CYCLIC CELLULAR AUTOMATA IN TWO DIMENSIONS

Dedicated to T.E. Harris on his 70th birthday

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Abstract

Start by randomly populating each site of the two-dimensional integer lattice with any one of N types, labeled $0, 1, \ldots, N-1$ ($N \geq 3$). The type $\zeta(y)$ at site y can *eat* the type $\zeta(x)$ at neighboring site x (i.e., replace the type at x with $\zeta(y)$) *provided* that $\zeta(y) - \zeta(x) = 1 \mod N$. We describe the dynamics of *cyclic cellular automata (c.c.a.)* ζ_t, discrete-time deterministic systems which follow the rule:

(•)   At any time t, each type $\zeta_t(y)$ eats *every* neighboring type that it can.

These systems have remarkably complex dynamics. As N becomes large they display a curious metastability leading to large-scale locally-periodic structure. This article contains a preliminary account of our findings. For the most part, we rely on computations and computer graphics produced by the Cellular Automaton Machine. However we are able to give a simple proof that the infinite system ζ_t is asymptotically locally periodic for any $N < \infty$. Moreover, we identify a number of regularity properties of rule (•), mostly topological in nature, that offer some hope for a more detailed rigorous analysis.

1. Introduction

Think of the two-dimensional integers $\mathbb{Z}^2$ as a terrain inhabited by creatures of *N types*, labeled $0,1,\ldots,N-1$ ($N \geq 3$). Let $\zeta_0(x)$, $x \in \mathbb{Z}^2$, denote an i.i.d. random initial configuration of types, the value at each site x being uniformly distributed over $[0,\ldots,N-1]$. The creatures proceed to compete for space by feeding on their neighbors, but each type can only eat the one immediately preceding it in a "cyclic food chain." More precisely, if $\| y - x \| = 1$, the type $\zeta(y)$ at site y can *eat* the type $\zeta(x)$ — that is, replace the type at x with its own — provided that $\zeta(y) - \zeta(x) = 1 \mod N$. This article describes the dynamics of discrete-time deterministic systems with local transitions of this sort. Namely, we study *cyclic cellular automata (c.c.a.)* determined recursively by the rule:

$$\zeta_{t+1}(x) = \zeta_t(y) \quad \text{if} \ \ \zeta_t(y) - \zeta_t(x) = 1 \mod N \ \text{ for } some \ y \text{ such that } \| y - x \| = 1,$$

(•)

$$= \zeta_t(x) \quad \text{otherwise.}$$

In other words, at any time t, each type eats *every* neighbor that it can. Note that this update rule makes sense since each type can only be eaten by one other.

The c.c.a. ζ_t determined by (•) exhibit unusual self-organization over time, leading to large-scale locally-periodic structure. We will describe four distinct phases of the evolution, characterized by *debris, droplets, defects* and *demons*. Each of these objects will be defined rigorously, and a series of propositions, essentially topological, will illuminate their roles in the dynamics. The c.c.a. are interesting not only as mathematically tractable prototypes for local periodicity, but also for the exotic *metastability* they display along the way to the limit.

Our goal here is to identify the basic qualitative behavior of these systems, give some background that motivates our work, present a few empirical findings, and lay the groundwork for a rigorous mathematical analysis. A more complete and systematic study will appear elsewhere [7]. A popular account of c.c.a. dynamics appeared in [4].

2. Four stages of complex dynamics

The c.c.a. ζ_t form a one-parameter family of random systems indexed by the number of types N. Noise in the i.i.d. initial configuration ζ_0 is subjected to the organizational logic of deterministic rule (•). As we shall see, this dynamic gives rise to coherent limiting structure which grows in size as N increases. Approximation of the infinite c.c.a. by finite lattice systems is therefore delicate. To obtain an accurate glimpse of a finite window, say of width L, in the infinite c.c.a., we must select finite L-systems with L a suitably chosen function of N. In particular, L must grow very rapidly with N.

In order to describe the essential qualitative features of c.c.a. evolution, let us first consider the 14-color model on a 256×256 finite sublattice of $\mathbb{Z}^2$, with wrap-around at the boundary. The choice $N = 14$ is large enough that all stages of the complex dynamics are plainly visible, but small enough that the finite system with $L = 256$ provides a faithful representation of the infinite 14-type system. Computer graphics, with types represented as *colors* effectively communicate the behavior of ζ_t. Most of our computations have been performed on a customized Cellular Automaton Machine (CAM) [12]. We use the word *computation* rather than *simulation* because pseudo-randomness only generates the initial state ζ_0. The scenario described below is remarkably insensitive to the statistics of ζ_0, at least in its qualitative aspects.

Sample trajectories of ζ_t, such as the one illustrated in Figures 1-4, are characterized by *four* distinct stages:

Stage 1   The vast majority of creatures have nothing to eat initially or quickly run out of food — i.e., within a short time there are only a very few types that have a neighbor they can eat. However, the rare remaining active areas form *critical droplets*.

Stage 2   These *critical droplets* remain active and, by generating wave activity, expand at a linear rate until they overrun all of the inactive *debris* that was present in the initial state.

Stage 3   *Defects* are formed, leading to the emergence of periodic spiral structures. Spirals overtake the Stage 2 wave activity at a linear rate, competing with one another for all of the available space.

Stage 4   Certain minimal defects, which we call *clocks*, give rise to optimally efficient *demons*. These demons displace less efficient spirals until every site in the lattice is regulated by a local, periodic structure of period *N*.

For $N = 14$, Stage 1 is over quite quickly; the droplets form and coalesce so rapidly that it is hard to distinguish one from another. (By merely increasing N to 20, as illustrated in Figure 5, the active droplets become quite distinct and "drop-like.") Figure 1 depicts the situation well into Stage 2: the lattice is about equally divided between fine-grained regions that consist mainly of debris from ζ_0, and waves of solid type.

Figure 2 occurs early in Stage 3. A *first spiral* has "popped up" in the center of the picture. Others appeared shortly thereafter, and still others are in the process of forming. We will explain later how these spiral formations are generated by topological defects.
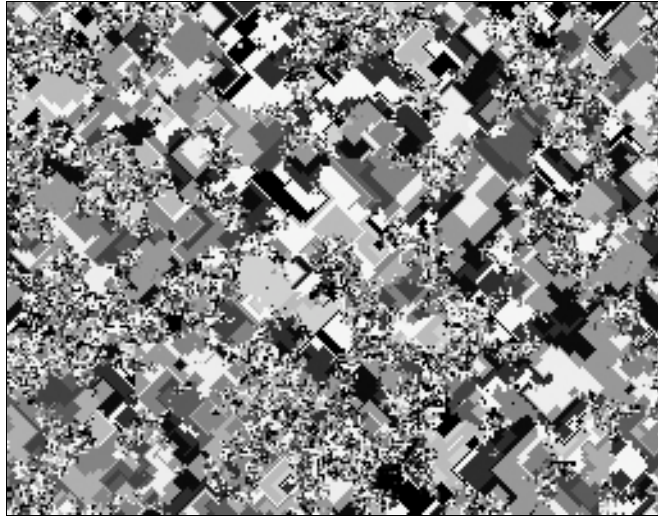
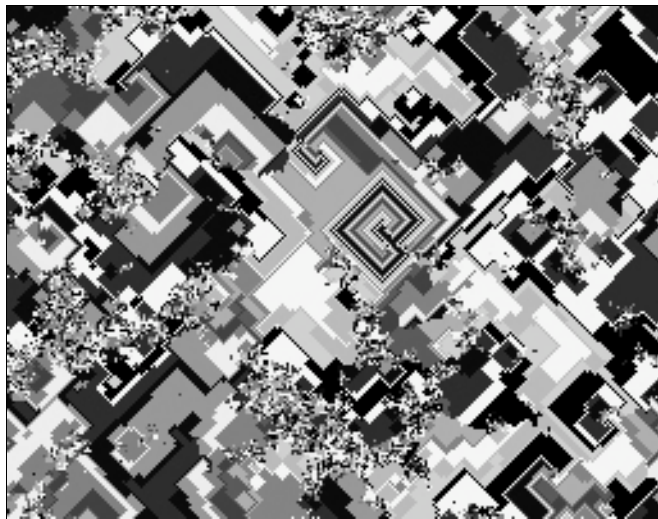***Fig. 1.*** **The 14-color c.c.a. at time 100 ($L = 256$)**

Fig. 2. **The 14-color c.c.a. at time 150 ($L = 256$)**

The state of ζ_t toward the end of Stage 3 is shown in Figure 3: all waves of solid color have been overwhelmed by spiral formations. Note that the first spiral is still evident in the center of the figure. But note also the spiral with center just to the north and slightly west. As chance would have it, this is a demon — a structure driven by a period 14 clock. The first spiral, on the other hand, turned out to have period 16. The eventual periodic state is reached after several thousand more updates. In it, the first spiral is almost entirely displaced by the nearby demon. Moreover, every site of the entire lattice becomes part of some demon, cycling through the 14 states in order. Different regions are regulated by different clocks that are out of phase with one another. This is *local periodicity*.
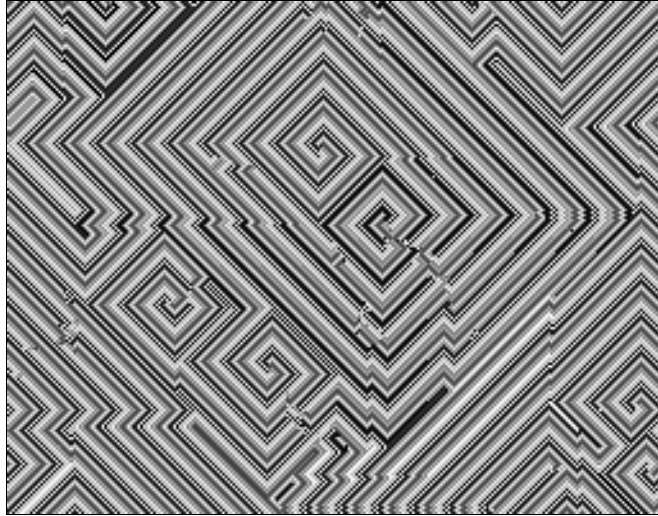
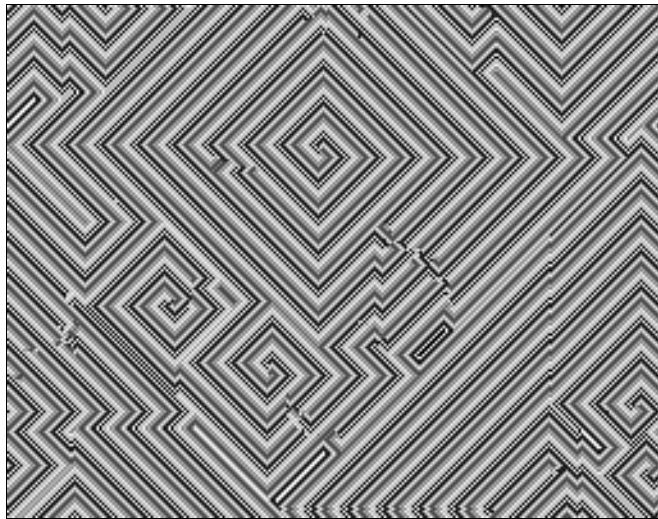***Fig.* 3. The 14-color c.c.a. at time 300 ($L = 256$)**

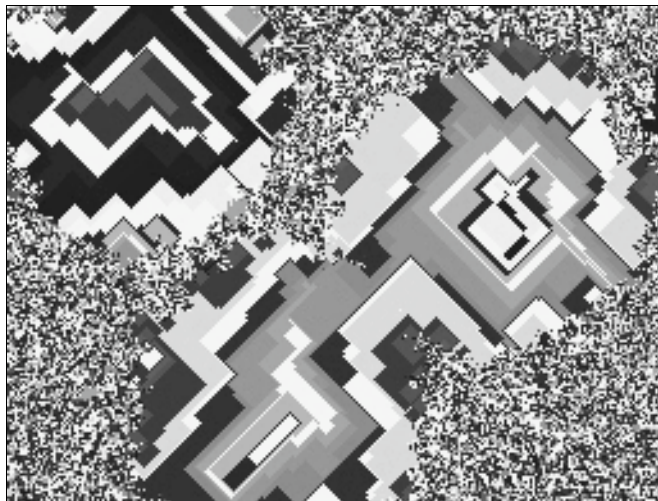***Fig.* 4. The 14-color c.c.a. at time 3000 ($L = 256$)**

***Fig.* 5. Droplet interaction in the 20-color c.c.a. ($L = 256$)**

3. Background and Motivation

Our interest in the c.c.a. ζ_t grew out of work on similar cyclic systems ξ_t with *random* dynamics, described in [2] and [8]. In those models, with time either discrete or continuous, the type at each site chooses a neighbor at random and eats that neighbor if it can. The main result of [2] deals with one-dimensional (continuous-time) ξ_t. It is proved that for $N \geq 5$ the system *fixates*. That is to say, each site attains a final random type (with probability one). If $N = 3$ or 4, on the other hand, ξ_t *clusters*, meaning that contiguous regions of all one type grow arbitrarily large (in distribution). Article [8], an expository piece written for a general audience, describes some Cellular Automaton Machine experiments on the two-dimensional ξ_t. To our great surprise, CAM gave compelling evidence that these systems neither cluster nor fixate for *any* value of *N*. Rather, after an initial metastable period of wave droplet formation and growth, the dynamic settles down to a remarkably stable statistical equilibrium consisting of large-scale spiraling wave formations, as illustrated in Fig. 6. When *N* increases, both the duration of the initial metastable phase, and the size of the eventual vortices, grow rapidly. For more details and a few pictures see [8].
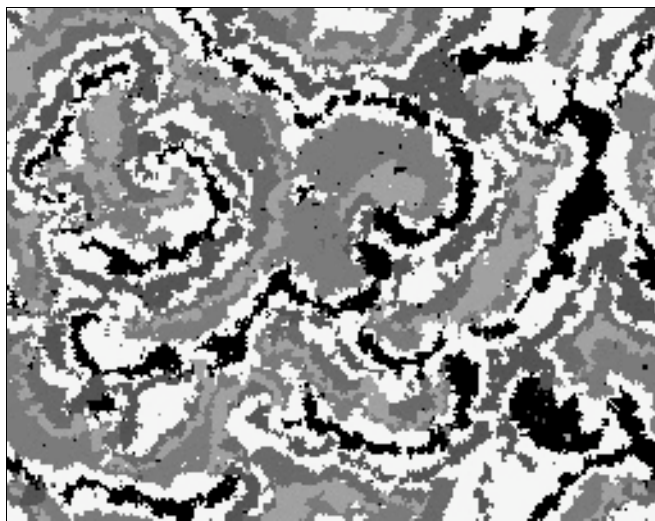

Fig. 6. **Stable spirals in the cyclic particle system ($L = 256$)**

In some cases, deterministic cellular automata are easier to analyze than their random counterparts. For instance, some rigorous results for the one-dimensional c.c.a. ζ_t are derived in [5] and [6]. Reference [5] establishes fixation for $N \geq 5$ and fluctuation for $N = 3, 4$, in direct analogy to the results of [2]. Paper [6] computes the exact asymptotic rate of clustering for the 3-color c.c.a. on $\mathbb{Z}$: as $t \to \infty$,

(†) $$\text{Average cluster size at time } t \quad \sim \quad \sqrt{\tfrac{3\pi t}{2}}$$

(with probability one); the corresponding problem for ξ_t is open. We note in passing that the calculation (†) is accomplished by considering the evolution of cluster *edges* as a deterministic particle system, one closely related to the deterministic model analyzed by Harris in [9]. Instead of reflecting, the particles in [6] annihilate upon collision.

We initiated our investigation of the two-dimensional c.c.a. in the hope that they might shed light on the random cyclic systems ξ_t. In spite of the fact that the deterministic rules lead to local periodicity and the random rules lead to stable random equilibria, many qualitative and quantitative features of the two dynamics are shared. In the next section we identify several regularity properties of ζ_t that improve the prospects for detailed rigorous analysis. Such an analysis may, in turn, aid our understanding of the more mysterious behavior of ξ_t.

In the general classification of cellular automata, both fixation and local periodicity are instances of "Class 2" behavior, according to the scheme of Wolfram [16], [11]. (It is less clear where *clustering* fits into that scheme.) Local periodic spiral formations are present in two-dimensional cellular automaton models for: atrial fibrillation [10], Belousov-Zhabotinsky chemical reactions [13], [15], neural networks with firing and recovery [3], [12], and related nonlinear wave dynamics. As far as we are aware, there are no substantial rigorous mathematical results for any of these systems. We propose the c.c.a. systems ζ_t as relatively tractable prototypes for this important variety of complex spatial phenomena.

4. Regularity Properties

Let us now define *debris, droplets, defects,* and *demons,* the principal features of c.c.a. dynamics. First, site x belongs to the *debris* D_t^0 at time t if its type $\zeta_t(x)$ differs from that of each of its neighbors by at least 2 (mod N). We set $D_t^1 = \mathbb{Z}^2 - D_t^0$, call the connected components of D_t^1 *droplets,* and for $x \in D_t^1$, let $D_t^1(x)$ denote the droplet containing x. Roughly speaking, the fine-grained regions in *Fig. 1* correspond to D_{100}^0, whereas the open fields comprise D_{100}^1. Note that by time 200, in *Figure 2*, very little debris remains. This reflects the first key property of ζ_t:

($\bigstar_1$) $\qquad\qquad\qquad\qquad\qquad\qquad D_t^1$ is increasing in t.

Given neighboring sites x, y, say that bond $\{x, y\}$ is *open* at time t if $\zeta_t(y) - \zeta_t(x) \in \{-1, 0, 1\}$ (mod N). Property ($\bigstar_1$) follows from the simple observation that an open bond stays open at all subsequent times. Call $\{x, y\}$ *active* at time t if $\zeta_t(y) - \zeta_t(x) \in \{-1, 1\}$ (mod N), *inert* if $\zeta_t(y) = \zeta_t(x)$, and *closed* otherwise. A droplet is *active* at time t if it involves at least one active bond; otherwise it is *inert*. Whereas the set of all droplets increases, over time it is quite possible for $D_t^1(x)$ to switch repeatedly between being active and inert.

Next, we introduce the *index* $\iota_t(\ell)$ of ζ_t restricted to an open loop ℓ in D_t^1. Let $\ell = z_0, z_1, \ldots, z_n$ be an *open loop* in D_t^1 (with $\{z_k, z_{k+1}\}$ an open bond for each k, and $z_n = z_0$)). Let S_m be the unit circle S^1 marked with m equidistant vertices, labeled $v_0, \ldots, v_{m-1}$ (counterclockwise). Given ℓ, take $\phi: S_n \to S_N$ to be the linear extension of the vertex map

$$\phi(v_k) = v_{\zeta_t(z_k)}.$$

Now define $\iota_t(\ell)$ to be the index (winding number) of ϕ. Figure 5 below shows loops of index 0 and index 1 in sample 8-type configurations; these examples should make the definition clear. A loop in D_t^1 with positive index is called a *defect*. We let D_t^2 denote the set of all sites that are part of some defect at time t, and put $D_t^2(x) = D_t^2 \cap D_t^1(x)$. Of course any defect must involve all N types, so in fact, D_t^2 is contained in the set of D_t^+ of active droplets at time t.

$$
\begin{array}{ccccccc}
0 & 1 & 2 & & & & \\
1 & & 3 & 4 & 5 & 6 & \\
2 & & & & & 7 & \\
3 & 4 & & & & 6 & \\
& 5 & 6 & 7 & 6 & 5 & \\
\end{array}
\qquad \text{\emph{An 8-type loop of index 0}}
$$

$$
\begin{array}{ccccccc}
0 & 1 & 2 & & & & \\
7 & & 3 & 4 & 5 & 4 & \\
6 & & & & & 3 & \\
6 & 5 & & & & 2 & \\
& 4 & 5 & 4 & 3 & 2 & \\
\end{array}
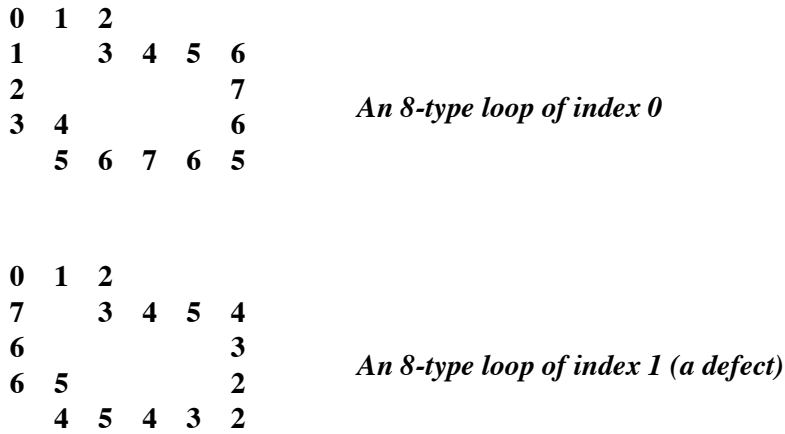\qquad \text{\emph{An 8-type loop of index 1 (a defect)}}
$$

Fig. 7. Sample loops of index 0 and 1 in the $N = 8$ c.c.a.

The characteristic c.c.a. spirals, such as those forming at time 150 in Figure 2, indicate the presence of defects. In fact, any loop contained within one of these formations and surrounding its "center" is a defect. The spread of spirals by time 300, evident in Figure 3, reflects a second key property of ζ_t:

($\bigstar_2$) $\qquad\qquad\qquad\qquad\qquad D_t^2$ is increasing in t.

To verify this claim, we make use of the formula

$$\iota(\ell) = n^{-1} \sum_{k=0}^{n-1} \left\{ \zeta_t(z_{k+1}) - \zeta_t(z_k) \right\},$$

where the terms in brackets are all chosen mod N to be either -1, 0, or 1 (since $\ell \subset D_t^1$), and the addition is ordinary (*not* mod N). It is easy to check, case by case, that this sum of differences cannot

change from time t to $t+1$. Thus, the index of any loop is invariant once that loop is open. In particular, defects are permanent, property ($\bigstar_2$) holds, and any defect present at time t belongs to D_u^+ for all $u \geq t$.

The transition of ζ_t from Stage 3 to Stage 4 is governed by *clocks* and *demons*. A *clock* at time t is a loop ℓ such that $\zeta_t(z_{k+1}) - \zeta_t(z_k) \equiv 1$ $(0 \leq k \leq n-1)$. Evidently the size of the loop must be a multiple of N. The set D_t^3 consists of all x that belong to a clock, or can be reached from one by a path of $+1$ bonds. More precisely, $x \in D_t^3$ provided there exists a path $\wp = (y_1,\ldots,y_m)$ of sites such that (i) $\{y_k, y_{k+1}\}$ is a bond for each k, (ii) $y_0 = x$ and y_m belongs to a clock at time t, and (iii) $\zeta_t(y_{k+1}) - \zeta_t(y_k)$ $\equiv 1$ $(1 \leq k \leq m-1)$. Connected components of D_t^3 are called *demons;* we denote the demon containing x by $D_t^3(x)$.

As explained earlier, the onset of local periodicity involves the displacement of "less efficient" spiral formations by demons. This process reflects yet a third key property of ζ_t:

($\bigstar_3$) $\qquad\qquad\qquad\qquad\qquad\qquad D_t^3$ is increasing in t.

Clocks are clearly stable under the c.c.a. dynamic. Since any site connected to a clock by a path $\wp$ of $+1$ bonds remains "in phase" with the clock at all later times, ($\bigstar_3$) follows. Such paths are clock-driven "tentacles" of the demon, which cycle at maximal speed, and latch on to neighboring sites unless their types also cycle in N steps.

Note that monotonicity properties ($\bigstar_1$)-($\bigstar_3$) apply to any *c.c.a.*, either on $\mathbb{Z}^2$ or a finite lattice of width L, starting from any initial configuration, either deterministic or random. Letting $T_t^k(x) = \inf \{ t \geq 0 : D_t^k(x) \neq \emptyset \}$ $(k = 1, 2, 3)$, it follows that

$$T_t^1(x) \leq T_t^2(x) \leq T_t^3(x)$$

in full generality. As we shall now see, existence of a single clock ensures periodicity at every x.

5. Local Periodicity

Say that a cellular automaton ζ_t with random initial state is *locally periodic* if

(i) for each x, $\zeta_t(x)$ is eventually periodic in t with probability one, and

(ii) $\displaystyle\inf_{t,\, x \neq y,\, a,\, b} P\left(\zeta_t(x) = a, \zeta_t(y) = b\right) > 0.$

Condition (i) asserts that the type at each site settles into a periodic orbit eventually (with period > 1). Condition (ii) requires that self-organization remain local: correlations between distinct sites must be uniformly controlled in space and time. In particular, the latter condition distinguishes local periodicity from *clustering,* where for each x, y,

$$\lim_{t \to \infty} P\left(\zeta_t(x) = \zeta_t(y)\right) = 1.$$

We now present a proof that, starting from a random initial configuration ζ_0, the c.c.a. on $\mathbb{Z}^d$, $d \geq 2$, is always locally periodic, with minimal period at every site, no matter how large the number of colors N. Thus the limiting behavior is quite different from that of one-dimensional cyclic systems ([2], [5], [6]), which either fixate or cluster.

Verification of (i) is based on the following simple observation.

Lemma. If there is a clock anywhere in ζ_0 (or if a clock forms at some later time t), then every site x is eventually periodic with period N.

Proof. Let $\mathcal{C}$ be the set of sites that have period N eventually in t. $\mathcal{C} \neq \emptyset$, since the c.c.a. contains a clock by hypothesis. Suppose $x \notin \mathcal{C}$, $y \in \mathcal{C}$, $\| y - x \| = 1$. Then the difference $\zeta_t(y) - \zeta_t(x) \in \{0, \ldots, N-1\} \pmod{N}$ is eventually non-increasing. Hence this difference is eventually constant. But then $x \in \mathcal{C}$, a contradiction. We conclude that every site x belongs to $\mathcal{C}$, as claimed. (We note that in CAM experiments with N large, the eventual constant difference is almost invariably 1 or 0. In the former case, x and y are part of the same demon; otherwise bond $\{x, y\}$ almost always belongs to the boundary between two distinct demons.)

Condition (ii) is even easier to check. Given any time t, distinct sites x and y, and types a and b, the event in question occurs provided that ζ_0 contains a pair of disjoint clocks through x and y with appropriate phases. The probability of this latter occurrence is at least

$$\left(\tfrac{1}{N}\right)^{4N} > 0.$$

6. Spirals: what, where, when, and why?

Although the argument just given determines the limiting behavior of any c.c.a. on $\mathbb{Z}^2$, it does not capture the true nature of the dynamics. CAM experiments clearly indicate that the first demon to reach a typical site x is *not* present in the initial state, but "pops up" at a later time. In other words, defects in ζ_0, used to prove local periodicity of the infinite system, occur on too large a scale.

Many intriguing problems concerning c.c.a. evolution are suggested by Figures 1-4. In this section we make some empirical observations that shed light on perhaps the four most fascinating questions:

What are the characteristic c.c.a. spirals, and where, when and why do they pop up?

What? —

When N is small, the spirals of ζ_t have noticeable "glitches." As these coherent structures form, there is enough variation in the environment that occasional "errors" occur. But as N increases, the individual spirals grow in final size *and* exhibit fewer errors. What is an *ideal spiral*? Here, at least, is an example. Suppose $N = 14$. Let ℓ be a "convex" loop of length 14 in $\mathbb{Z}^2$, say the boundary of a 5×4 rectangular array of sites. Specify ζ on ℓ to be a clock. Choose types for the interior of the array so that all interior sites are connected to the clock by a path of +1 edges. Now define ζ outside of ℓ by the formula:

$$(\S) \qquad \zeta(x) \;=\; \zeta(x^*) + \rho(x, x^*), \qquad \text{where } \rho(x, x^*) \;=\; \min\{\rho(x, y) : y \in \ell\},$$

and $\rho(x, y)$ denotes the length of a shortest path in $\mathbb{Z}^2$ between x and y. We invite the reader to color a piece of graph paper according to ($\S$) in order to confirm the connection with Figures 2 and 3. Recipe ($\S$) also generates representative spirals when ζ on ℓ is a defect but not a clock. If ℓ is not convex, then, roughly speaking, the recipe should be applied to a suitable "convexification" of ℓ.

Where? —

CAM experiments strongly suggest that spirals can only pop up on the interfaces between droplets and debris. In fact, this is a topological necessity. Recall that any loop of sites surrounding the center of a spiral has index at least one, and that the index of any loop is invariant once that is open. It follows that if a defect ℓ forms at time $t > 0$ (i.e., $\ell \not\subset D_{t-1}^1$, $\ell \subset D_t^1$), then ℓ must involve at least one closed bond at time $t-1$. Either some site of ℓ belongs to the debris, or ℓ involves two distinct droplets, at that previous time.

When? —

Suppose we lay down a uniform random distribution of N types on $\mathbb{Z}^2$, and then construct a starting configuration ζ_0 by modifying the values on a suitable loop ℓ to produce a clock. For instance, if $N = 14$, we could change the values on the boundary of a 5×4 array of sites. What happens when the c.c.a. starts from such an initial state? With the aid of a microcomputer, the enterprising reader can verify that ζ_t does *not* give rise to well-formed spirals. A roughly circular disturbance spreads out from the clock, but it is fragmented and incoherent, with remarkably little resemblance to the spirals encountered before. This is a striking indication of the fact, already mentioned, that characteristic c.c.a. spirals are produced by the dynamic, not the initial noise. Aided by CAM we discover that at typical sites x, all of the random times $T^k(x)$ increase rapidly with N. In particular, since the distance between critical droplets

becomes very large, and since the formation of defects seems typically to involve interaction between distinct droplets, spirals only tend to pop up after most of the debris has been displaced.

Why? —

How does the c.c.a. evolve if ζ_0 consists of a clock on a background of all one type? We encourage the reader to solve this riddle, either by hand or by computer. Again, surprisingly, waves emanating from the clock do *not* make a spiral. So what *is* the proper environment for spiral creation? In an attempt to solve this mystery we examined a few trajectories of the 18-color c.c.a. (with $L = 256$), carefully watching for the appearance of the first spiral.

In one typical instance the first spiral "popped up" at about time 800. It contained a clock ℓ consisting of 18 sites, and by time 850 had formed a perfect spiral satisfying (§) without error for a distance of about 50 bonds in all directions. We then worked *backwards* through the (deterministic) history of ζ_t, in order to discover the time

$$T^2 \;\; = \;\; \min_x T^2(x)$$

of the first defect anywhere in the 256×256 lattice. In this case we determined $T^2 = 727$. Many different defects formed at that time, since various routes across numerous wide bands of solid type were possible. As shown in Fig. 8, all these routes needed to wind through a narrow passage of debris, leading to a "pivotal" bond that completed the loops by opening at time T^2. The first defects were remarkably long, involving over 100 sites that had previously belonged to three different large droplets. At time 727 there was no real indication of a spiral. After T^2, however, shorter interior defects rapidly formed, feeding on remanants of debris, until the clock appeared. This process of consolidation, it would seem, created the proper environment for a perfect spiral.
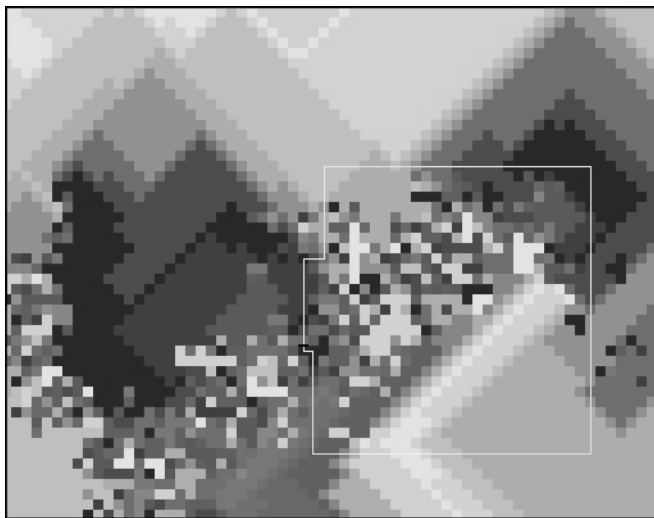

Fig. 8. **Defect formation in the 14-color c.c.a.at time 727 ($L = 256$)**

7. Additional challenging problems

Our preliminary investigation of the c.c.a. rule on $\mathbb{Z}^2$ suggests many challenging mathematical problems. In this final section we mention a few.

The early stages of c.c.a. dynamics are *metastable*. To understand their quantitative aspects one should study finite systems of size L^2, as N and L increase appropriately, along the lines of the asymptotic analysis for *bootstrap percolation* presented in [1]. However the lack of monotonicity and manifest complexity of cyclic systems make the corresponding agenda more challenging. We have therefore collected some empirical data as a first step. Here we present only a sampling of our findings; a more complete account will appear in [7].
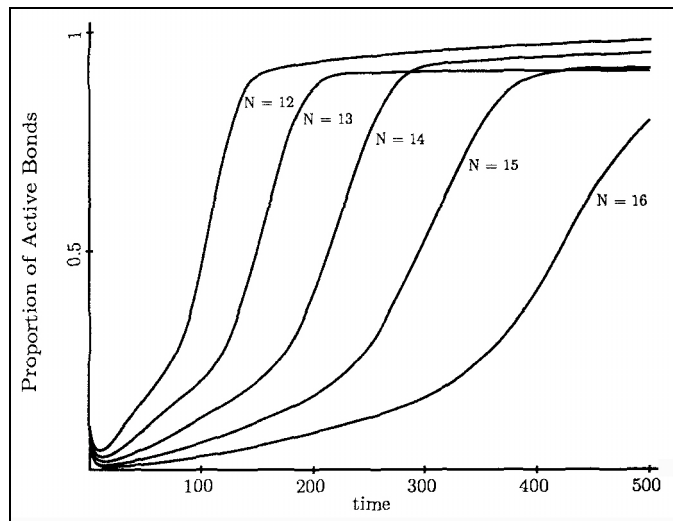

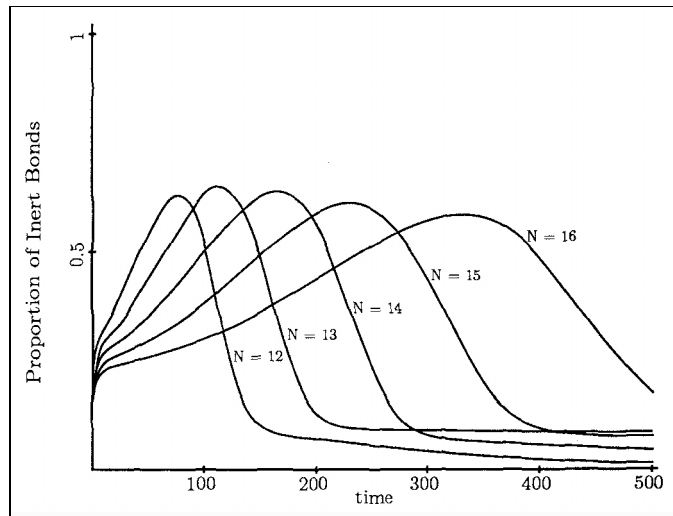*Fig. 9.* **Density of active bonds in c.c.a. ($L = 256$)**

Fig. 10. **Proportion of inert bonds in c.c.a. ($L = 256$)**

Figures 9 and 10 show the frequencies of active and inert bonds over time in the *N*-color c.c.a. on a 256×256 lattice with wrap-around, for *N* ranging from 12 to 16. Each plot tracks the average values obtained from 50 trajectories. In Figure 9, first note the rapid initial decrease. This corresponds to Stage 1, when most creatures run out of food. But droplet growth signals the onset of Stage 2 and causes the density of active bonds to increase. The density then continues increasing to a limit: roughly, the probability that adjacent sites belong to a common demon. In Figure 10, the proportion of inert bonds at first increases very rapidly, as initially active bonds form small inert clusters. There follows a longer period of slower but steady increase, signaling the presence of waves of solid type in the growing droplets. At the end of Stage 2 a maximum is reached, and then the freqency begins to dip as bonds are reactivated by defects and demons. The eventual limit is, roughly, the (very small but positive) probability that adjacent sites belong to different demons.

We note that the sum of the corresponding frequencies in Figures 9 and 10 (i.e., the density of open bonds) increases monotonically in accordance with ($\bigstar_1$). In our data the limit of the sum is very close to 1; in particular, none of our samples fixated. Evidently, the theoretical probability of fixation for $L = 256$ and $N \leq 16$ is extremely small. With *L* fixed and *N* increasing, fixation continues to be very rare until an apparent critical value $N_c(L)$ when critical droplets are too rare to appear in the *L*-system. Although we have not checked, we expect that $N_c(256)$ lies in the low 20's. For values of *N* beyond that point, the curves corresponding to Figure 6 will appear, for all practical purposes, to decrease exponentially to 0. As beautifully illustrated in [1], this "pseudo phase transition" should become more pronounced as *L* increases, but $N_c(L)$ will increase *very slowly* to $+\infty$.

At what rate should $N_c(L)$ increase with *L*? Based in part on weak analogy with bootstrap percolation, and in part on heuristic arguments, we make the following conjecture, which suggests *logarithmic* growth:

Conjecture. Let $p(N, L)$ denote the probability that the *N*-color c.c.a. on an $L \times L$ lattice (with wrap-around) fixates. Then there are constants *c* and *C* ($0 < c < C < \infty$) such that

$$\lim_{N \to \infty} p(N, L_N) = 1 \qquad \text{whenever } L_N < e^{cN},$$

$$= 0 \qquad \text{whenever } L_N > e^{CN}.$$

We suspect that the first assertion of the conjecture may follow from percolation considerations. To establish the second claim, one will need a better understanding of critical droplets and how they grow. A *localized* c.c.a. rule may prove helpful in this connection. Let ζ_0^0 be a uniform random configuration of *N* types on $\mathbb{Z}^2$ as before. Inductively define ζ_t^0 and a sequence A_t of finite subsets of $\mathbb{Z}^2$ by: $A_0 = \{0\}$ (the origin), and

$$\zeta_{t+1}^0(x) = \zeta_t^0(y) \quad \text{if} \ \ \zeta_t^0(y) - \zeta_t^0(x) = 1 \bmod N$$
$$\text{for some } y \text{ such that } \|y - x\| = 1, \text{ and } (x \in A_t \text{ or } y \in A_t),$$

(o)
$$= \zeta_t^0(x) \quad \text{otherwise};$$

$$A_{t+1} = A_t \cup \{x : \zeta_{t+1}^0(x) \ne \zeta_t^0(x)\}.$$

Roughly speaking, algorithm (o) isolates the droplet at the origin. This droplet will stop growing with overwhelming probability when N is large. But if it reaches a certain critical size, then it almost invariably survives forever, acquiring an asymptotic shape as it spreads out at a linear rate. The three large droplets in Figure 4 evolve in essentially the same manner until they collide. This localized rule is interesting in its own right, as a deterministic dynamic with behavior reminiscent of stochastic growth models.

Finally, most of our discussion also applies to corresponding dynamics in three or more dimensions. Winfree and Strogatz [14] have made a rather thorough investigation of the locally periodic formations that can arise from systems of interacting cyclic waves in three-dimensional space. We plan to investigate $\mathbb{Z}^3$ counterparts of the ideal spirals in Figures 2-3, and compare our findings with those of [14]. Fig. 11 shows a c.c.a. configuration for a rule having 12 nearest neighbors instead of 4.
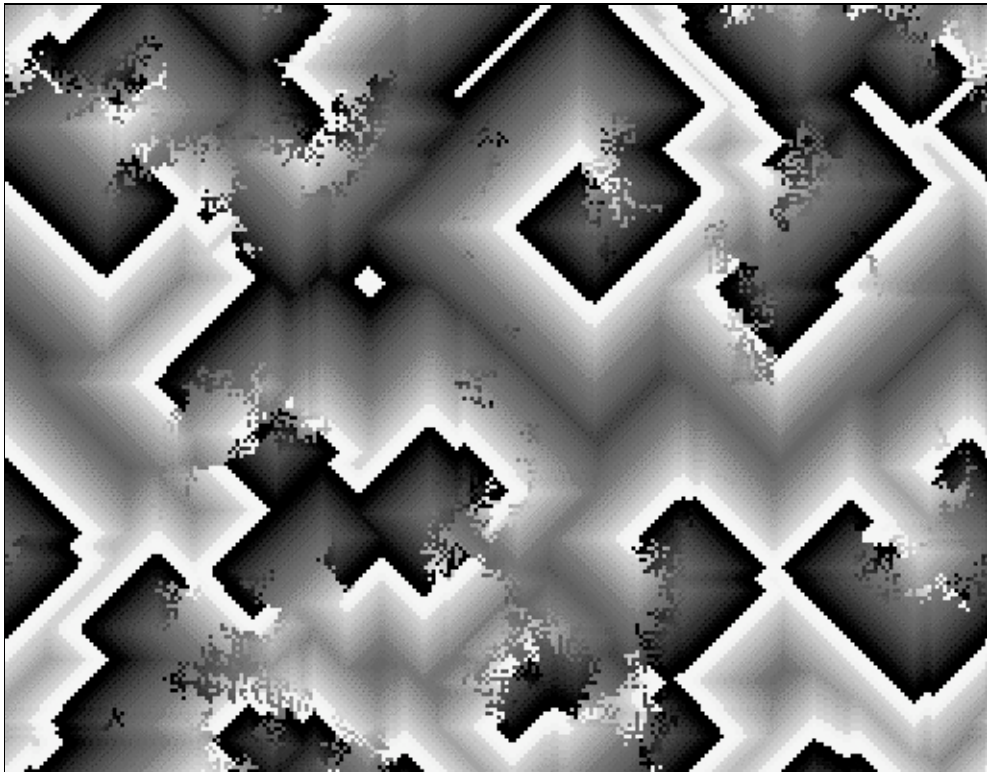

Fig. 11. A c.c.a. with 12-cell neighborhood

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