RECURRENT RING DYNAMICS IN TWO–DIMENSIONAL EXCITABLE CELLULAR AUTOMATA

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Abstract. The *Greenberg–Hastings model (GHM)* is a simple cellular automaton which emulates two properties of excitable media: excitation by contact and a refractory period. We study two ways in which external stimulation can make *ring* dynamics in the GHM recurrent. The first scheme involves initial placement of excitation centers which gradually lose strength, i.e. each time they become inactive (and then stay so forever) with probability $1 - p_f$. In this case, the density of excited sites must go to 0; however, their long–term connectivity structure undergoes a phase transition as p_f increases from 0 to 1. The second proposed rule utilizes continuous nucleation: new rings are started at every rested site with probability p_s. We show that, for small p_s, this dynamics makes a site excited about every $p_s^{-1/3}$ time units. This result yields some information about the asymptotic shape of a closely related random growth model.

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1. Introduction.

One of the simplest mathematical models for excitable media is the 3-color *Greenberg-Hastings model (GHM)* γ_t. The state space of this cellular automaton is $\{0,1,2\}^{\mathbf{Z}^2}$, where 1 represents the excited state, 0 the rested state, and 2 the refractory (or recovering) state. We assume that the local *neighborhood* of a site x is given by $\mathcal{N}_x = x + \mathcal{N}$, where $\mathcal{N}$ is a finite set containing the origin. Excitation by contact and the refractory period are thus incorporated into the update rule:

$$(1.1)\qquad \gamma_{t+1}(x) = \begin{cases} 2, & \text{if } \gamma_t(x) = 1, \\ 1, & \text{if } \gamma_t(x) = 0 \text{ and } 1 \in \gamma_t(\mathcal{N}_x), \\ 0, & \text{otherwise.} \end{cases}$$

Since this model was introduced in [WR], and resurrected in [GH], it has been studied by a number of researchers from various fields, and underwent many generalizations (see [DG], [DN], [DS], [FGG1], [FGG2], [FK], [Gra1], [Gri2], [Ste], [WTW] and many further references contained in these papers).

We will assume, throughout the paper, that $\mathcal{N} = B_\infty(0,1)$ is the Moore neighborhood (we use the standard notation for balls: $B_r(x,R) = \{y : ||y-x||_r \le R\}$). This is an assumption of convenience; our results remain true, with minor modifications, for general symmetric irreducible neighborhoods ([Gra1]). If the initial state of the GHM dynamics is given by a positive density of 1's on a background of 0's, then the update rule (1.1) generates square rings of excitation, with a transient percolation property. To make this more precise, fix a finite set $\mathcal{D} \subset \mathbf{Z}^2$ and say that a set $G \subset \mathbf{Z}^2$ *$\mathcal{D}$–percolates* if there exists an infinite sequence $x_1, x_2, \dots$ of distinct sites in G such that $x_{k+1} - x_k \in \mathcal{D}$ for $k = 1, 2, \dots$. The usual *ℓ^∞–percolation* is given by $\mathcal{D} = B_\infty(0,1)$, while the *$\ell^1$–percolation* is given by $\mathcal{D} = B_1(0,1)$. The following theorem is contained in [Gra1].

Theorem 1. *Assume that γ_0 is a product measure with $P(\gamma_0(x) = 1) = 1 - P(\gamma_0(x) = 0) = p$ for every x. Each site x has a unique time t at which $\gamma_t(x) = 1$. Moreover, for each p there exists a time $T(p)$ such that the set $\{\gamma_t = 1\}$ ℓ^∞–percolates for $t = T(p)$, but does not for $t \notin [T(p), T(p)+1]$. Finally, $T(p)\sqrt{p}$ converges to a positive constant λ_c as $p \to 0$.*

Loosely put, then, the GHM dynamics makes very short–lived connected rings. In this paper, we propose two simple models which have recurrence properties absent in the basic GHM. These models are simple caricatures of a physical excitable medium which is externally or spontaneously excited due to presence of catalysts, thermal or electrical stimuli, or impurities.

In the first rule, referred to as *annihilating nested rings (ANR)*, we envision every site either in one of the ordinary states 0,1,2 or in one of the externally excited states e_0, e_1, e_2. The externally excited states go through excitation cycle automatically, but they turn into ordinary states with probability $1 - p_f$ every excitation period; the normal states behave as in (1.1). To be more precise, we define the ANR process $\tilde{\gamma}_t$ as a discrete-time Markov chain with state space $\{0, 1, 2, e_0, e_1, e_2\}^{\mathbf{Z}^2}$ and the following transition rule at a site x:

$$
\begin{array}{ll}
\left.\begin{array}{l} 1 \to 2 \\ 2 \to 0 \\ e_0 \to e_1 \\ e_2 \to e_0 \end{array}\right\} & \text{automatically,} \\[4pt]
e_1 \to e_2 & \text{with probability } p_f, \\
e_1 \to 2 & \text{with probability } 1 - p_f, \\
0 \to 1 & \text{if there is either a 1 or an } e_1 \text{ in } \mathcal{N}_x, \\
0 \to 0 & \text{otherwise.}
\end{array}
\tag{1.2}
$$

To understand the nature of these dynamics, start first with a single e_1 surrounded by 0's. This e_1 proceeds to generate a geometrically distributed number of concentric expanding square rings before it finally turns into a 2. The created nested rings then keep expanding forever. If we start with two e_1's, their rings annihilate upon collision, but only along their intersection. We will assume throughout that the ANR is started from a fully excited state: $\tilde{\gamma}_0(x) = e_1$ for every x (Theorems 2, 3 and 5 hence implicitly assume this).

In the applied literature, ring dynamics such as the one given by (1.2) are usually referred to as *target states*. They have been long known to arise in a variety of biological and chemical contexts (see, for example, the special issue of Physica D 49 (1991), titled *Waves and patterns in chemical and biological media*). Early investigations of the Belousov–Zhabotinsky reaction, for example, have demonstrated that target states are ubiquitous in the petri dish experiments and explained their appearance by modeling with reaction–diffusion PDE's ([RKM], [TF]). Often, the centers which create expanding rings are spatial heterogeneities, such as catalytic particles or energy sources. Here we study global properties of such systems in the case when the centers are not permanent, but their energy gradually dissipates until it falls below the level necessary to induce excitation, at which point they effectively vanish from the medium. In our view, the ANR model (2.1), started from a translation invariant initial state, is the simplest probabilistic model of this process, and our hope is that an investigation into its global behavior will shed some light on the macroscopic properties of physical excitable media.

Originally, our interest in the ring dynamics arose from the threshold–range GHM dynamics in the "ball" regime ([FGG1]). To explain this, consider a generalization of the rule (1.1) in which an x changes its state from 0 to 1 iff the number of 1's in the neighborhood exceed a given *threshold* θ. In some parameter regimes, the only structures which emerge from a disordered initial state with a chance of indefinite survival are expanding rings, which can be nested, with arbitrary multiplicity. These are extraordinarily rare creatures, impossible to obtain by simulations and cumbersome to study rigorously, thus a need for a simpler model in which expanding nested rings are provably the dominant feature.

Our final motivation for studying ANR comes from some interesting phenomena which can be rigorously established for these dynamics. Perhaps the most surprising fact is that, depending on the regime, excited sites can stay permanently connected (in a sense) or their connectivity may oscillate through time. We are not aware of any other interacting spatial process with similar properties. We start, however, with a result which estimates the density of 1's in $\tilde{\gamma}_t$ and shows that the system experiences slow relaxation.

Theorem 2. *There exists constants $C_1 = C_1(p_f), C_2 = C_2(p_f) \in (0, \infty)$ such that for $t > 0$*

$$\frac{C_1}{t} \leq P(\tilde{\gamma}_t(x) = 1) \leq \frac{C_2}{t} \tag{1.3}$$

Moreover, $\tilde{\gamma}_t$ dies out weakly, i.e. for every $x \in \mathbf{Z}^d$ $P(\tilde{\gamma}_t(x) = 1 \text{ i.o.}) = 1$, but

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t} 1_{\tilde{\gamma}_s(x)\neq 0} = 0 \qquad a.s. \tag{1.4}$$

It is perhaps worth noting that the fluctuations in this model are strong enough so that $t \cdot P(\tilde{\gamma}_t(x) = 1)$ does not converge as $t \to \infty$.

As already suggested, our main aim is to understand how connectivity properties of the set of 1's evolve through time. We use the following quantity as a measure of the extent to which percolation at time t fails:

$$\mathit{PercFail}_t = \sup\{a : \{\tilde{\gamma}_t = 1\} \text{ does not } B_\infty(0, a)\text{–percolate}\}.$$

Due to the ergodic theorem, $\mathit{PercFail}_t$ is a deterministic quantity. The next theorem simply says that 1's in $\tilde{\gamma}_t$ sometimes ℓ^∞–percolate and sometimes do not.

Theorem 3. *The events $\{\mathit{PercFail}_t = 0 \text{ i.o.}\}$ and $\{\mathit{PercFail}_t > 0 \text{ i.o.}\}$ both happen a.s.*

Our main result about the ANR establishes a phase transition in the asymptotic properties of $\mathit{PercFail}_t$, as p_f changes from small to large: if p_f is close to 0 then the set $\{\tilde{\gamma}_t = 1\}$ is occasionally further and further away from percolation (linearly in t away, in fact), while if if

p_f is close to 1, the system stays close to percolation at all large times. Unfortunately, we are not able to prove that the phase transition is sharp. To understand where the obstacle lies, and to formulate the theorem, we need several definitions. The main point that we wish to make here is that there exist some critical values $\lambda'_{lc}, \lambda'_{uc} \in (0, \infty)$ which are conjectured equal and used in the statement of Theorem 5. The peculiar fashion in which they are defined will become important only in Section 3.

We will say that a subset of $\mathbf{R}^2$ *percolates* if it contains an unbounded connected set. Let $\mathcal{P}$ be a Poisson point location with intensity λ in $\mathbf{R}^2$ and let $W(\lambda, r) = \mathcal{P} + B_\infty(0, r)$. Take two independent such sets $W_1(\lambda, r)$ and $W_2(1, r)$ (by which we mean that the corresponding Poisson point locations are independent) and define the following critical values:

$$\begin{aligned}
\lambda'_{lc} &= \sup\{\lambda : \exists \epsilon > 0, \forall r > 0, W_1(\lambda, r+\epsilon) \setminus W_2(1, r-\epsilon) \text{ does not percolate, while} \\
&\qquad (W_1(\lambda, r+\epsilon) \setminus W_2(1, r-\epsilon))^c \text{ percolates}\}, \\
\lambda'_{uc} &= \inf\{\lambda : \exists \epsilon > 0, \exists r > 0, W_1(\lambda, r-\epsilon) \setminus W_2(1, r+\epsilon) \text{ percolates, while} \\
&\qquad (W_1(\lambda, r-\epsilon) \setminus W_2(1, r+\epsilon))^c \text{ does not percolate}\}, \\
\lambda'_c &= \sup\{\lambda : W_1(\lambda, r) \setminus W_2(1, r) \text{ percolates for no } r \in (0, \infty)\}.
\end{aligned}$$

The next conjecture is quite natural, but the techniques necessary to turn it into a theorem seem to be lacking (one can, however, obtain some bounds, see Lemma 3.1).

Conjecture 4. *The equalities* $\lambda'_{lc} = \lambda'_c = \lambda'_{uc}$ *hold.*

We are now in the position to state our main theorem about the ANR. The reader is referred to [Gra2] for a computer–generated illustration.

Theorem 5. *If* $p_f < 1/(1 + \lambda'_{uc})$*, then a.s.*

$$\limsup_{t\to\infty} \frac{PercFail_t}{t} \in (0, \infty), \tag{1.5}$$

whereas, if $p_f > 1/(1 + \lambda'_{lc})$ *then a.s.*

$$\limsup_{t\to\infty} PercFail_t \in (0, \infty). \tag{1.6}$$

We conclude our introductory discussion on ANR by mentioning two works where connections between continuous and discrete percolation models also play a crucial role: the basic reference [MR], and the paper [Pen] on the large–range threshold contact process.

Our second model, *digital boiling (DB)* is perhaps the simplest recurrent ring model one can concoct. The state of this system is $\tilde{\gamma}_t \in \{0, 1, 2\}^{\mathbf{Z}^2}$ (we use the same notation $\tilde{\gamma}_t$ as for ANR, as there is no possibility for confusion) and the evolution is governed by

$$(1.7)\qquad \begin{array}{ll} \left.\begin{array}{l} 1 \to 2 \\ 2 \to 0 \end{array}\right\} & \text{automatically,} \\ 0 \to 1 & \text{automatically if a 1 is in } \mathcal{N}_x, \text{ with probability } p_s \text{ otherwise,} \\ 0 \to 0 & \text{otherwise.} \end{array}$$

Assume also that the system is started from the quiescent state $\tilde{\gamma}_0 \equiv 0$. Visual features of DB dynamics (see Figure 2 of [Gra2] or Feb. 12, 1996 Recipe of [Gri2]) resemble bubble formation, growth and annihilation in a boiling liquid, hence the name for this model.

The straightforward reason for our interest in DB dynamics is that it models an excitable medium in the presence of persistent random spontaneous excitation (which can have external causes, of course). Another motivation is the fact that this model represents contour (constant height) lines for one of the simplest models for growing connected interface in three dimensions. The precise definition of the interface model $\tilde{\xi}_t$ is in Section 4, here we only mention two similar systems which have previously appeared in the literature. The first is a continuous–time relative analyzed in [KS], where an "approximate shape" result was proved; we will have more to say on this subject in Section 5. The second is the synchronization dynamics described in Section 9.6 of [TM], and discussed in [Gri1] and [EG]. In fact, the coupling method from [EG] can be used to analyze the one–dimensional version of DB dynamics, whereas the corresponding coupling in two dimensions is much more elusive ([Gra2]). Nevertheless, the connection between the DB and the interface model can be utilized to prove the following result.

Theorem 6. *Let* $N_t(0) = \sum_{i=0}^{t} 1_{\{\tilde{\gamma}_i(0)=1\}}$ *be the number of times the origin is excited in the time interval* $[0, t]$.

(1) For every fixed p_s, $\frac{N_t(0)}{t}$ *a.s. converges to a constant* $\nu(p_s) \in (0, 1/3)$ *as* $t \to \infty$.

(2) $\nu(p_s) p_s^{-1/3} \to \nu^* \in (0, \infty)$ *as* $p_s \to 0$.

Since the origin gets excited about once per every $1/(\nu^* p_s^{1/3})$ time steps, we would expect that the system approaches a unique equilibrium measure with density of excited sites on the order of $p_s^{1/3}$. This of course does not follow from Theorem 6, and the problem remains open (see [Gra2] for a discussion on this issue).

We proceed with a few remarks on related and more general models. The fact that rings are the dominant feature of the systems discussed here depends crucially on synchronicity; an asynchronous GHM–type system would either die out or approach a spiral equilibrium ([DN], [FGG2]). We expect that the same conclusion holds for externally stimulated threshold–range synchronous systems, as failed nuclei will break the rings in this case ([FGG1]). Higher–dimensional versions of our results seem more promising; in fact, the proof of Theorem 6 can be easily extended to handle any dimension $d \geq 1$, the only difference in the statement being that $p_s^{1/3}$ needs to be replaced by $p_s^{1/(d+1)}$. Finally, we have not discussed percolation properties of the DB model. A natural conjecture would be that $PercFail_t$ oscillates between 0 and about $p_s^{-1/3}$, but it is far from clear to us how one would confirm this.

We present detailed proofs in the rest of the paper. Sections 2 and 3 deal with the ANR dynamics: Section 2 proves Theorem 2, while Section 3 contains proofs to Theorems 3 and 5. In Section 4, we turn to the DB dynamics and the proof of Theorem 6. In Section 5, we prove a few large deviation estimates and obtain an approximate shape result for a three–dimensional growth model related to bootstrap percolation.

2. Decay of density in the ANR dynamics.

We start by defining a cellular automaton ξ_t which has state space $\{0,1,2,\dots\}^{\mathbf{Z}^2}$ and one of the simplest rules we can think of:

$$\xi_{t+1}(x) = \max\{\xi_t(y), y \in \mathcal{N}_x\}.$$

We will assume that ξ_0 is a product measure and denote $p_k = P(\xi_0(x) = k)$ and $r_k = \sum_{i=k}^{\infty} p_i$, $k = 0,1,2,\dots$ For obvious reasons, we call this automaton *expanding squares (ES)*; note that $\{\xi_t = k\} = (\{\xi_0 = k\} + B_\infty(0,t)) \setminus (\{\xi_0 \geq k+1\} + B_\infty(0,t))$. The values taken by ξ_t will often be referred to as *colors*. As we have already seen in Lemma 2.1 of [Gra1], the behavior of boundaries between colors in the ES ξ_t can be connected to the behavior of the rings of 1's in the ANR $\tilde{\gamma}_t$. To make this correspondence more precise, assume that $\tilde{\gamma}_t$ is a realization of the ANR dynamics with $\tilde{\gamma}_0 \equiv e_1$ and, for every site $x \in \mathbf{Z}^2$, define $T_x = \inf\{t : \tilde{\gamma}_t(x) = 2\}$ and $\xi_0(x) = (T_x + 2)/3$. This makes $\xi_0(x)$ a geometric random variable with $p_0 = 0$ and $p_k = p_f^{k-1}(1-p_f)$ for $k > 0$.

Lemma 2.1. *Under the specified coupling we have, for all $t \geq 0$,*

$$\{x : \tilde{\gamma}_{t+1}(x) = 1\} = \bigcup_{\substack{s=0,1,\dots,t:\\ (t-s) \bmod 3 = 0}} \{x : \xi_s(x) \leq \frac{t-s}{3} < \xi_{s+1}(x)\}. \tag{2.1}$$

It is important to note that, as $\xi_t(x)$ is non-decreasing in t, the sets which form the union in (2.1) are disjoint.

Proof. Denote the set on the right of (2.1) as S_t. Our first step is to prove $S_t \cap S_{t-1} = \emptyset$ and $S_t \cap S_{t-2} = \emptyset$. Let $x \in S_t \cap S_{t-1}$. Then there exist s_1, s_2 such that $s_1 \neq s_2$ and $\xi_{s_1}(x) \leq (t-1-s_1)/3 < \xi_{s_1+1}(x)$, $\xi_{s_2}(x) \leq (t-s_2)/3 < \xi_{s_2+1}(x)$. If $s_1 < s_2$, then $(t-1-s_1)/3 < \xi_{s_1+1}(x) \leq \xi_{s_2}(x) \leq (t-s_2)/3$. This implies that $s_1 \geq s_2$, a contradiction. The assumption $s_1 > s_2$ leads to a similar contradiction, and the proof that $S_t \cap S_{t-2} = \emptyset$ is just as simple.

We now proceed to prove (2.1) by induction on t. To start, note that both S_0 and $\{\tilde{\gamma}_1 = 1\}$ are empty. Make now the induction hypothesis that $S_u = \{\tilde{\gamma}_{u+1} = 1\}$ for every $u < t$.

First, we prove that $S_t \subset \{\tilde{\gamma}_{t+1} = 1\}$. Assume that $x \in S_t$. The induction hypothesis and $x \notin S_{t-1}$ imply that $\tilde{\gamma}_t(x) \in \{0, 2, e_i\}$, and then induction hypothesis and $x \notin S_{t-2}$ imply that either $\tilde{\gamma}_t(x) = 0$ or $\tilde{\gamma}_{t-1}(x) \in \{e_i\}$. The second case implies that $\xi_0(x) \geq (t+1)/3$, hence $\xi_s(x) > t/3$ for $s \geq 0$, which contradicts $x \in S_t$. It therefore follows that $\tilde{\gamma}_t(x) = 0$. Moreover, find the $s \in \{0, 1, \ldots, t\}$ such that $(t-s) \bmod 3 = 0$ and $\xi_s(x) \leq (t-s)/3 < \xi_{s+1}(x)$. Then there must exist a $y \in \mathcal{N}_x$ with $\xi_s(y) > (t-s)/3$. It must also be true that $\xi_{s-1}(y) \leq (t-s)/3$ (otherwise $\xi_s(x) > (t-s)/3$). Hence $y \in S_{t-1}$ and, by the induction hypothesis, $\tilde{\gamma}_t(y) = 1$. Thus $\tilde{\gamma}_{t+1}(x) = 1$.

Finally, we prove that $\{\tilde{\gamma}_{t+1} = 1\} \subset S_t$. If $\tilde{\gamma}_{t+1}(x) = 1$, then $\tilde{\gamma}_t(x) = 0$ and there is a $y \in \mathcal{N}_x$ such that either $\tilde{\gamma}_t(y) = 1$ or $\tilde{\gamma}_t(y) = e_1$.

We deal with the second case first. Note that it implies that $t \bmod 3 = 0$, and we claim that x is in the $s = 0$ set of the union (2.1). For this, we have to check that $\xi_0(x) \leq t/3 < \xi_1(x)$. However, since $\xi_0(y) \geq t/3+1$, $\xi_1(x) > t/3$. Moreover, since $\xi_0(x) > t/3$ would imply $\tilde{\gamma}_t(x) = e_1$, $\xi_0(x) \leq t/3$. Therefore, $x \in S_t$.

Assume now that $\tilde{\gamma}_t(y) = 1$. Hence $y \in S_{t-1}$, and there exists an s with $\xi_s(y) \leq (t-1-s)/3 < \xi_{s+1}(y)$. We claim that $s+1$ works to show that $x \in S_t$. Because $\xi_{s+1}(y) > ((t-1)-s)/3$, it follows that $\xi_{s+2}(x) > (t-(s+1))/3$. If $\xi_{s+1}(x) > (t-(s+1))/3$, then $\xi_s(x) > (t-1-s)/3$ (since $x \notin S_{t-1}$) and then $\xi_{s-1}(x) > ((t-2)-(s-1))/3$ (since $x \notin S_{t-2}$), in contradiction with $\xi_s(y) \leq (t-s-1)/3$. Hence $\xi_{s+1}(x) \leq (t-(s+1))/3$, and $x \in S_t$. □

Proof of (1.3). In this proof, and the ones to follow, we use the standard notation of C as a "generic constant," whose value may change from line to line. By Lemma 2.1,

$$
\begin{aligned}
P(\tilde{\gamma}_t(x) = 1) &= \sum_{u=0}^{\lfloor t/3 \rfloor} P(\xi_{t-3u}(x) \leq u < \xi_{t-3u+1}(x)) \\
&= \sum_{u=0}^{\lfloor t/3 \rfloor} (1 - r_{u+1})^{(2t-6u+1)^2} (1 - (1 - r_{u+1})^{8(t-3u+1)}) \\
&\leq \sum_{u=0}^{\lfloor t/3 \rfloor} e^{-r_{u+1}(2t-6u+1)^2} 8(t-3u+1) r_{u+1} \\
&\leq \frac{8}{t} \cdot \frac{p_f}{1-p_f} \sum_{u=0}^{C \log t} e^{-r_{u+1} t^2} t^2 (r_{u+1} - r_{u+2}) + C \cdot \frac{1}{t^2}.
\end{aligned} \tag{2.2}
$$

To prove the last line in (2.2), first note that $r_{u+1} = p_f^u < 1/t^4$ if $u > C \log t$ for a large constant C. Hence we can restrict the summation to u's smaller than $C \log t$; for a large enough t and a

u in this range, $t-3u > t/2$. The sum in the last line of (2.2) is smaller than $\int_0^\infty e^{-t}\,dt$, proving the upper bound in (1.3).

For the lower bound, choose

$$u = \left\lfloor \frac{\log(t^2)}{\log(1/p_f)} \right\rfloor$$

and use this term as a lower bound for the sum in the second line of (2.2). Since $1 \le r_{u+1}t^2 \le 1/p_f$, $2(t-3u)+1 \le 2t$, and tr_{u+1} goes to 0 as $t \to \infty$, it follows that:

$$P(\tilde{\gamma}_t(x) = 1) \ge e^{-2r_{u+1}t^2} \cdot tr_{u+1} \ge e^{-2/q} \cdot \frac{1}{t},$$

for large t. This ends the proof. □

Assume for simplicity that $p_f^{-1/2}$ is an integer and take $t_k = t_0 p_f^{-k/2}$. Then

$$\lim_{k\to\infty} t_k P(\tilde{\gamma}_{t_k}(x) = 1) = 8t_0^2 \sum_{n=-\infty}^{\infty} p_f^n e^{4t_0^2 p_f^n}.$$

As this expression depends on t_0, $\lim_{t\to\infty} tP(\tilde{\gamma}_t(x) = 1)$ can not exist.

Proof that $\tilde{\gamma}_t$ dies out weakly. If $\xi_s(x) < \xi_{s+1}(x)$ and $t = 3\xi_s(x) + s$, then by Lemma 2.1, $\tilde{\gamma}_{t+1}(x) = 1$. Since ξ_t changes infinitely often, so does $\tilde{\gamma}_t(x)$.

To prove (1.4), we have to show that $X_t = t^{-1}\sum_{s=0}^t 1_{\tilde{\gamma}_s(x)\neq 0} \to 0$ a.s. as $t \to \infty$. By (1.3), $E(X_t) \le Ct^{-1}\log t$, therefore $E(\sum_{n=1}^\infty X_{n^2}) < \infty$ and $X_{n^2} \to 0$ a.s. If t is any integer, take $n = n(t)$ such that $n^2 < t \le (n+1)^2$ and observe that $X_t \le \left(\frac{n+1}{n}\right)^2 X_{(n+1)^2} \to 0$ a.s. □

3. Percolation properties of the ANR dynamics.

Again, our starting point is the comparison automaton ξ_t. Assume first that ξ_0 contains only 3 colors and that the initial measure is a product measure with

$$P(\xi_0(x) = 2) = p,\ P(\xi_0(x) = 1) = q,\ P(\xi_0(x) = 0) = 1 - p - q,$$

for each $x \in \mathbf{Z}^2$.

Lemma 3.1. *We have $1 \le \lambda'_{lc} \le \lambda'_{uc} < \infty$. Moreover, if $\lambda < \lambda'_{lc}$, $q \le \lambda p$, and p is small enough, $\{\xi_t \in \{0,2\}\}$ ℓ^∞–percolates for every t, while $\{\xi_t = 1\}$ never ℓ^∞–percolates. Finally, if $\lambda > \lambda'_{uc}$ and $q \ge \lambda p$, then there exist constants $r > 0$ and $\epsilon > 0$, so that if p is small enough, then at time $t = \lfloor r/\sqrt{p} \rfloor$ the set $\{\xi_t = 1\}$ ℓ^∞–percolates, while $\{\xi_t = 1\} + B_\infty(0, \epsilon/\sqrt{p})$ does not.*

Proof. We start by noting that $\lambda'_{lc} \le \lambda'_{uc}$ by definitions. Next, we prove that $1 \le \lambda'_{lc}$. Assume that $\lambda < 1$. Then, by results in Chapter 4 of [MR], there exist $\epsilon > 0$ and $r_0 > 0$ so that both $W_2(1, r_0 - \epsilon)$ and $W_1(\lambda, r_0 + \epsilon)^c$ percolate, while neither $W_2(1, r_0 - \epsilon)^c$ nor $W_1(\lambda, r_0 + \epsilon)$ percolate. This clearly implies that $\lambda \le \lambda'_{lc}$.

Next, we prove that $\lambda'_{uc} < \infty$. Fix a positive real number $b > 0$ (to be specified later) and declare a site $x \in \mathbf{Z}^2$ to be *open* if the following two conditions hold:

(1) $B_\infty(2bx, b)$ is covered by $W_1(\lambda, b/2)$, and

(2) $B_\infty(2bx, b) \cap W_2(1, b) = \emptyset$.

Then, since $B_\infty(2bx, b)$ contains 16 squares of side $b/2$,

$$P(x \text{ is open}) \ge \left(1 - e^{-\lambda b^2/4}\right)^{16} \cdot e^{-16b^2}.$$

Now choose first a small b and then a large λ so that $P(x \text{ is open}) > 0.99$. Since two sites x, y with $||x - y||_\infty \ge 2$ are open independently, a standard contour argument can be utilized to show that the open sites ℓ^1–percolate, while the non–open sites do not ℓ^∞–percolate; thus $\lambda \ge \lambda'_{uc}$.

To prove the remaining statements, start by the following coupling between ξ_0 and two independent Poisson point locations: $\mathcal{P}_1$ with intensity λ and $\mathcal{P}_2$ with intensity 1. For $x \in \mathbf{Z}^2$, let $\xi_0(x) = 2$ if $B_\infty(\sqrt{p}x, \sqrt{p}/2) \cap \mathcal{P}_2 \ne \emptyset$, let $\xi_0(x) = 1$ if $B_\infty(\sqrt{p}x, \sqrt{p}/2) \cap \mathcal{P}_2 = \emptyset$ and $B_\infty(\sqrt{p}x, \sqrt{p}/2) \cap \mathcal{P}_1 \ne \emptyset$, and let $\xi_0(x) = 0$ otherwise. Then ξ_0 is a product measure with $P(\xi_0(x) = 2) = 1 - e^{-p} \approx p$ and $P(\xi_0(x) = 1) = e^{-p}(1 - e^{-\lambda p}) \approx \lambda p$. Moreover,

$$\begin{aligned} W_1(\lambda, (t-1)\sqrt{p}) \setminus W_2(1, (t+1)\sqrt{p}) &\subset \sqrt{p}\{\xi_t = 1\} \\ &\subset W_1(\lambda, (t+1)\sqrt{p}) \setminus W_2(1, (t-1)\sqrt{p}). \end{aligned} \tag{3.1}$$

Assume that $\lambda > \lambda'_{uc}$. Let r and ϵ be as in the definition of λ'_{uc}. Now, (3.1) implies that if we choose $t = \lfloor r/\sqrt{p} \rfloor$, and p is so small that $2\sqrt{p} < \epsilon$, then $\{\xi_t = 1\}$ ℓ^∞–percolates. Moreover, (3.1) also implies that $\sqrt{p}(\{\xi_t = 1\}^c + B_\infty(0, 0.2\epsilon/\sqrt{p})$ is for small enough p included in $(W_1(\lambda, r - 0.2\epsilon) \setminus W_2(1, r + 0.2\epsilon))^c$, which does not ℓ^∞–percolate. A similar application of (3.1) proves the remaining statement. □

Corollary 3.2. *Assume that ξ_t contains infinitely many colors, and let again $p_k = P(\xi_0(x) = k)$, $r_k = P(\xi_0(x) \ge k)$. If $\limsup_{k\to\infty} p_k/r_{k+1} > \lambda'_{uc}$, then, almost surely, there are infinitely many times at which a single color ℓ^∞–percolates. On the other hand, if $\limsup_{k\to\infty} p_k/r_{k+1} < \lambda'_{lc}$, then, almost surely, at sufficiently large times no single color percolates.*

We now proceed to prove Theorems 3 and 5. We will use many of the techniques introduced in [Gra1], especially those of Section 5 of that paper. For the readers' convenience, we summarize some notation at this point. For set $A \subset \mathbf{Z}^d$, we define $\partial^\infty A = \{x \in A : \text{there exists a } y \in A^c$ with $||x-y||_\infty = 1\}$ and $\partial_o^\infty A = \partial^\infty(A^c)$. We make it a convention that a GHM γ_t is always started from an initial product measure containing only 0's and 1's. Then, the entire evolution of γ_t is specified by the set $\{\gamma_0 = 1\}$. Assume for a moment that $p = P(\gamma_0(x) = 1)$. For $k = 0, 1, \dots$, we define $p_c^{(k)}$ (resp. $\tilde{p}_c^{(k)}$) to be the critical density p for ℓ^1–percolation (resp. ℓ^∞–percolation) of the set $\{\gamma_t = 1\} + B_\infty(0,k)$.

Proof of Theorem 3. Fix an integer $k \geq 0$. To the ANR $\tilde{\gamma}_t$ associate the GHM γ_t with $\{\gamma_0 = 1\} = \{\xi_0 \geq k\}$. Let ξ_t' be the two color ES model associated with γ_t (which simply means $\xi_0' = \gamma_0$). Thus $\{\xi_0' = 1\} = \{\xi_0 \geq k\}$ and hence $\{\xi_t' = 1\} = \{\xi_t \geq k\}$ for all $t \geq 0$. Now it follows from Lemma 2.1 and [Gra1, Lemma 2.1] that, for each $t \geq 0$,

$$\begin{aligned} \{x : \gamma_{t+1}(x) = 1\} &= \{x : \xi_t'(x) = 0, \xi_{t+1}'(x) = 1\} \\ &= \{x : \xi_t(x) \leq k-1 < \xi_{t+1}(x)\} \\ &\subset \{x : \tilde{\gamma}_{t+3(k-1)+1}(x) = 1\}. \end{aligned} \tag{3.2}$$

Theorem 1 implies that, for every k, t_k can be chosen so that $\{\gamma_{t_k+1} = 1\}$ ℓ^∞–percolates. Therefore, $\{\tilde{\gamma}_{t_k+3(k-1)+1} = 1\}$ ℓ^∞–percolates and thus $PercFail_t = 0$ i.o.

Next, we claim that there exists a time t so that the set $\{\gamma_t = 0\}$ does not ℓ^∞–percolate, no matter what the initial density $p = P(\gamma_0(x) = 1)$ is. This will imply, by virtue of (3.2), that for every k there exists a t_k such that $\{\tilde{\gamma}_{t_k+3(k-1)+2} \in \{2,0\}\}^c \subset \{\tilde{\gamma}_{t_k+3(k-1)+1} \in \{1,2\}\}^c$ does not ℓ^∞–percolate, hence $PercFail_t > 0$ i.o.

To demonstrate the claim, observe first that, by Theorem 1, $\gamma_{t+1}(x) = 0$ if and only if either $\gamma_s(x) = 0$ for all $s \leq t+1$ or $\gamma_s(x) = 1$ for an $s \leq t-1$. Thus, by [Gra1, Lemma 2.1] $\{\gamma_{t+1} = 0\} = \{\xi_{t+1}' = 0\} \cup \{\xi_{t-1}' = 1\}$. Not only is this a disjoint union, but the $||\cdot||_\infty$–distance between the two sets is at least 3. Hence it is enough to show that for a suitably chosen time neither of them ℓ^∞–percolates. If $t = T(p)$ (from Theorem 1) is the first time at which $\{\gamma_t = 1\}$ ℓ^∞–percolates, then the proof of Theorem 2 in [Gra1] shows that $\{\xi_{t-1}' = 1\}$ does not ℓ^∞–percolate, and the same is true for $\{\xi_{t+1}' = 0\}$. $\square$

Proof of (1.6). We start by picking an integer $k \geq 1$ and defining three auxiliary models: a 3-color ES model ξ_t' (with colors 0, 1, and 2) and two GHM's γ_t and γ_t'. The initial state of ξ_t' is given by $P(\xi_0'(x) = 1) = P(\xi_0(x) = k)$ and $P(\xi_0'(x) = 2) = P(\xi_0(x) > k)$, while γ_t and γ_t' are given by $\{\gamma_0 = 1\} = \{\xi_0' > 0\}$, and $\{\gamma_0' = 1\} = \{\xi_0' = 2\}$. It follows from Lemma 3.1 that if $p_f > 1/(1+\lambda_{lc}')$, and k is large enough, then $\{\xi_t' \in \{0,2\}\}$ ℓ^∞–percolates for every t.

Fix a time t and denote by H_t the event that $\{\xi_t' = 2\}$ does not ℓ^1–percolate, that $\{\xi_t' = 0\}$ does not ℓ^∞–percolate, but that $\{\xi_t' \in \{1,2\}\}$ does ℓ^∞–percolate. Our first task is to prove that $H_t \subset \{PercFail_{t+3k-2} \leq 3\}$.

To this end, assume that H_t happens. Let $\mathcal{C}_t$ be the infinite cluster of $\{\xi'_t \in \{1,2\}\}$. Let $v_0, v_1, \dots$ be an infinite self–avoiding ℓ^∞–path which consists of sites in $\{\xi'_t \in \{0,2\}\}$. Then, either both 0's and 2's have infinitely many representatives on the path, or else the path consists of 2's (after a finite segment is discarded). We deal with the first case first.

Let $i_1, i_2, i_3, \dots$ be such indices that v_{i_n}, $n = 1, 2, \dots$ are the successive sites in $\partial^\infty_o \mathcal{C}_t$. (Such indices must exist since the path goes into and out of $\mathcal{C}_t$ infinitely many times.) Pick an n and assume that $i_{n+1} > i_n + 1$. If v_{i_n} and $v_{i_{n+1}}$ are in the same ℓ^∞–connected component of $\mathcal{C}_t^c$, then by Lemma 5.1 in [Gra1], there exists an ℓ^∞–path $v_{i_n} = w_0, w_1, \dots, w_m = v_{i_{n+1}}$ such that $w_i \in \partial^\infty_o \mathcal{C}_t$ for $i = 0, \dots, m$.

If v_{i_n} and $v_{i_{n+1}}$ are in separate ℓ^∞–connected components of $\mathcal{C}_t^c$, then $\xi'_t(v_i) = 2$ and $v_i \in \mathcal{C}_t$ for $i = i_n + 1, \dots, i_{n+1} - 1$. Let A be the (finite) ℓ^1–connected component of $\{\xi'_t = 2\}$ including v_{i_n+1}. Let $j > i_n + 1$ be the smallest index such that $v_j \in \partial^\infty A$. In this case, Lemma 5.1 of [Gra1] implies that we can find an ℓ^∞–path included in $\partial^\infty A$ that connects v_{i_n} and v_j. Continuing in this fashion, we construct an ℓ^∞–path which connects v_{i_n+1} and $v_{i_{n+1}-1}$ and is included in $\partial^\infty\{\xi'_t = 2\}$.

The last paragraph in fact also shows how to deal with the case when $\xi'_t(v_i) = 2$ for every i. In this case the ℓ^∞–path can be deformed into a path contained entirely in $\partial^\infty\{\xi'_t = 2\}$.

By combining these cases, we conclude that there exists a self-avoiding ℓ^∞–path $v'_0, v'_1, v'_2, \dots$ such that every i falls into two cases: either $\xi'_t(v'_i) = 0$ and $\xi'_{t+1}(v'_i) > 0$, or $\xi'_{t-1}(v'_i) < 2$ and $\xi'_t(v'_i) = 2$. In the first case $\gamma_{t+1}(v'_i) = 1$, while in the second case $\gamma'_t(v'_i) = 1$. It follows from (3.2) that in the first case $\tilde\gamma_{t+3k-2}(v'_i) = 1$, while in the second case $\tilde\gamma_{t+3k}(v'_i) = 1$. In either case, each v'_i is at $||\cdot||_\infty$-distance no larger than 2 from $\{\tilde\gamma_{t+3k-2} = 1\}$. Therefore, $\mathit{PercFail}_{t+3k-2} \le 3$.

Now fix an arbitrary time t. Let $I_k = (p_f^k, p_f^{k-1})$ and I_k^- the closure of this interval. Note that H_t happens (see Section 4 in [Gra1]) provided $P(\xi_0(x) \ge k) > p_c^{(t)}$ and $P(\xi_0(x) \ge k+1) < p_c^{(t)}$ or, equivalently, $p_c^{(t)} \in I_k$. Observe also that, if t is large enough, $p_c^{(t)} \in I_k^-$ implies that $p_c^{(t+1)} \in I_k \cup I_{k+1}$. (Otherwise $p_c^{(t)} \ge p_f^k$ and $p_c^{(t+1)} \le p_f^{k+1}$ for some k, which implies $p_c^{(t+1)}/p_c^{(t)} \le p_f < 1$. This is in contradiction with the fact that $p_c^{(t)} t^2$ converges to a finite non-zero constant as $t \to \infty$.)

Let us assume that $p_c^{(t)} \in I_k$. Then, as we have proved, H_t happens and, consequently, $\mathit{PercFail}_{t+3k-2} \le 3$. There are three possibilities for the location of $p_c^{(t+1)}$. If $p_c^{(t+1)} \in I_k$, then H_{t+1} happens and $\mathit{PercFail}_{t+3k-1} \le 3$. If $p_c^{(t+1)} \in I_{k+1}$, then (replacing t by $t+1$ and k by $k+1$) $\mathit{PercFail}_{t+3k+2} \le 3$. The final possibility is that $p_c^{(t+1)}$ is the left endpoint of I_k^-. However, in this case $p_c^{(t+2)} \in I_{k+1}$ and hence $\mathit{PercFail}_{t+3k+3} \le 3$.

By iterating this procedure, one proves that, if t is large enough, $\mathit{PercFail}_s \le 3$ for some

$s \in [t, t+4]$. If s_t is the minimal time $s \geq t$ such that $PercFail_s \leq 3$, then every site in $\{\tilde{\gamma}_{s_t} = 1\}$ is at $||\cdot||_\infty$–distance no larger than 4 from $\{\tilde{\gamma}_t = 1\}$. Since $\{\tilde{\gamma}_{s_t} = 1\}$ $B_\infty(0,3)$–percolates, $\{\tilde{\gamma}_{s_t} = 1\}$ $B_\infty(0,11)$–percolates and $PercFail_t \leq 10$. Therefore, $\limsup PercFail_t \leq 10$. □

Proof that $\limsup PercFail_t/t < \infty$ *for all* q. For each t, let

$$k = \left\lfloor \frac{2\log(t/(\alpha\sqrt{p_f}))}{\log(1/p_f)} \right\rfloor + 1.$$

If α is chosen small enough, then, for large t, $\tilde{p}_c^{(t)} > P(\xi_0(x) \geq k)$. Let γ_t be the GHM determined by $\{\gamma_0 = 1\} = \{\xi_0 \geq k\}$. Then, let $T = T(p_f^{k-1})$ (from Theorem 1) be the first time at which 1's in γ_t ℓ^∞–percolate. For large t we have (by Theorem 1) $t < T \leq 2\lambda_c p_f^{-(k-1)/2} \leq t \cdot 2\lambda_c/(\alpha\sqrt{p_f})$.

Now, by (3.2), $\{\tilde{\gamma}_{T+3k-3} = 1\}$ ℓ^∞–percolates. Also, every site in $\{\tilde{\gamma}_{T+3k-3} = 1\}$ is at $||\cdot||_\infty$–distance at most $T + 3k - 3 - t$ from $\{\tilde{\gamma}_t = 1\}$. Therefore, $PercFail_t \leq 2(T + 3k - 3 - t)$ and $\limsup PercFail_t/t \leq 4\lambda_c/(\alpha\sqrt{p_f}) - 2$. □

Proof of (1.5). Assume that $p_f < 1/(1+\lambda'_{uc})$. Then, by Lemma 3.1 there exist constants $r > 0$, and $\epsilon > 0$ so that for k large enough and $t = r/\sqrt{p_k}$, the set $\{\xi_t = k\}^c + B_\infty(0, \epsilon/\sqrt{p_k})$ does not ℓ^∞–percolate.

Pick an $x \in \{\xi_t = k\} \setminus (\{\xi_t = k\}^c + B_\infty(0, 3k+6))$ (i.e. x has color k and is far away from the boundaries). We claim that $\tilde{\gamma}_{t+3k+4}(x) \neq 1$. Otherwise, Lemma 2.1 would imply existence of an $s \in \{0, \dots, t+3k+4\}$ so that $\xi_s(x) \leq (t-s)/3 + k + 4/3 < \xi_{s+1}(x)$. Since $\xi_s(x) = k$ for $t \leq s \leq t+3k+5$, this would imply that $s < t$, but then $\xi_{s+1} > k$, a contradiction. This implies that

$$PercFail_{t+3k+4} \geq \frac{\epsilon}{\sqrt{p_k}} - 2(3k+5),$$

and therefore

$$\frac{1}{t+3k+4} PercFail_{t+3k+4} \geq \frac{\epsilon - 2(3k+5)\sqrt{p_k}}{r + (3k+4)\sqrt{p_k}}.$$

This last expression converges to ϵ/r as $k \to \infty$, proving that $\limsup PercFail_t/t \geq \epsilon/r$. □

4. Excitation times in the DB dynamics.

We begin by introducing a model for randomly growing interface $\tilde{\xi}_t$ in three dimensions. The state space of $\tilde{\xi}_t$ is $\{0, 1, \dots\}^{\mathbf{Z}^2}$; here $\tilde{\xi}_t(x) = k$ means that the height of the interface above the site x is k. The update rule is as follows:

(I1) If there is at least one $y \in \mathcal{N}_x$ with $\tilde{\xi}_t(y) > \tilde{\xi}_t(x)$, then $\tilde{\xi}_t(x)$ advances automatically by 1.

(I2) Otherwise, if $\tilde{\xi}_t(x) = \tilde{\xi}_{t-1}(x) = \tilde{\xi}_{t-2}(x)$, then $\tilde{\xi}_t(x)$ advances by 1 with probability p.

(I3) In other cases $\tilde{\xi}_t(x)$ stays the same.

We will assume throughout that $\tilde{\xi}_t \equiv 0$ for $t \leq 0$. The connection between $\tilde{\xi}_t$ and $\tilde{\gamma}_t$ will be established in Lemma 4.1 below, but we point out immediately that the awkward condition (I2) involving the previous two times stems from the fact that only 0's can be externally excited in the DB. In fact, the two models will be coupled using a space–time percolation structure on sites of $\mathbf{Z}^3$: interpret $\mathbf{Z}^2 \times \mathbf{Z}_+$ as space×time and make any site $(x,t) \in \mathbf{Z}^2 \times \mathbf{Z}_+$ a *nucleus* independently, with probability p_s. The random set of nuclei will be denoted by $\Pi = \Pi(p_s)$. Then, the DB $\tilde{\gamma}_t$ can be equivalently defined by declaring that, for $t \geq 0$, $\tilde{\gamma}_{t+1}(x) = 1$ iff $\tilde{\gamma}_t(x) = 0$ and either $x \in \{\tilde{\gamma}_t = 1\} + \mathcal{N}$ or $(x,t) \in \Pi$. The interface dynamics $\tilde{\xi}_t$ can also be be defined this way: simply replace (I2) by

(I2') Otherwise, if $\tilde{\xi}_t(x) = \tilde{\xi}_{t-1}(x) = \tilde{\xi}_{t-2}(x)$ and $(x,t) \in \Pi(p)$, then $\tilde{\xi}_{t+1}(x) = \tilde{\xi}_t(x) + 1$.

The percolation structure is most useful because it allows us a last passage interpretation, similar to the one described in [Gri1] for the synchronization dynamics and in [CGGK] for the PERT networks. We introduce several definitions, some of which are directly linked to the DB, while others are useful for approximation purposes. We denote by $\mathcal{P}$ the unit–intensity Poisson point location in $\mathbf{R}^3$.

Fix a space–time point $(x,t) \in \mathbf{Z}^2 \times \mathbf{Z}_+$. A sequence of points $(x_1,t_1),(x_2,t_2),\dots,(x_n,t_n) \in \Pi$ is called a *discrete path ending at* (x,t) if $0 \leq t_1 < t_2 < \cdots < t_n < t$, $||x_i - x_{i-1}||_\infty \leq t_i - t_{i-1}$ for $i = 2,\dots,n$, and $||x - x_n||_\infty \leq t - t_n$. An *admissible path* π ending at (x,t) is a discrete path as above such that $||x_i - x_{i-1}||_\infty + 3 \leq t_i - t_{i-1}$ for $i = 2,\dots n$, and $||x - x_n||_\infty + 1 \leq t - t_n$. A *continuous* path differs from a discrete one in the requirement that all $(x_i,t_i) \in \mathcal{P}$, $i = 1,\dots,n$. In either case, the *length* a path π is $\text{Len}(\pi) = n$. Now define, for $x \in \mathbf{Z}^2$ and $0 \leq s < t$:

$$\tilde{L}((x,t),s) = \max\{\text{Len}(\pi) : \pi \text{ is an admissible path contained in } \mathbf{Z}^2 \times [s,t) \text{ ending at } (x,t)\}.$$

$$L((x,t),s) = \max\{\text{Len}(\pi) : \pi \text{ is a discrete path contained in } \mathbf{Z}^2 \times [s,t) \text{ ending at } (x,t)\}.$$

$$\mathcal{L}((x,t),s) = \max\{\text{Len}(\pi) : \pi \text{ is a continuous path contained in } \mathbf{R}^2 \times [s,t) \text{ ending at } (x,t)\}.$$

The properties of the described coupling of $\tilde{\gamma}_t$, $\tilde{\xi}_t$ and the last passage problem is contained in our next lemma. We denote by $N_t(x)$ the number of times x is excited in the time interval $[0,t]$.

Lemma 4.1. *At any time $t > 0$, $N_t(x) = \tilde{\xi}_t(x) = \tilde{L}((x,t),0)$.*

Proof. We first establish the connection between the interface and the last passage problem. We prove, by induction on t, that if $x \in \mathbf{Z}^2$ and $t > 0$ are such that t is the first time at which $\tilde{\xi}_t(x) = k$, then an admissible path π ending at (x,t) with $\text{Len}(\pi) = k$ exists. This will show that $\tilde{\xi}_t(x) \le \tilde{L}((x,t),0)$. Now, $\tilde{\xi}_t(x)$ may become k in two ways. One possibility is that $\tilde{\xi}_{t-1}(x) = \tilde{\xi}_{t-2}(x) = \tilde{\xi}_{t-3}(x) = k-1$, and $(x,t-1) \in \Pi$. In this case, add the nucleus $(x,t-1)$ to the admissible path of length $k-1$ which ends at $(x,t-3)$. The other possibility is that $\tilde{\xi}_{t-1}(y) > k-1$ for some $y \in \mathcal{N}_x$. Then there exists an admissible path of length k ending at $(y,t-1)$, and just replace that final point by (x,t).

On the other hand, if there exists an admissible path π with $\mathrm{Len}(\pi) = k$, then $\tilde{\xi}_t(x) \ge k$ by obvious monotonicity ($\tilde{\xi}_t$ can only be increased by adding more nuclei). This shows that $\tilde{L}((x,t),0) \le \tilde{\xi}_t(x)$ and ends the proof of the second identity.

If $x - y \in \mathcal{N}$, then $|\tilde{L}((x,t),0) - \tilde{L}((y,t),0)| \le 1$, therefore we know at this point that $|\tilde{\xi}_t(x) - \tilde{\xi}_t(y)| \le 1$.

To prove the first equality, it is easiest to once again identify 1's in $\tilde{\gamma}_t$ with boundaries in $\tilde{\xi}_t$. This time, the connection is simply

$$\{x : \tilde{\gamma}_{t+1}(x) = 1\} = \{x : \tilde{\xi}_{t+1}(x) > \tilde{\xi}_t(x)\}. \tag{4.1}$$

We prove (4.1) by induction. If $\tilde{\gamma}_{t+1}(x) = 1$, then $\tilde{\gamma}_t(x) \ne 1$ and $\tilde{\gamma}_{t-1}(x) \ne 1$, hence (by the induction hypothesis) $\tilde{\xi}_t(x) = \tilde{\xi}_{t-1}(x) = \tilde{\xi}_{t-2}(x)$. Moreover, either $(x,t) \in \Pi$ or else there exists a $y \in \mathcal{N}_x$ such that $\tilde{\gamma}_t(y) = 1$. In the first case, $\tilde{\xi}_{t+1}(x) = \tilde{\xi}_t(x) + 1$ by (I2') above. In the second case, $\tilde{\xi}_t(y) > \tilde{\xi}_{t-1}(y)$ and $\tilde{\xi}_t(y) > \tilde{\xi}_t(x)$ (since $\tilde{\xi}_{t-1}(y) \ge \tilde{\xi}_{t-2}(x)$), hence $\tilde{\xi}_{t+1}(x) = \tilde{\xi}_t(x) + 1$ as well.

Conversely, if $\tilde{\xi}_{t+1}(x) > \tilde{\xi}_t(x)$, then the situation is either as in (I2') in which case clearly $\tilde{\gamma}_{t+1}(x) = 1$, or else there exists a $y \in \mathcal{N}_x$ such that $\tilde{\xi}_t(y) > \tilde{\xi}_t(x)$. In the second case, $\tilde{\xi}_t(y) > \tilde{\xi}_{t-1}(y)$, so by the induction hypothesis $\tilde{\gamma}_{t-1}(y) = 1$, and hence $\tilde{\gamma}_{t-2}(y) \ne 1$ and $\tilde{\gamma}_{t-3}(y) \ne 1$, so again by the induction hypothesis $\tilde{\xi}_{t-3}(y) = \tilde{\xi}_{t-1}(y)$. It follows that $\tilde{\xi}_{t-2}(x) \ge \tilde{\xi}_{t-3}(y) = \tilde{\xi}_{t-1}(y) = \tilde{\xi}_t(x)$, and so $\tilde{\xi}_{t-2}(x) = \tilde{\xi}_t(x)$. Using once again the induction hypothesis, we get $\tilde{\gamma}_t(x) = 0$ and $\tilde{\gamma}_t(y) = 1$, so that $\tilde{\gamma}_{t+1}(x) = 1$. □

Lemma 4.2. *As $t \to \infty$, $\frac{\tilde{L}((0,t),0)}{t}$ converges to a constant $\nu(p_s) \in (0,1/3)$ a.s. and in L^1.*

Proof. We define random variables $X_{s,t}$ for $0 \le s \le t$. If $s = t$ then we declare $X_{s,t} = 0$. Otherwise, take a path π with the maximal length $\tilde{L}((0,t),s)$ (with some arbitrary convention in cases when there is more than one maximizer). Let (x_1,t_1) be the first point on π. Then declare $X_{s,t} = \tilde{L}((x_1,t_1),0)$; assume the maximum is achieved at a path π'. By concatenation of π and π' and, if necessary, omission of (x_1,t_1), one gets

$$X_{0,t} \ge X_{0,s} + X_{s,t} - 1.$$

By the subadditive ergodic theorem ([Lig, p. 277]), $X_{0,t}/t$ converge as $t \to \infty$, a. s. and in L^1, to $\nu(p_s) = \sup_{t \ge 1} E(X_{0,t} - 1)/t = \sup_{t \ge 1} E(\tilde{L}((0,t),0) - 1)/t$. Since $E(\tilde{L}((0,t),0) > 1$ for large

t, $\nu(p_s) > 0$. It is obvious that $\nu(p_s) \leq 1/3$. Strict inequality follows from an argument similar to the one on p. 51 of [Gri1]; as this is not relevant to further discussion, we omit the detailed proof. □

Lemma 4.3. *As $t \to \infty$, $\frac{L((0,t),0)}{t}$ converges to a constant $\nu'(p_s) \in (0,1]$ a.s. and in L^1. Moreover, $\nu'(p_s) \leq 2.3 \cdot p_s^{1/3}$ for a small enough $p_s \in (0,1)$.*

Proof. We skip the proof of convergence as $t \to \infty$, as it is even easier to establish than in the case of admissible paths. To prove the upper bound for $\nu'(p_s)$, assume that if $L((0,t),0) \geq n$. Then there must exist n times $0 \leq t_1 < \cdots < t_n < t_{n+1} = t$ and sites (x_i, t_i) such that $(x_i, t_i) \in \Pi$ and $||x_{i+1} - x_i||_\infty \leq t_{i+1} - t_i$, $i = 1, \ldots n$. (Here we declare $x_{n+1} = 0$.)

Let Δ_i be the time differences, i.e. $\Delta_i = t_{i+1} - t_i$, $i = 1, \ldots, n$. The number of ways to choose the times t_i is at most $\binom{t}{n}$. After the t_i are chosen, the probability that the nuclei (x_i, t_i) exist is bounded above by $(2\Delta_1 + 1)^2 \ldots (2\Delta_n + 1)^2 p_s^n$. This product is maximized when Δ_i are equal, thus

$$P(L((0,t),0) \geq n) \leq \binom{t}{n} \left(2\frac{t}{n} + 1\right)^{2n} p_s^n \leq \left(\frac{ep_s(2t+n)^2 t}{n^3}\right)^n,$$

which decreases exponentially in n as soon as $n/(tp_s^{1/3}) \geq 2.3 > (4e)^{1/3}$ and p_s is small enough. An application of the Borel–Cantelli lemma ends the proof. □

Of course, $\nu(p_s)$ and $\nu'(p_s)$ are quite different for large p_s (in fact, $\nu'(p_s) = 1$ for p_s close to 1), but, as the next lemma demonstrates, they have the same scaling law near $p_s = 0$.

Lemma 4.4. $\limsup_{p_s \to 0} p_s^{-1/3} |\nu(p_s) - \nu'(p_s)| = 0.$

Proof. Of course, $\nu(p_s) \leq \nu'(p_s)$. Fix an $\epsilon > 0$. Let H_t be the event that some discrete path ending at $(0,t)$ exceeds the lengths of all admissible paths ending at $(0,t)$ by at least $\epsilon p_s^{1/3} t$. What we will prove is that, given that p_s is small enough, $P(H_t)$ converges to 0 exponentially fast as $t \to \infty$. Let $n = 2\lfloor \epsilon p_s^{1/3} t/2 \rfloor$

If $t_i < t_{i+1}$, and (x_i, t_i), (x_{i+1}, t_{i+1}) are successive nuclei on a path which violate admissibility, then (x_i, t_i) must be one of only $24(t_{i+1} - t_i - 1)$ points. If H_t happens, there must exist a discrete path of length at least n on which at least every other nucleus violates admissibility. With t_i

and Δ_i as in the proof of Lemma 4.3, we then get that

$$\begin{aligned} P(H_t) &\le \binom{t}{n}(24\Delta_1-1)(2\Delta_2+1)^2(24\Delta_3-1)(2\Delta_4+1)^2\dots(24\Delta_{n-1}-1)(2\Delta_n+1)^2p_s^n \\ &\le \binom{t}{n}(18)^n\left(\frac{2t}{n}\right)^{3n/2}p_s^n \\ &\le \left(\frac{50t^{5/2}p_s}{n^{5/2}}\right)^n \\ &\le (100\cdot p_s^{1/6}\epsilon^{-5/2})^n. \end{aligned}$$

The proof is concluded by choosing $p_s < 10^{-12}\epsilon^{15}$. □

Lemma 4.5. *As $t\to\infty$, $\frac{\mathcal{L}((0,t),0)}{t}$ converges to a constant $\nu^*\in(0.75,2.3)$ a.s. and in L^1. Moreover,*

$$p_s^{-1/3}\nu'(p_s)\to\nu^* \qquad \text{as } p_s\to 0. \tag{4.1}$$

Proof. Again, we skip the proof of the existence of the limit as $t\to\infty$, as it is the same as the proof of Lemma 4.2. It is not immediately clear why $\nu^*<\infty$ though, and to see this we prove the small p_s approximation result first.

For a realization of $\mathcal{P}$, couple $\mathcal{P}$ and $\Pi=\Pi(1-e^{-p_s})$ in the standard way, by declaring $(x,t)\in\mathbf{Z}^2\times\mathbf{Z}_+$ a nucleus iff $B_\infty((x,t),1/2)\cap p_s^{-1/3}\mathcal{P}\ne\emptyset$. (Note that the box $B_\infty(\cdot,\cdot)$ is three–dimensional here.) Fix an $\epsilon>0$. We will prove that, under this coupling, and for a sufficiently small p_s there exists an $\alpha>0$ so that

$$P(|\mathcal{L}((0,p_s^{1/3}t),0)-L((0,t),0)|>\epsilon p_s^{1/3}t)\le e^{-\alpha t}. \tag{4.2}$$

Once we have (4.2), it immediately follows that $p_s^{-1/3}\nu'(1-e^{-p_s})\to\nu^*$ as $p_s\to 0$, which is clearly enough to prove (4.1).

To prove (4.2), note first that $\mathcal{L}((0,p_s^{1/3}t),0)\ge L((0,t),0)-1$ (nuclei at time 0 "feel" negative times). On the other hand, let H_t be the event that $\mathcal{L}((0,p_s^{1/3}t),0)\ge L((0,t),0)+\epsilon p_s^{1/3}t$. Denote by $Y_{(x,t)}$ the cardinality of $B_\infty((x,t),1/2)\cap p_s^{-1/3}\mathcal{P}$ and let $n=\epsilon p_s^{1/3}t$. (Hence $Y_{(x,t)}$ are independent Poisson random variables with mean p_s.) Moreover, let F_m be the event that there exists a discrete path $(x_1,t_1),\dots,(x_m,t_m)\in\Pi$ of length $m\le n$, such that $Y_{(x_i,t_i)}\ge 2$ for every i and $Y_{(x_1,t_1)}+\dots Y_{(x_m,t_m)}\ge n-m$. We now divide H_t into three events:

$$\begin{aligned} H_t^1 &= \{Y_{(0,t)}\ge n/3\}, \\ H_t^2 &= \cup_{m\ge n/3}F_m, \\ H_t^3 &= \cup_{m\le n/3}F_m. \end{aligned}$$

Since $H_t \subset H_t^1 \cup H_t^2 \cup H_t^3$, we need to show that all three events are exponentially unlikely for small p_s. For starters, $P(H_t^1)$ goes to 0 as $t \to \infty$ at a faster then exponential rate. Now abbreviate $Y_i = Y_{(x_i,t_i)}$. Since $P(Y_i \geq 2) = \mathcal{O}(p_s^2)$, the proof of Lemma 4.3 immediately implies that $P(H_t^2)$ must go to 0 exponentially as $t \to \infty$. Finally, (see the Appendix in [KS]), $\sup_{m \leq n/3} P(Y_1 + \cdots + Y_m \geq 2n/3) \leq C^n p_s^{2n/3}$, therefore a similar argument as in the proof of Lemma 4.3 yields that $P(H_t^3)$ is exponentially small in t. This ends the proof of (4.2) and hence (4.1).

Finally, we need to prove the bounds for ν^*. The upper bound follows from Lemma 4.3 and (4.1). We now obtain the lower bound. If $P(\mathcal{L}(0,c),0) \geq 1) = \alpha$, then $E(L(0,c),0)) \geq \alpha$ and, by superadditivity, $\nu^* \geq \alpha/c$. Since the volume of $\{(x,t) \in \mathbf{R}^3 : 0 \leq t < c, ||x||_\infty \leq t - c\}$ is $\frac{4}{3} \cdot c^3$, $\alpha = 1 - e^{-4c^3/3}$. Therefore, for small p_s, ν^* is bounded below by the maximum of $c^{-1}(1 - e^{-4c^3/3})$ over $c > 0$. This maximum is about 0.756. □

We should mention that the lower bound on ν^* in Lemma 4.4 can certainly be improved with some work. A substantial improvement of the upper bound seems to present a much bigger challenge. Computer simulations indicate that ν^* is somewhere in the neighborhood of 1.3.

Proof of Theorem 6. Lemma 4.2 deals with (1), while (2) is proved by Lemmas 4.3–4.5. □

5. A shape result for a related growth model.

In this section, we prove a large deviation estimate for the convergence in Theorem 6(1), and then apply it to obtain an "approximate shape" result similar to the one in [KS], but, due to simplicity of the discrete–time dynamics, a little more precise. To this end, define a three–dimensional discrete–time random growth model $\eta_t \in \{0,1\}^{\mathbf{Z}^3}$, in which a 1 never changes, a 0 at x changes into a 1 automatically if the 6 nearest–neighbor sites of x contain 2 or more 1's, and with probability p_s if these 6 sites contain exactly one 1. Assume that these growth dynamics are started from a single occupied site, say $\eta_0(x) = 1$ iff $x = 0$. Standard subadditivity arguments imply that there exist a convex set $\mathcal{A}(p_s) \subset \mathbf{R}^3$ with non–empty interior such that,

$$\frac{1}{t}\{\eta_t = 1\} \to \mathcal{A}(p_s) \qquad \text{a.s.}$$

Here, the convergence holds in the Hausdorff metric (see [KS], [GG] for discussions on such convergence issues). Below is our main result of this section.

Theorem 5.1. *As $p_s \to 0$, in Hausdorff metric,*

$$p_s^{-1/3}\mathcal{A}(p_s) \to B_\infty(0, \nu^*/\sqrt{2}).$$

Our first step in proving Theorem 5.1 is to analyze what happens if $\{\eta_0 = 1\}$ is precisely the half–space $H_{e_3} = \{x \in \mathbf{Z}^3 : \langle x, e_3\rangle \le 0\}$. We can do this by only slightly changing the setting from Section 4: let now $\mathcal{N}_x = B_1(x, 1)$, and define the interface dynamics $\bar\xi_t$ by (I1), (I3) and

(I2") Otherwise, if $(x,t) \in \Pi(p_s)$, then $\bar\xi_{t+1}(x) = \bar\xi_t(x) + 1$.

Moreover, define an *η–admissible path π ending at (x,t)* to be a sequence of space–time points $(x_1,t_1),\dots,(x_n,t_n) \in \Pi(p_s)$, $(x_{n+1},t_{n+1}) = (x,t)$ such that $0 \le t_1 < t_2 < \cdots < t_{n+1}$ and $||x_{i+1} - x_i||_1 + 1 \le t_{i+1} - t_i$, $i = 1,\dots,n$. Again, $n = \mathrm{Len}(\pi)$, and $\bar L((x,t),s) = \max\{\mathrm{Len}(\pi) : \pi$ is an η–admissible path contained in $\mathbf{Z}^2 \times [s,t)$ ending in $(x,t)\}$. The following two lemmas could now be proved similarly as Lemmas 4.1–4.5.

Lemma 5.2. *At any time $t > 0$, $\bar\xi_t(x) = \bar L((x,t),0)$.*

Lemma 5.3. *As $t \to \infty$, $\frac{1}{t}\bar L((0,t),0)$ converges a.s. and in L^1 to a constant $\bar\nu(p_s) \in (0,1)$. Moreover, $p_s^{-1/3}\bar\nu(p_s) \to \nu^*/\sqrt{2}$ as $p_s \to 0$.*

To justify the last statement, note that the ℓ^1 version of $\mathcal{L}$ is obtained from the ℓ^∞ version by a 45 degrees rotation and a $\sqrt{2}$ scaling. The fact that $\bar\nu(p_s) < 1$ again follows from a Peierls argument, very similar to the one on p. 51 of [Gri1].

By Lemma 5.3, $\mathcal{A}(p_s)$ should intersect the coordinate axes in about $[-p_s^{1/3}\nu^*/\sqrt{2}, p_s^{1/3}\nu^*/\sqrt{2}]$. Since the growth proceeds in other directions with greater ease, $\mathcal{A}(p_s)$ should be close to a cube. This intuitive argument makes Theorem 5.1 plausible, all that remain are some technical details. Before proceeding, we add a remark about a continuous–time version of these growth dynamics introduced by Kesten and Schonmann ([KS]), in which a 0 changes to 1 at rate 1 if it has 2 or more occupied neighbors, and at rate p_s if it has exactly one occupied neighbor. Though details are much more elusive, we suspect that the asymptotic shape $\mathcal{A}_c(p_s)$, multiplied by $p_s^{-1/3}$, converges to $B_\infty(0,\nu_c^*)$. Here, ν_c^* is obtained as in Lemma 4.5, except that in the definition of $\mathcal{L}$ the norm $||\cdot||_\infty$ is replaced by the Minkowski functional of the asymptotic shape of the two–dimensional Richardson growth model.

Lemma 5.4. *For every $\epsilon > 0$,*

$$P(|\bar L((0,t),0) - E(\bar L((0,t),0)| > \epsilon p_s^{1/3} t) \le 2\exp(-\frac{1}{2}\epsilon^2 p_s^{2/3} t).$$

Proof. We use the method of bounded differences (see [McD] for an accessible introduction to the method and its many applications). For $s = 0,\dots,t$, let $\mathcal{F}_s$ be the σ–algebra generated by $\Pi(p_s) \cap (\mathbf{Z}^2 \times [0, s-1])$ ($\mathcal{F}_0$ is the trivial σ–algebra), and define $X_s = E(\bar L((0,t),0)|\mathcal{F}_s)$. For

$s = 1, \dots, t$, let $\bar{L}_s$ be the length of the longest η–admissible path ending at (x,t) and contained in $\mathbf{Z}^2 \times ([0,t-1] \setminus \{s-1\})$. Then $\bar{L}_s \le \bar{L}((0,t),0) \le \bar{L}_s + 1$ and therefore $|X_s - X_{s-1}| \le 1$, so that the Hoeffding's inequality ([Hoe]) implies that, for every $x \ge 0$, $P(|X_t - X_0| \ge x) \le 2\exp(-\frac{1}{2}x^2/t)$. □

Lemma 5.5. *Assume that $X_1, X_2, \dots$ are i.i.d. symmetric random variables taking values in $[-M, M] \cap \mathbf{Z}^2$, and $S_n = X_1 + \cdots + X_n$. Then, for any $n \ge 1$, $P(\max_{1\le k\le n} |S_k| \ge Mn^{3/4}) \le 8e^{-\sqrt{n}/32}$.*

Proof. By basic martingale inequalities (e.g. Theorem 22.5 in [Bil]), the probability in question is bounded by $8\max_{1\le k\le n} P(S_k \ge Mn^{3/4}/4)$. But for every $k = 1, \dots n$ and every $\lambda > 0$,

$$\begin{aligned} P(S_k \ge Mn^{3/4}/4) &\le e^{-\lambda M n^{3/4}/4} E(e^{\lambda X_1})^n \\ &\le e^{-\lambda M n^{3/4}/4} (\cosh(\lambda M))^n \\ &\le e^{-\lambda M n^{3/4}/4 + n(\lambda M)^2/2}. \end{aligned}$$

The choice of $\lambda M = n^{-1/4}/4$ hence finishes off the proof. □

Lemma 5.6. *For every p_s and a small enough $\epsilon > 0$, there exists a time $t_0 = t_0(p_s, \epsilon)$ so that, for $t \ge t_0$, the probability that there exists an η–admissible path ending at $(0,t)$, with the space coordinate of the first point (x_1, t_1) satisfying $||x_1||_\infty \le t^{7/8}$ and length at least $(\bar{\nu}(p_s) - 2\epsilon p_s^{1/3})t$, is at least $1 - 3\sqrt{t}\exp(-\frac{1}{10}\epsilon^2 p_s^{2/3} t^{1/4})$.*

Proof. Let $T = \lfloor \sqrt{t} \rfloor$, and $n = \lfloor t/T \rfloor$. By Lemmas 5.3–5.4, there exists a t_0 so that for $t \ge t_0$,

$$P(\bar{L}((0,T),0) \in [T(\bar{\nu}(p_s) - \epsilon p_s^{1/3}), T(\bar{\nu}(p_s) + \epsilon p_s^{1/3})]) \ge 1 - 2\exp(-\frac{1}{8}\epsilon^2 p_s^{2/3} T). \tag{5.1}$$

Let $Y_1 = \bar{L}((0,t), t-T)$ and choose, from all the maximizing η–admissible paths with this length, one uniformly at random. The starting point (x_1', t_1') has its space coordinate x_1 distributed symmetrically with respect to switching signs of either coordinate. In the next step, $Y_2 = \bar{L}((x_1', t-T), t-2T)$, determine the starting point (x_1'', t_1'') of the randomly chosen maximal η–admissible path, and let $Y_3 = \bar{L}((x_1'', t-2T), t-3T)$. By continuing in this fashion, $Y_1, Y_2, \dots, Y_n$ are defined recursively, with (x_1, t_1) being the starting point on the last, n'th such η–admissible path (which has length Y_n). Note that the concatenation of such paths produces an η–admissible path with length $Y_1 + \cdots + Y_n$.

To apply Lemma 5.5, let $M = T$ and let $X_1, X_2, \dots$ from Lemma 5.5 be the first coordinates of $x_1', x_1'', \dots$, then repeat the argument with second coordinates. The conclusion is that $||x_1||_\infty \le t^{7/8}$ with probability at least $1 - 16\exp(-n^{1/4}/32)$. Moreover, by (5.1), the probability that

$Y_1+\cdots+Y_n \geq nT(\bar{\nu}(p_s)-\epsilon p_s^{1/3})$ is at least $1-2T\exp(-\frac{1}{8}\epsilon^2 p_s^{2/3}T)$. Hence the proof is concluded by suitably increasing t_0, if necessary. □

In the next two proofs, we use the "lattice" ball $B_2(x,r)$, the set of all points $y \in \mathbf{Z}^3$ such that $||x-y||_2 \leq r$. We will also fix an $\alpha < \nu^*/\sqrt{2}$ and a $\beta > \nu^*/\sqrt{2}$ for the rest of this section. Thus all constants (in particular, those denoted by C) will be allowed to depend on them.

Lemma 5.7. *Assume that* $\{\eta_0 = 1\} = B_2(0,r)$. *For a small enough* p_s, *there exists a* $r_0 = r_0(p_s)$ *so that, for* $r \geq r_0$, $P(B_2(0,r) + B_\infty(0, \alpha p_s^{1/3}\sqrt{r})) \subset \{\eta_{\sqrt{r}} = 1\}) \geq 1 - \exp(-Cp_s^{2/3}r^{1/8})$.

Proof. Let $\alpha' = \frac{1}{2}(\alpha + \nu^*/\sqrt{2})$. Note first that, as remarked in [KS],

$$(B_2(0,r) + B_1(0,\sqrt{r})) \cap B_\infty(0,r) \subset \{\eta_{\sqrt{r}} = 1\}. \tag{5.2}$$

For any unit vector $u \in S^3$, there is an $i \in \{1,2,3\}$ so that $|\langle u, e_i\rangle| \geq 1/2$. Without loss of generality we will assume, from now on, that $\langle u, e_3\rangle \geq 1/2$. It follows from (5.2) that if r is large enough

$$[0, r + 2^{-1/2}\sqrt{r}\cdot\left(\frac{1}{\langle e_3, u\rangle} - 1\right)]u \cap \mathbf{Z}^3 \subset \{\eta_{\sqrt{r}} = 1\}. \tag{5.3}$$

On the other hand, Lemma 5.6 implies that

$$[0, r + \sqrt{r}\alpha' p_s^{1/3}\langle e_3, u\rangle - Cr^{7/16}]u \cap \mathbf{Z}^3 \subset \{\eta_{\sqrt{r}} = 1\}, \tag{5.4}$$

with probability at least $1 - \exp(-Cp_s^{2/3}r^{1/8})$. Since there are at most Cr^3 vectors $u \in S^2$ such that $[0,2r]u \cap \mathbf{Z}^2 \neq \emptyset$, $P((5.4)$ holds for all $u) \geq 1 - \exp(-Cp_s^{2/3}r^{1/8})$.

Now if $\langle e_3, u\rangle \geq (\alpha + \alpha')/2\alpha'$, $r \geq \left(\frac{1}{2}(\alpha' - \alpha)p_s^{1/3}\right)^{-16}$, and (5.4) holds, then

$$[0, r + \alpha p_s^{1/3}\sqrt{r}\cdot\frac{1}{\langle e_3, u\rangle}]u \subset \{\eta_{\sqrt{r}} = 1\}. \tag{5.5}$$

If $\langle e_3, u\rangle < (\alpha + \alpha')/2\alpha'$, (5.5) is deterministically true for sufficiently small p_s by (5.3). □

Proof of Theorem 5.1. By Lemmas 5.2–5.4, for a small enough p_s, there exist a $t_0 = t_0(p_s)$ so $P(\bar{\xi}_t(x) \geq \beta p_s^{1/3}t) \leq 2\exp(-Cp_s^{2/3}t)$. Therefore, for $t \geq t_0$,

$$P(\{\eta_t = 1\} \cap B_\infty(0, \beta p_s^{1/3}t)^c \neq \emptyset) \leq Cp_s^{2/3}t^2 e^{-Cp_s^{2/3}t},$$

and so, with probability 1, $\{\eta_t = 1\} \subset B_\infty(0, \beta p_s^{1/3}t)$ eventually. After dividing by t, and sending $t \to \infty$, one gets that $\mathcal{A}(p_s) \subset B_\infty(0, \beta p_s^{1/3})$, which proves the upper bound in Theorem 1.1.

To prove the lower bound, fix a small $\delta > 0$. Define the Euclidean set $\mathcal{B}_\delta = B_\infty(0, 1-\delta) + B_2(0,\delta) \subset \mathbf{R}^3$ and note that $r\mathcal{B}_\delta = \cup_{x\in B_\infty(0,r(1-\delta))} B_2(x, r\delta)$. By Lemma 5.7, there exists a $r_0 = r_0(p_s,\delta)$ so that whenever $r \ge r_0$, and $r\mathcal{B}_\delta \cap \mathbf{Z}^3 \subset \{\eta_{t_0} = 1\}$

$$P((r + \alpha p_s^{1/3}\sqrt{\delta r})\mathcal{B}_\delta \cap \mathbf{Z}^3 \subset \{\eta_{t_0+\sqrt{\delta r}} = 1\}) \ge 1 - e^{-C(p_s)r^{1/8}}. \tag{5.6}$$

Define the sequence r_k by $r_1 = r$, $r_{k+1} = r_k + \alpha p_s^{1/3}\sqrt{\delta r_k}$. Note first that $r_k \ge r + (k-1)\sqrt{\delta}\alpha p_s^{1/3}\sqrt{\delta r}$ and

$$\lim_{k\to\infty} \frac{r_{k+1}}{\sqrt{r_1} + \ldots \sqrt{r_k}} = \lim_{k\to\infty} \frac{r_{k+1} - r_k}{\sqrt{r_k}} = \sqrt{\delta}\alpha p_s^{1/3}. \tag{5.7}$$

Iteration of (5.6) and monotonicity of the dynamics yield (recall that r is large)

$$P(r_{k+1}\mathcal{B}_\delta \cap \mathbf{Z}^3 \subset \{\eta_{t_0+\sqrt{\delta r_1}+\ldots\sqrt{\delta r_k}} = 1\} \text{ for all } k = 0,1,\ldots) \ge 1 - \sum_k e^{-Cp_s^{2/3}r_k^{1/8}} > 0. \tag{5.8}$$

Since $t_0 < \infty$ a.s., (5.7) and (5.8) imply that

$$\alpha p_s^{1/3}\mathcal{B}_\delta \subset \mathcal{A}(p_s),$$

which establishes the lower bound and ends the proof. □

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