathcal{N})\cap C_{t-1}| + |(z+\mathcal{N})\cap (C_t\backslash C_{t-1})| + |(z+\mathcal{N})\cap (C_m\backslash C_t)|\\ &\geq |(z+\mathcal{N})\cap C_{t-1}| + 1 + |(z+\mathcal{N})\cap C_{t-1}| + \Delta_z\\ &\geq 2\theta + 1 + \Delta_z\end{aligned}$$

and

$$\Delta_z \leq |(z+\mathcal{N})\cap C_m| - (2\theta+1). \tag{17}$$

Inequalities (16) and (17) yield

$$\sum_{z\in X_r} |(z+\mathcal{N})\cap C_m| - (2\theta+1)|X_r| \geq -(2\rho+1)\rho|B_r|. \tag{18}$$

We use the function φ_{C_m} to partition X_r into subsets. In particular, we divide X_r into sets of the form $\varphi_{C_m}^{-1}(h)$ where $\varphi_{C_m}^{-1}(h) \subset B_\infty(x,r)$. Since, for an arbitrary $z \in X_r$, $\varphi_{C_m}^{-1}(\varphi_{C_m}(z))$ is not necessarily contained in X_r, this division does not account for all of X_r. Therefore, we need a set of 'leftovers' to complete the partition. Let

$$H_r = \{h \in C_m\backslash \mathbf{Z}^2 : \varphi_{C_m}^{-1}(h) \subset X_r\}.$$

and

$$B_r' = \{z \in X_r : \varphi_{C_m}(z) \notin H_r\}$$

(B_r' is the set of 'leftovers'). We clearly have

$$\sum_{z\in X_r} |(z+\mathcal{N})\cap C_m| = \sum_{h\in H_r}\sum_{z\in\varphi_{C_m}^{-1}(h)} |(z+\mathcal{N})\cap C_m| + \sum_{z\in B_r'} |(z+\mathcal{N})\cap C_m|. \tag{19}$$

Consider $z \in B_\infty(x, R-3\rho^3)$. It follows from (14) that

$$\varphi_{C_m}(z) \in \mathcal{C}_z. \tag{20}$$

By symmetry, this gives

$$\varphi_{C_m}^{-1}(h) \subset \mathcal{C}_h$$

for all $h \in B_\infty(x, R-3\rho^3)\backslash C_m$. So we can apply Lemma 14 to $\varphi_{C_m}^{-1}(h)$ for all $h \in H_r$. When we do so (19) becomes

$$\sum_{z\in X_r} |(z+\mathcal{N})\cap C_m| \leq \sum_{h\in H_r} 2\theta|\varphi_{C_m}^{-1}(h)| + \sum_{z\in B_r'} |(z+\mathcal{N})\cap C_m|.$$

It also follows from (20) that

$$B_r' \subset B_r'' := C_m \cap (B_\infty(x,r)\backslash B_\infty(x, r-2\rho)).$$

For $z \in B'_r$ we use the trivial bound $|(z+\mathcal{N}) \cap C_m| \le |\mathcal{N}| \le 9\rho^2$. This gives

$$\begin{aligned}\sum_{z \in X_r} |(z+\mathcal{N}) \cap C_m| &\le \sum_{h \in H_r} 2\theta |\varphi^{-1}_{C_m}(h)| + 9\rho^2 |B'_r| \\ &\le 2\theta |X_r| + 9\rho^2 |B''_r|.\end{aligned} \tag{21}$$

From (21) and (18) we get

$$2\theta |X_r| + 9\rho^2 |B''_r| - (2\theta+1)|X_r| \ge -3\rho^2 |B_r|$$

which simplifies to

$$|X_r| \le 9\rho^2(|B_r| + |B''_r|). \tag{22}$$

To finish the proof we consider the sequence of radii $r_0, r_1, \ldots$ where $r_i = \rho i$. Clearly,

$$|X_{r_0}| = 1$$

and

$$|X_{r_{3i}}| = |X_{r_{3(i-1)}}| + |B_{r_{3i-1}}| + |B''_{r_{3i-1}}|.$$

It follows from (22) that

$$|X_{r_{3(i-1)}}| \le |X_{r_{3i-1}}| \le 9\rho^2(|B_{r_{3i-1}}| + |B''_{r_{3i-1}}|).$$

Thus

$$\begin{aligned}|X_{r_{3i}}| &\ge |X_{r_{3(i-1)}}| \left(1 + \frac{1}{9\rho^2}\right) \\ &\ge |X_{r_0}| \left(1 + \frac{1}{9\rho^2}\right)^i \\ &> \exp\left(\frac{i}{18\rho^2}\right).\end{aligned}$$

But, we also have the trivial

$$|X_{r_{3i}}| \le |B_\infty(x, r_{3i})| = (2r_{3i}+1)^2 = (6\rho i + 1)^2.$$

This leaves us with

$$(6\rho i + 1)^2 \ge \exp\left(\frac{i}{18\rho^2}\right)$$

which is a contradiction for $i \ge 330\rho^2(\log \rho + 1)$.

5 Convergence to the asymptotic shape

We prove only Theorem 2 as Theorem 6 is proved in exactly the same way (except that Theorem 5 is used in place of Theorem 3).

As usual, we start by introducing a coupling of the random dynamics started from all initial sets. This will be done by choosing independent random vectors $(Y_0^{x,t}, \dots, Y_N^{x,t})$, $x \in \mathbf{Z}^2$, $t = 0, 1, \dots$, such that $Y_0^{x,t} \le \dots \le Y_N^{x,t}$ and $P(Y_k^{x,t} = 1) = p_k = 1 - P(Y_k^{x,t} = 0)$. We will drop the superscripts when we are only interested in the distributions, e.g., $Y_k^{x,t} = Y_k$. Then we can define simply

$$A_{t+1} = A_t \cup \{x \in \mathbf{Z}^2 : Y^{x,t}_{|A_t \cap (x+\mathcal{N})|} = 1\}.$$

Note that this coupling is monotone: enlargement of the initial set enlarges the growing droplet at all times.

Proof of Theorem 2. We will shorten $p = p_\theta$ and let τ stand for a generic geometric random variable with parameter p (that is, $P(\tau = k) = p(1-p)^{k-1}$ for $k = 1, 2, \dots$).

Fix an r large enough so that $B_\infty(0, r)$ generates the plane. We will assume that $A_0 = B_\infty(0, r)$. This entails no loss of generality, as the following simple argument demonstrates. Let A'_t be the random threshold growth starting from an arbitrary set A_0 that generates the plane. Then there exists a (random) finite time T_0 such that $A_0 \subset A'_{T_0}$ and $A'_0 \subset A_{T_0}$. Hence

$$A_{t-T_0} \subset A'_t \subset A_{t+T_0} \qquad \text{for } t \ge T_0.$$

Now define

$$\begin{aligned}
T(x) &= \inf\{t : x \in A_t\} \\
T'(x) &= \inf\{t : B_\infty(x, r) \subset A_t\}, \\
T'(x, y) &= \inf\{t : B_\infty(y, r) \subset A_{t+T'(x)}(B_\infty(x, r), T'(x))\}.
\end{aligned}$$

Here, $A_{t+s}(B, s)$ is the notation for the state of the dynamics at time $t + s$ if it is re-started with the set B at time s. Let $A'_t = \{x : B_\infty(x, r) \subset A_t\}$. Our first step is to prove that A_t is not too far ahead of A'_t.

Theorem 3 implies that every $x \in A_t$ has a set that generates the plane included in $B_\infty(x, 2R)$ (recall $R = 1000\rho^3(\log\rho + 1)$). For every set $G \subset B_\infty(x, 2R)$ that generates the plane, define

$$\begin{aligned}
T_G^d(x) &= \inf\{n : B_\infty(x, r) \subset \mathcal{D}^n(G)\}, \\
T_G(x) &= \inf\{t : B_\infty(x, r) \subset A_{t+T(x)}(G, T(x))\}.
\end{aligned}$$

At time $T_G^d(x)$, the deterministic dynamics occupies at most $(2(2R + \rho T_G^d(x)) + 1)^2$ sites, and this number is bounded above by a constant $M = M(\rho)$, independent of x

and G. Therefore, $T_G(x)$ is bounded above by the sum of M independent copies of τ. The following crude bounds then hold for any $s \geq 0$,

$$\begin{aligned} P(T'(x) - T(x) \geq s) &\leq M \cdot P(\tau \geq s/M) \\ &\leq M \cdot e^{-ps/M}. \end{aligned} \tag{23}$$

Of course, $A_t \subset \mathcal{D}^t(A_0)$ and so A_t includes at most $(2(r+t\rho)+1)^2$ sites. Therefore, for a constant $C = C(\rho, p)$,

$$P(T'(x) - T(x) > C \log t \text{ for at least one } x \in A_t) \leq Ct^{-2}$$

and hence there exists a random T_0 such that $T'(x) - T(x) \leq C \log t$ for every $x \in A_t$ as soon as $t \geq T_0$. It follows that

$$A'_t \subset A_t \subset A'_{t+C\log t} \qquad \text{for } t \geq T_0. \tag{24}$$

The last step is to use subadditivity to show that A'_t has a limiting shape. By monotonicity, $T'(y) \leq T'(x) + T'(x,y)$; in addition, the two summands are independendent and $T'(x,y) \overset{d}{=} T'(x-y)$. Moreover, if $e_1 = (1,0)$, then the same arguments as those leading to (24) show that there is a constant $M = M(\rho) > 30$ such that, for every $s \geq 0$, $P(T'(e_1) > s) \leq M \cdot \exp(-ps/M)$. In particular, it follows that $E(T'(x)) < \infty$ for every x. Moreover, if $\delta = \delta(p,\rho) = p/(12M\log(2M))$, and $T_1, T_2, \ldots$ are i.i.d. versions of $T'(e_1)$, then

$$\begin{aligned} P(B_\infty(0,\delta t) \not\subset A'_t) &\leq (2\delta t+1)^2 P\left(\sum_{i=1}^{2\delta t} T_i \geq t\right) \\ &= (2\delta t+1)^2 P\left(\exp\left(\frac{p}{2M}\sum_{i=1}^{2\delta t} T_i - \frac{p}{2M}t\right) \geq 1\right) \\ &\leq (2\delta t+1)^2 \exp\left(-\frac{p}{2M}t\right) \cdot E\left(\exp\left(\frac{p}{2M}T'(e_1)\right)\right)^{2\delta t} \\ &\leq e^{4\delta t} \exp\left(-\frac{p}{2M}t\right) \cdot (2M)^{2\delta t} \\ &\leq \exp\left(-pt/(4M)\right), \end{aligned} \tag{25}$$

so that, with probability 1, $A'_t + I$ eventually includes $B_\infty(0,\delta t)$.

From now on, completely standard arguments take over (Chapter 1 of [D] and [CD]). Namely, if one extends $T'(x)$ to $x \in \mathbf{R}^2$ by $T'(x) = \inf\{t : x \in A'_t + I\}$, then, by the Subadditive Ergodic Theorem, for every $x \in \mathbf{R}^2$

$$T'(nx)/n \to \mu(x) \text{ a.s. as } n \to \infty, \tag{26}$$

where $\mu(x)$ is a deterministic constant. This function is a norm on $\mathbf{R}^2$; the a.s. convergence (26) for all x in an appropriate finite set and the lower bound (25) imply that the unit ball $S = \{x : \mu(x) \leq 1\}$ is the limiting shape: $(A'_t + I)/t \to S$. This fact, together with (24), ends the proof. $\square$

References

[A] K. S. Alexander, Approximation of subadditive functions and convergence rates in limiting shape results. *Ann. Probab.* 25 (1997), 30–55.

[B] T. Bohman, Discrete threshold growth dynamics are omnivorous for box neighborhoods. *Transactions of the AMS*, to appear.

[BGG] T. Bohman, J. Gravner, D. Griffeath, Shapes in random threshold growth models. In preparation.

[BCG] E. Berlekamp, J. Conway, R. Guy, *Winning Ways for your Mathematical Plays.* Academic Press, 1982.

[CD] J. T. Cox, R. Durrett, Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* 9 (1981), 583–603.

[D] R. Durrett, *Lecture Notes on Particle Systems and Percolation.* Wadsworth & Brooks/Cole, 1988.

[DL] R. Durrett, T. Liggett, The shape of the limit set in Richardson growth model. *Ann. Probab.* 9 (1981), 186–193.

[ES] P. Erdős and J. Selfridge, On a combinatorial game. *J. Comb. Theory A* 14 (1973), 298-301.

[GG1] J. Gravner, D. Griffeath, Threshold growth dynamics. *Transactions of the AMS* 340 (1993), 837-869.

[GG2] J. Gravner, D. Griffeath, First passage times for threshold growth dynamics on $\mathbf{Z}^2$. *Ann. Probab.* 24(1996), 1752-1778.

[GG3] J. Gravner, D. Griffeath, Nucleation parameters for discrete threshold growth. *Experimental Mathematics* 6(1997), 207-220.

[K] H. Kesten, On the speed of convergence in first-passage percolation. *Ann. Appl. Prob.* 4 (1994), 76–107.

[KS] J. Krug, H. Spohn, *Kinetic roughening of growing surfaces.* Pages 478–582 of *Solids Far From Equilibrium* (G. Godrèche, ed.), Cambridge Univ. Press, 1991.

[NP] C. M. Newman, M. S. T. Piza, Divergence of shape fluctuations in two dimensions. *Ann. Probab.* 23 (1995), 977-1005.

[S] K. Schürger, On the asymptotical geometrical behavior of a class of contact interaction processes with a monotone infection rate. *Z. Wahrsch. Verw. Gebiete* 48 (1979), 35–48.

[W] S. J. Willson, On convergence of configurations. *Discrete Mathematics* 23 (1978), 279–300.