GROWTH PHENOMENA IN CELLULAR AUTOMATA

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(Preliminary version, January 1999)

Abstract. We illustrate growth phenomena in two–dimensional cellular automata (CA) by four case studies. The first CA, which we call *Obstacle Course*, describes the effect that obstacles have on such features of simple growth models as linear expansion and coherent asymptotic shape. Our next CA is random–walk based *Internal Diffusion Limited Aggregation*, which spreads sublinearly, but with a shape which can be explicitly computed due to hydrodynamic effects. Then we propose a simple scheme for characterizing CA according to their growth properties, as indicated by two *Larger than Life* examples. Finally, a very simple case of *Spatial Prisoner's Dilemma* illustrates nucleation analysis of CA.

1991 *Mathematics Subject Classification*. Primary 60K35.

Keywords: cellular automaton, growth model, nucleation, shape theory.

Acknowledgements. I wish to express my gratitude to David Griffeath and other organizers of the CA 98 workshop, as well as the Santa Fe Institute, for the opportunity to participate in this wonderful event. In addition, I thank David Griffeath and Jeremy Quastel for help during the preparation of my talk and this paper. The research presented here was partially supported by grant J1-8542-0101-97 from Slovenia's Ministry of Science and Technology and NSF grant DMS–9703923.

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1. Introduction.

In essence, analysis of growth models is an attempt to study properties of physical systems far from equilibrium (e.g., [KS], [Mea] and more than 1300 references cited in the latter). CA growth models, by virtue of their simplicity and amenability to computer experimentation ([Gri2]), have become particularly popular in the last 20 years, especially in physics research literature ([TM], [Vic]). Needless to say, precise mathematical results are hard to come by, and many basic questions remain completely open at the rigorous level. The purpose of this paper, then, is to outline some successes of the mathematical approach and to identify some fundamental difficulties.

We will mainly address three themes which can be summarized by the terms: aggregation, nucleation, and constraint–expansion transition. These themes also provide opportunities to touch on the roles of randomness, monotonicity, and linearity in CA investigations. We choose to illustrate these issues by particular CA rules, with little attempt to formulate a general theory. Simplicity is often, and rightly, touted as an important selling point of cellular automata. We have therefore tried to choose the simplest models which, while being amenable to some mathematical analysis, raise a host of intriguing unanswered questions. The next few paragraphs outline subsequent sections of this paper.

Aggregation models typically study properties of growth from a small initial seed. Arguably, the simplest dynamics are obtained by adding sites on the boundary in a uniform fashion. In fact, such examples were among the first studied. It soon became clear that they expand linearly in time and, properly rescaled, obtain a characteristic limiting shape. What if the space over which such growth spreads is not uniform, but instead contains a field of obstacles? Stationary obstacles do not complicate the analysis much, but the situation becomes murkier once one allows the obstacles to move. In fact, the literature appears to contain conflicting claims in the case of moving obstacles. Section 2 presents a detailed discussion of this class of models, including some rigorous results and conjectures.

Properties of asymptotic shape for linearly spreading growth can be notoriously hard to elucidate. By contrast, symmetric random walks progress through space more slowly (diffusing as the square root of time), and have an isotropic continuum space–time limit. For these reasons, growth models based on such walks often yield sublinear growth and circular asymptotic shape. One such example is presented in Section 3.

Section 4 is more theoretical in nature. It proposes a general classification scheme which, simply put, provides a precise way to divide CA into those which grow and those which do not. This taxonomy may be viewed as a simple alternative to Langton's approach based on the frequency of transitions to non-quiescent states (the λ parameter, see [Lan]). Especially for CA which depend on a parameter, it provides an alternative strategy to search for complex rules on the boundary between qualitatively distinct regimes. Section 5 then provides two illustrative examples from a four parameter rule space of general Life–like non–monotone rules.

If the initial state is disordered, how do droplets which generate persistent growth emerge from random "soup" and with what frequency? Nucleation analysis addresses such questions. Nucleation effects can sometimes be very tricky to discern by computer, but we will show in our final Section 6 how a mathematical analysis with an essential experimental component aids in understanding self–organization of a simple competition model.

2. Obstacle Course (OC).

Before describing our first models, let us emphasize that in this section and the next, the neighborhood of a site consists of its nearest 4 points. To define the OC CA, start by assuming that the state of every site in $\mathbf{Z}^2$ can be either 0 (empty), 1 (occupied) or 2 (an obstacle). In the simplest version of the OC rule, called *static OC*, 1's never change, a point in state 0 changes to 1 as soon as some neighbor is in state 1, and finally a site in state 2 with a neighboring 1 changes to 0 with a fixed probability $q \in [0,1]$. (In epidemics terms, 1's, 0's, and 2's could be interpreted as infected, and more and less susceptible individuals, respectively.) This rule is applied synchronously and independently at all sites in $\mathbf{Z}^2$ at every step of discrete time $t = 0, 1, 2, \dots$. As for the initial state, we assume that the origin contains the only 1, while every other site is independently 2 with probability p and 0 with probability $1-p$. Our attention will focus on the set A_t of 1's at time t.

Say that $L_{p,q}$ is the *(linear) asymptotic shape* of A_t if

$$\frac{A_t}{t} \to L_{p,q} \tag{2.1}$$

as $t \to \infty$. It is easiest to define this convergence in terms of the *Hausdorff metric*. That is, define the ϵ–fattening of a set $B \subset \mathbf{R}^2$ to be $B^\epsilon = B + B_2(0,\epsilon) = \cup_{x \in B} B_2(x,\epsilon)$. Then say that

(2.1) holds if, for any $\epsilon > 0$,

$$L_{p,q} \subset \left(\frac{A_t}{t}\right)^\epsilon \text{ and } \frac{A_t}{t} \subset L_{p,q}^\epsilon,$$

for a large enough t.

The case $p = 0$ is simple: A_t is merely the diamond $\{(x,y) \in \mathbf{Z}^2 : |x| + |y| \leq t\}$. Therefore, we can explicitly compute

$$L_{0,q} = \{(x,y) \in \mathbf{R}^2 : |x| + |y| \leq 1\}.$$

Equally clearly, $L_{1,1} = L_{0,q}/2$.

Assume now that $p > 0$ and $q \in (0,1]$. This model fits into a general class of dynamics known as *first passage percolation (FPP)*. To explain the correspondence, we assign to every site $x \in \mathbf{Z}^2$ an independent random variable ξ_x with $P(\xi_x = 1) = 1 - p$ and $P(\xi_x = k) = p(1-q)^{k-1}q$ for $k = 2, 3, \ldots$ Assuming only x is initially occupied by a 1, the time $T_{x,y}$ when y becomes occupied is given by

$$\inf\{\sum_{i=0}^{n} \xi_{x_i} : n \geq 1 \text{ and } x = x_0, x_1, \ldots, x_n = y \text{ is a nearest–neighbor path}\}.$$

If ξ_x is interpreted as the time needed for x to become 1 after it has a neighboring 1, then a short induction gives $A_t = \{x : T_{0,x} \leq t\}$. The next crucial observation is *subadditivity*: $T_{x,y} \leq T_{x,z} + T_{z,y}$. A fair amount of mathematical theory and technical machinery ([CD], [Kes]) then yields existence of a deterministic convex set $L_{p,q} \subset L_{0,q}$ with non–empty interior, such that $A_t/t \to L_{p,q}$ almost surely.

The $q = 0$ case is similar, but we need to allow for the possibility that the set A_∞, consisting of the origin and any sites with state 0 to which the origin is connected (by a nearest neighbor path), is finite. This happens a.s. if $p \geq 1 - p_c \approx 0.407$ and otherwise with probability strictly less than 1. (Here, p_c is the critical density for site percolation in the plane.) Thus $L_{p,0}$ is a random set if $p < 1 - p_c$: there exists a nontrivial deterministic convex set $L'_{p,0}$ such that $L_{p,0}$ equals $\{0\}$ on $\{|A_\infty| < \infty\}$ and $L'_{p,0}$ on $\{|A_\infty| = \infty\}$. The top left frame of Figure 1 provides an example with $p = 0.3$. Sites in A_t are gray and obstacles black in all four frames.

Thus the existence of $L_{p,q}$ is established, but what more can we say about these sets? It is possible to show that $L_{p,q} \to L_{0,q}$ as $p \to 0$, and that $L_{1,q} \to L_{1,1}$ as $q \to 1$ (using techniques from [DL]), but more detailed aspects of $L_{p,q}$ are not easy to discern. For instance, there are presently no rigorous methods available to show that $L_{p,q}$ is never a circle. More discussion on existence and properties of asymptotic shapes appears in [BoGr] and [GG3], while [GrMc] addresses a combination of bootstrap percolation ([AL]) and OC rules.

A more complex CA called *moving OC* results if we allow the obstacles to diffuse. The easiest way to achieve this effect is to view 2's as particles which move freely on 0's and are forbidden to jump onto a 1. More precisely, the state of the CA consists of sites in state 1, sites in state 0, and sites containing one or more 2–particle. The following steps are then performed in succession:

(1) Every 2-particle randomly and independently chooses a neighbor. If the chosen neighbor is not a 1 it jumps onto it. If the chosen neighbor is a 1, the jump is suppressed and the particle is killed (removed from the system) with probability q.

(2) Every 0 with a neighboring 1 becomes a 1.

As before, start with a single 1 at the origin, surrounded by sites filled independently with a random number of particles from a distribution which is the same for all sites. We will also assume that this random number is bounded. Let p stand for the initial density of 2–particles, that is, the average number of 2's per site.

We should mention that, alternatively, one could restrict the number of 2's to at most 1. In this case, 2's would perform simple exclusion outside A_t, in which case synchronous dynamics would require a scheme such as Margolus neighborhood updating ([TM]). Phenomenologically, there should be little difference between our moving OC and the exclusion OC ; this is easy to believe when p is small, while exclusion effects for $p \approx 1$ should correspond to those for $p \approx \infty$ in the version above. Furthermore, simple exclusion outside A_t ensures that 0's perform the same dynamics there as 2's, so the exclusion OC rule with $q = 0$ is a variant of *lattice–gas DLA* ([TM], [Vos], [SU]). One motivation for the agenda of this section is to take a step towards rigorous study of that still–mysterious rule. Additional details will appear in [GG8].

In what follows, we will say that a sequence A_t, $t = 0, 1, 2, \ldots$ of subsets of $\mathbf{Z}^2$ *expands linearly* if there exists an open ball $G \neq \emptyset$ and a $\rho_0 > 0$ such that, for every open ball $B \subset G$,

$$|A_t \cap (tB)| \geq \rho_0 t^2 \text{area}(B) \text{ for all large enough } t, \tag{2.2}$$

hence A_t/t covers G with density at least ρ_0. In practice, G is most often centered at the origin.

As with the static OC, let us start with the simpler case $q > 0$ and outline the proof that A_t expands linearly, almost surely. Linear expansion implies that the number of particles in A_t is on the order of the square of its diameter, eliminating the possibility that A_t is fractal (as extensive experimental evidence suggests to be the case in "ordinary" DLA, to which much of [Mea] is devoted). The top right frame of Figure 1 is a snapshot of A_t with $q = 0.1$ and an initial set consisting of three 2–particles per site. In light of this picture, (2.2) is no surprise. In fact, A_t/t seems to converge to a deterministic limit, but techniques for investigating this issue are completely lacking.

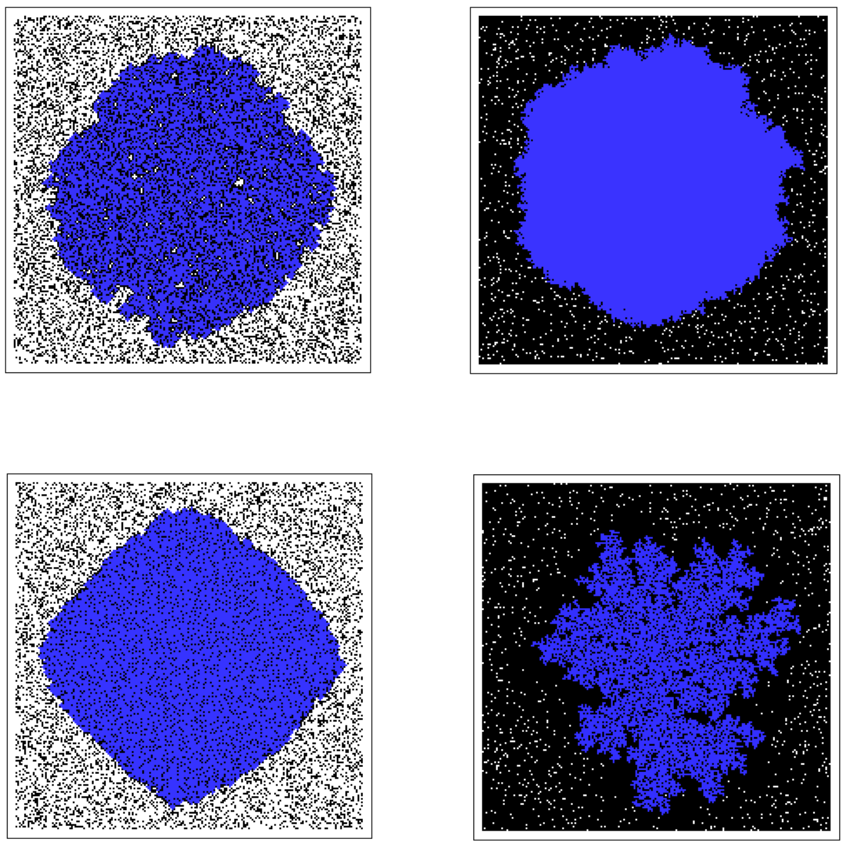

Figure 1. Growth in the static and moving OC dynamics.

To prove (2.2), attach to every 2–particle w a random variable ζ_w, which simply measures the number of times w attempts to jump onto A_t. Thus, for example, $\zeta_w = 1$ iff the particle is killed on its first attempt. It is also clear that ζ_w are independent geometric random variables with mean $1/q$. Moreover, the position of w at any time t before extinction is, in ℓ^1–distance, at most ζ_w from where w would be if it moved freely. A standard computation with random walks then shows that the density (i.e., the expected number of 2–particles) is bounded uniformly in space and time. After a substantial extra argument, this property ultimately suffices to establish (2.2). It is also worth noting that at every time t a particle w which has come in contact with A_t is either killed, in which case it contributes nothing to the total density, or else is still alive, in which case it contributes at most $2\zeta_w^2 + 2\zeta_w + 1$ to the density. Hence the density is always bounded by p if q is sufficiently close to 1.

What happens with (2.2) in the most interesting case $q = 0$? For $p \in (0, 1)$, there could conceivably be three scenarios: either linear expansion persists for all values of p, or there is no linear expansion for any p, or there is a phase transition at some value of p. (Recall that this last is the state of affairs in the case of static OC.) The empirical literature on exclusion OC concurs that the second scenario is impossible, but there appear to be conflicting claims about linear expansion for high p ([Vos], [SU]).

In fact, it seems natural to conjecture that, for very small p, the density of obstacles which remain active, in the sense that they are not captured within A_t, is uniformly bounded by p, and this property should imply linear expansion (2.2). At present, a rigorous argument still appears elusive, so we simply illustrate the result by means of the bottom left frame of Figure 1 (which has $p = 0.3$). If p is high, however, judging from computer simulations, the density of 2–particles at the boundary of A_t increases substantially above p, and the possibility that it increases without bounds cannot be eliminated. See [SU] for more discussion on this thorny issue, and the bottom right frame of Figure 1, where $p = 3$, for an illustration.

3. Internal Diffusion Limited Aggregation (IDLA).

The IDLA dynamics was first introduced in [BGL], where its basic asymptotic shape theory is developed. For a different perspective, see [MM] which contains an analysis of complexity properties on this rule. The synchronous version of IDLA we present is specified by the *occupied set* A_t and the behavior of a collection of random walks. Initially, $A_0 = \{0\}$. At every time $t = 1, 2, \ldots$, each site in A_t contains one or more particles. To get A_{t+1}, together with a new particle configuration, execute (in succession) the following three steps:

(1) One particle at each site is frozen, while the others execute one step of a symmetric nearest–neighbor random walk.

(2) A_{t+1} is obtained by adjoining to A_t all sites which are being visited by a particle for the first time.

(3) One particle is added at the origin.

To understand the behavior of this process, one approximates it by a *continuum–valued* CA on $\mathbf{Z}^2$. This CA, determined by $u_t : \mathbf{Z}^2 \to \mathbf{R}^+$, $t = 0, 1, \ldots$, is obtained by simply replacing the true particle configuration at time $t+1$ by the expected number of particles at every site. If we set $\lambda(u) = \max\{u-1, 0\}$ and

$$\Delta_d(f)(x) = \sum_{y \in \partial\{x\}} (f(y) - f(x))$$

(where $\partial\{x\}$ is the set of nearest neighbors of x), we obtain

$$\begin{aligned} u_{t+1}(x) &= \max\{u_t(x), 1\} + \frac{1}{4} \sum_{y \in \partial\{x\}} \lambda((u_t(y)) + \mathbf{1}_{\{0\}} \\ &= u_t(x) + \frac{1}{4}\Delta_d(\lambda(u))(x) + \mathbf{1}_{\{0\}}. \end{aligned} \tag{3.1}$$

To get the *diffusion scaling* limit, one defines, for $x' \in \mathbf{R}^2$ and $t' \geq 0$, $u'(x', t') = u(\epsilon^{-1}x', \epsilon^{-2}t')$, writes (3.1) in terms of the new variables t', x', u', then divides it by ϵ^2, and computes the limit as $\epsilon \to 0$ by (formal) Taylor expansion to obtain (omitting primes)

$$\frac{\partial u}{\partial t} = \frac{1}{4}\Delta\lambda(u) + \delta_0 \tag{3.2}$$

In either (3.1) or (3.2), the occupied set is given simply by $\{u \geq 1\}$.

Although not obvious at first glance, it turns out that (3.2) is equivalent (in the proper weak interpretation), to the famous *Stefan problem*, a model for ice melting in the presence of heat sources. It is also true that (3.2) has an explicit unique solution given in polar coordinates by $\lambda(u) = v(rt^{-1/2})$, where

$$\begin{aligned} &v(s) = \int_s^K \frac{2}{\pi\sigma} e^{-\sigma^2}\, d\sigma, \\ &e^{-K^2} = \pi K^2. \end{aligned}$$

Hence $A_t/\sqrt{t}$ converges to a circle with radius $K \approx 0.498$, and the proportion of non–frozen particles is $1 - \pi K^2 \approx 0.22$. Figure 2 gives a snapshot of an IDLA simulation at $t \approx 38,000$ on a 200×200 array. For many more details, and to see how the above heuristic can be turned into a proof for the very similar asynchronous version of this model, see [GQ].

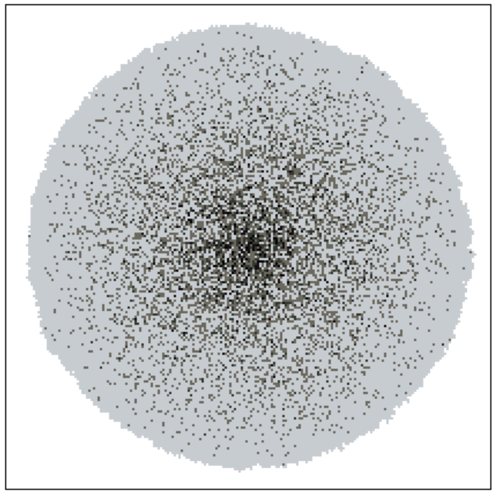

Figure 2. A snapshot of IDLA at a large time.

Assume now that one adds $c(t)$ particles at the origin at time t, where $c(t)$ is no longer 1, but an increasing function of t. An interesting question is how quickly c must increase for the shape of the occupied set to no longer be circular. Since the normal approximation for binomial probability $\binom{n}{k}2^{-n}$ holds up to $k = o(n)$, one would expect that the set A_t needs to expand linearly. Furthermore, during the time interval $[0, t]$, a random walk started at 0 will visit sites of distance order t from the origin with exponentially small probability. We therefore expect that $c(t)$ must increase exponentially fast to initiate the transition away from circular shape. More precisely, let us assume that $c(t) = e^{\gamma t}$, and write

$$L_\gamma = \lim_{t\to\infty} \frac{A_t}{t}.$$

Then we conjecture that L_γ exists almost surely, and approaches a circular shape as $\gamma \to 0$, while as $\gamma \to \infty$ it approaches the unit diamond $\{(x, y) : |x| + |y| \le 1\}$. In general, the shapes L_γ should be determined by large deviation rates; this model is therefore similar in spirit to branching random walks ([Big]).

An exponentially increasing number of particles makes verification of the above conjecture by direct simulation of the particle system prohibitively slow. On the other hand, the continuum–valued CA (3.1) simply becomes

$$u_{t+1}(x) = u_t(x) + \frac{1}{4}\Delta_d(\lambda(u))(x) + c(t)\mathbf{1}_{\{0\}}. \tag{3.3}$$

The two frames in Figure 3 show the resulting growths for $\gamma = 0.1$ and $\gamma = 5$, both stopped when they reached a radius of about 50. (Gray shading is logarithmic to make the density profile visible.) A more general discussion on applications of continuum valued CA may be found in [Ruc].

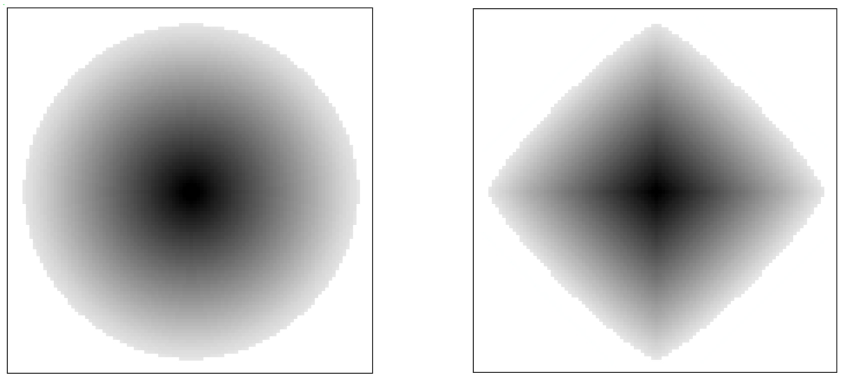

Figure 3. The continuum–valued CA (3.3) with $\gamma = 0.1$ and $\gamma = 5$.

4. Growth properties of CA: a general framework.

The setup we now introduce is essentially the same as in [GG4]. Let us start by describing the neighborhoods we most often consider. The neighborhood for the origin will be a finite set denoted by $\mathcal{N}$, its translation $x+\mathcal{N}$ then being the neighborhood of the point x. By convention, we assume that $\mathcal{N}$ contains the origin. Most typical is the range ρ Box neighborhood, in which case $\mathcal{N}$ is the $(2\rho+1)\times(2\rho+1)$ box centered at the origin, and the range ρ Diamond neighborhood, when $\mathcal{N}$ consist of points with ℓ^1–norm at most ρ. In particular, range 1 Diamond and Box neighborhoods are also known as von Neumann and Moore neighborhoods, respectively.

Let ξ_t be a general probabilistic CA, which, for simplicity, only has states 0 and 1. By this we mean first that $\xi_t : \mathbf{Z}^2 \to \{0,1\}$ describes the configuration at time t; as usual, 1's will be thought of as occupied sites, and ξ_t and the set of occupied sites $\{\xi_t = 1\}$ will be identified. Moreover, the synchronous transition rule is given by a neighborhood $\mathcal{N}$ and a set of probabilities $\pi(S) \in [0,1]$, $S \subset \mathcal{N}$, specifying that $\xi_{t+1}(x) = 1$ with probability $\pi((\xi_t - x)\cap\mathcal{N})$ independently at every time t and every spatial location x. Such CA rules are called *monotone* (or *attractive*) if $S_1 \subset S_2$ implies $\pi(S_1) \le \pi(S_2)$. Deterministic CA of course have $\pi(S)$ only 0 or 1. Finally, in *solidification* CA every set S which contains 0 has $\pi(S) = 1$.

Assume that a two–state CA fixes the all 0's state, that is, $\pi(\emptyset) = 0$. Let B_n be the $(2n +$

$1) \times (2n+1)$ box around the origin, and construct initial state ξ_0 by filling B_n with density 1/2 product measure, and B_n^c with 0's.

We will call a CA *expansive* if $P(\xi_t$ expands linearly$) \to 1$ as $n \to \infty$ (recall the definition of linear expansion from Section 2). On the other hand, we say that a CA is *constrained* if there exist positive constants c_1 and c_2 so that $P(B_{c_1 n}$ ever includes an occupied site$) \le \exp(-c_2 n)$. Finally, we classify as *equivocal* those CA which do not fit into either previous category. (Various subcategories of equivocal may also be of interest, e.g. CA in which the number of occupied sites is likely to grow without limit, those in which linear spread occurs along a subsequence of times, and so on.)

The main motivation for these definitions comes from *oriented percolation*, in which $\mathcal{N}$ is, say, the von Neumann neighborhood, and the monotone rule declares that $\pi(S) = p > 0$ as soon as $S \neq \emptyset$. Then there exists a critical probability $p_c \in (0,1)$ such that ξ_t is expansive in the supercritical regime (that is, when $p > p_c$). On the other hand, $P(\xi_t = \emptyset$ for some $t) = 1$ for every n as soon as $p \le p_c$, and the subcritical ($p < p_c$) regime leads to a constrained dynamics, while the critical ($p = p_c$) oriented percolation is equivocal ([Dur], [BeGr]).

Assume now that a CA rule is monotone and deterministic, and, for the sake of simplicity, that $\pi(S)$ does not change if S is reflected around either coordinate axis. It can then be proved that the model is expansive if and only if it enlarges every half–plane $H_u = \{x \in \mathbf{Z}^2 : \langle x, u \rangle \le 0\}$ ($u \in \mathbf{R}^2$ is an arbitrary unit vector):

$$\xi_0 = H_u \Rightarrow \xi_0 \subset \xi_1 \text{ and } \xi_0 \neq \xi_1.$$

One direction is easy: if this last condition is violated, then the CA is constrained, so in fact there are no equivocal CA in this class. (For much more on monotone CA growth, see [GG1,3,5,6].)

Outside the realm of monotonicity, there are very few available rigorous techniques ([GG3]), but it is worth mentioning a few. First, it is not hard to prove that any *linear* deterministic CA, where $\pi(S) = |S| \bmod 2$, is equivocal: it grows, but also repeatedly collapses to a set of $|\mathcal{N}| \cdot |\xi_0|$ points at exponentially spaced times. (For a discussion of the replication properties of such rules, see the July 15-21, 1996 recipe at [Gri2].) This example also illustrates how equivocal dynamics are typically fragile – by analogy with the one dimensional case ([BN]), one expects the random perturbation $\pi(S) = p \cdot (|S| \bmod 2)$ to be expansive for $p < 1$, as is strongly suggested by simulations. Figure 4 provides a range 1 Box example with $n = 20$ at time $t = 64$ with $p = 1$ and $p = 0.999$. By contrast, expansive dynamics typically seem to be robust with respect to small changes in p.

Some CA growth models can be analyzed by finding an embedded one–dimensional linear rule ([GG7]). One such case is Exactly 1 solidification with von Neumann neighborhood $\mathcal{N}$ and $\pi(S) = 1_{|S|=1}$ if $0 \notin S$. To see how this works, fix a site x at distance exactly $t+1$ from the initial seed. (This distance is measured in "light speed", in this case via the ℓ^1, metric.) The state of x at time $t+1$ is obtained by an *xor* of the states at time t of its two neighbors at distance t. While not immediate, it is possible to use this property to prove that this system is expansive, and in fact show that the final density of occupied sites is 2/3 starting from any finite initial seed.

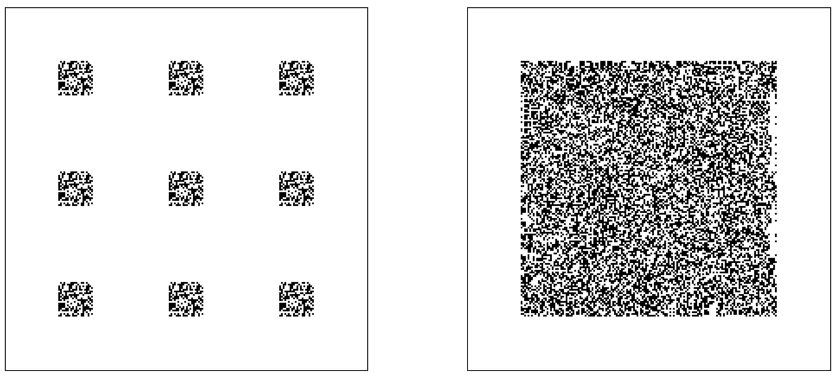

Figure 4. A linear CA and its random perturbation.

Finally, the "edge of the light cone" technique described above, in conjunction with analysis of one–dimensional random CA from [BN], shows that, for example, random Exactly 1 solidification, where $\pi(S) = p \cdot 1_{|S|=1}$ if $0 \neq S$, is expansive when $p < 1$ and $\mathcal{N}$ is either the von Neumann or Moore neighborhood.

For the vast majority of CA, however, one must resort to computer simulation to get an indication of their growth properties. Deterministic Moore neighborhood Exactly 1 solidification, for instance, is evidently expansive, although no rigorous argument for this is known at present.

In any case, one must be very cautious about conclusions from simulation, since it is always possible that a very large n (corresponding to a very large initial random set) is necessary before limiting behavior as $n \to \infty$ kicks in. For example, it is easy to convince oneself that Life is equivocal, as gliders are likely to emerge from initial soup, while the rest apparently settles into a periodic state. In fact, it seems more likely that Life is expansive, since a large box will likely contain space–filling structures on its boundary. But none of these structures is known to have appropriate self–defense properties against destabilizing influences from outside. Despite the spectacular advances in understanding the mechanisms of Life's growth, as described elsewhere in this volume, prospects for proving its expansiveness still seem quite remote.

Problems with simulation notwithstanding, one can use the computer to look for interesting CA rules on the border between "metastably" expansive and constrained cellular automata. If a CA depends on a parameter, and is apparently expansive for one value but constrained for another, then the values near the transition offer prospects not only for equivocal dynamics, but also for signature properties of complex dynamics. For example, one dimensional objects such as *gliders*, *bugs*, and *ladders* (see next section for the meaning of last two terms) typically cannot persist in "robust" expansive rules, as they "explode" into growth in all directions. (See [Boh] and [BoGr] for some rigorous results in this direction.)

5. Larger than Life (LtL).

This rule was introduced in [Gri1] and studied in [Eva]. Assume that the neighborhood $\mathcal{N}$ is a range ρ box. The deterministic LtL rule is given by

$$\pi(S) = 1_{0\notin S, \beta_1 \le |S| \le \beta_2} + 1_{0\in S, \delta_1 \le |S| \le \delta_2}.$$

In words, birth of a 1 occurs at a site if the number of occupied neighbors is between β_1 and β_2, while for survival of a 1 this number must be between δ_1 and δ_2. For example, Life is given by $\rho = 1$ and $(\beta_1, \beta_2, \delta_1, \delta_2) = (3, 3, 3, 4)$.

Of particular interest is the *threshold–range* regime, when ρ is large and $(\beta_1, \beta_2, \delta_1, \delta_2) = \rho^2 \cdot (\tilde\beta_1, \tilde\beta_2, \tilde\delta_1, \tilde\delta_2)$. Two reasons for significance of this regime are outlined below.

Assume that space is scaled by $1/\rho$. As $\rho \to \infty$, the LtL rule converges to an analogous Euclidean rule in which cardinalities are replaced by areas. This leads to limiting geometry of various objects of interest, such as *bugs*, which are finite sets with the property that the dynamics exactly translates them in finitely many steps – large–range versions of gliders, in short. Moreover, the boundary between expansive and constrained dynamics appears to form a three–dimensional subset of the four–dimensional parameter space $\{\Sigma = (\tilde\beta_1, \tilde\beta_2, \tilde\delta_1, \tilde\delta_2)\} \subset [0, 4]^4$. In other words, if a one parameter subfamily $\Sigma_{\tilde\alpha}$ experiences a transition between expansive and constrained dynamics, then it is very likely (unless $\Sigma_{\tilde\alpha}$ happens to move on the critical surface) to experience a *sharp* transition: for some $\tilde\alpha_c$, the dynamics is expansive if $\tilde\alpha < \tilde\alpha_c$ and ρ large enough, and constrained if $\tilde\alpha > \tilde\alpha_c$ and ρ large enough.

The study of phase transition in deterministic CA is generally hampered by the fact that the rule space is inherently discrete. But in models such as LtL with intrinsic threshold–range scaling, there is a natural way to introduce continuously varying rules. Excitable media modeling provides other examples ([FGG], [DG]).

An example when sharp transition can be proved is monotone LtL, that is, for $\beta_2 = \delta_2 = (2\rho+1)^2$ and $\delta_1 - 1 \le \beta_1$. These CA are often referred to as monotone *Biased Voter Automata (BVA)*. In the threshold–range regime, expansive BVA dynamics is characterized by $\tilde\beta_1 < 2$, and constrained by $\tilde\beta_1 > 2$ ([GG1]).

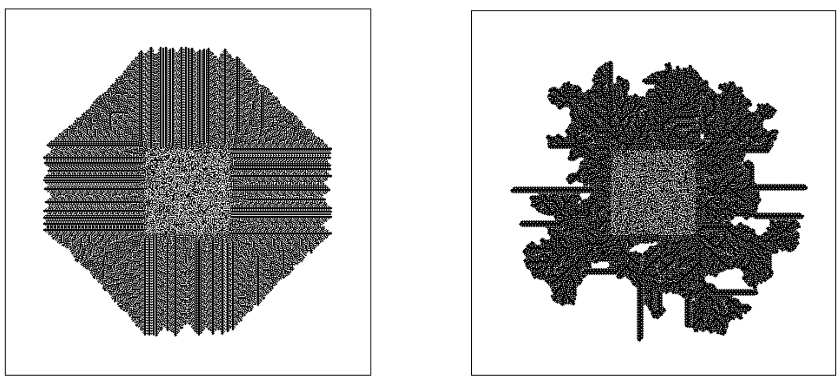

Figure 5. Exactly θ CA with $\theta = 2$ and $\theta = 3$.

Typically, one does not need to go all the way to the limit $\rho = \infty$ to experience interesting phenomena near critical points ([DG], [Eva]). Fairly small neighborhoods may already contain some ingredient of critical behavior. For example, a variety of interesting scaling phenomena occur in monotone BVA rules with $\delta_1 = 0$ near $\tilde\beta_1 = 2$ ([GG5]). Two non–monotone examples are given below. In the simulations of Figures 5 and 6 we have chosen $n = 100$, on a 400×400 array.

The first example is range 1 Exactly θ solidification: $\rho = 1$ and $\pi(S) = 1_{|S|=\theta}$ if $0 \notin S$. Note that this rule is obviously constrained for $\theta = 4$; in fact nothing outside the initial box B_n gets occupied. On the other hand, overwhelming experimental evidence (e.g., see Figure 5) suggests that the $\theta = 2$ rule is expansive. The $\theta = 3$ case, called *Life without Death*, also seems supercritical, but close enough to critical for existence of *ladders*. These objects, which grow but are restricted to a strip, actually appear quite frequently within the chaotic spread. Papers

[GrMo] and [GG3] analyze this rule in some detail, proving its P–completeness ([GrMo]), and showing that its growth is sensitive to small perturbations of the initial seed ([GG3]).

Our second example is from [Eva]. Consider range 3 LtL with parameters $(14, 19, 14, \delta_2)$. This rule seems constrained for $\delta_2 \leq 22$ (the periodic state in Figure 6 was achieved by $t = 40$) and expansive for $\delta_2 \geq 24$ (although the growth at $\delta_2 = 24$ is slow – the state in Figure 6 was achieved at $t = 300$). The intermediate case $\delta_2 = 23$ is not so easy to decipher; in fact it gives rise to a rich menagerie of bugs, and is otherwise remarkably similar to Life. (Note also that the proportions of neighborhood size, $(14, 19, 14, 23)/49$ and $(3, 3, 3, 4)/9$, are not too far apart, suggesting an interesting threshold–range critical point nearby.) The snapshot in Figure 6 was taken at $t = 1000$, by which time the dynamics has neither settled into a periodic state, nor conquered much space.

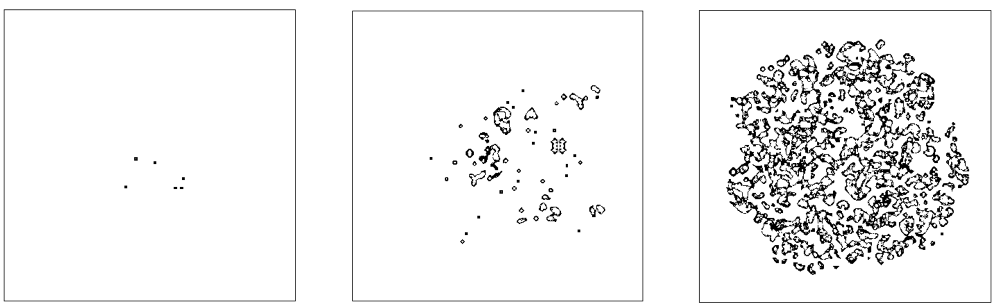

Figure 6. LtL with $\delta_2 = 22, 23$, and 24.

In light of this last example, it is an interesting open question whether one might devise a general scheme to design gliders guns and other fundamental building blocks of universal computation for large–range LtL in some parameter regime.

6. Spatial Prisoner's Dilemma (SPD).

Prisoner's Dilemma is a game in which either player chooses strategy 1 (cooperation) or 0 (defection), and the player with strategy i receives payoff a_{ij} when playing against a player who chooses strategy j.

The basic assumption $a_{01} > a_{11} > a_{00} > a_{10}$ leads to a well–known paradox: the defection strategy is clearly the optimal choice for either player, but making this choice leaves them with lower payoffs than mutual cooperation. This has lead to a large number of papers investigating strategies in tournaments with repeated rounds between players (see [Grim] and other papers in the same volume of BioSystems); in this case, one usually makes the additional assumption that $a_{10} + a_{01} < 2a_{11}$ to make the cooperating strategy better than out of phase flip–flopping by two players.

As an alternative approach, Nowak and May ([MN], [BMN]) investigated self–organizing properties of the *spatial* version of the game above. A later paper with a point of view somewhat closer to ours is [LN]. A version of the Nowak–May SPD rule is as follows. Start with a configuration of 0's and 1's on $\mathbf{Z}^2$. The player at each site x plays against every player in its neighborhood $x + \mathcal{N}$ (excluding itself). After all payoffs are computed, the player at each site x changes its state to the state associated with with the largest total payoff in $x + \mathcal{N}$. Moreover, with *mutation* probability p the player adopts a random state. Under indefinite iteration of this rule, let $\eta_t : \mathbf{Z}^2 \to \{0, 1\}$ be the state of the system at time t.

Without loss of generality, we will assume that $a_{10} = 0$ and $a_{00} = 1$. Therefore, the SPD parameters are $\mathcal{N}$, p, and the two remaining payoffs.

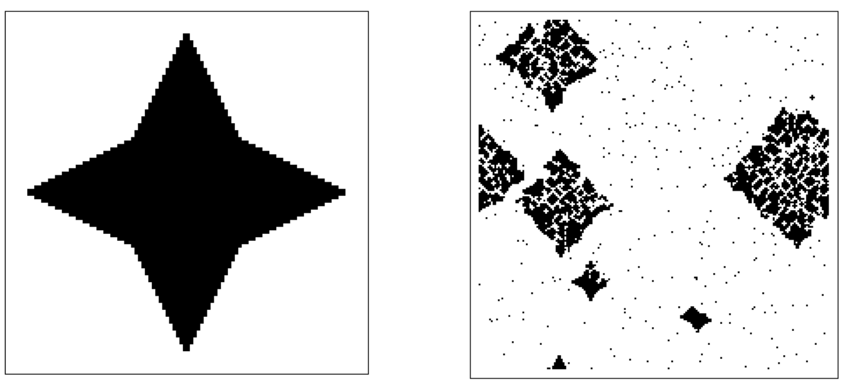

Figure 7. SPD with $p = 0$ and with $p = 0.01$.

Clearly, the most interesting issue is how regions of substantial cooperation may emerge from a sea of defectors in this rule. Let us start with the observation that if $p = 0$, then all 0's is a fixed state, and then investigate what happens under a small p perturbation.

To reiterate, we will assume $\eta_0 \equiv 0$ from now on. Also, for simplicity, let us assume, unless specified otherwise, that $\mathcal{N}$ is the von Neumann neighborhood. This makes SPD a CA with range 2 Diamond neighborhood, although writing out the associated π by hand would take some time.

If $a_{01} + 3 > 3a_{11}$, then every 1 with a neighboring 0 changes into 0 with probability $1 - p$, so the set of 0's compares favorably to supercritical oriented percolation CA (see Section 4). Thus

$$\lim_{p\to 0} \limsup_t P(\eta_t(0) = 1) = 0, \tag{6.1}$$

and there is a very low level of cooperation.

In fact, simulation suggests that (6.1) persists when $a_{01} + 3 > 2a_{11}$, but the situation changes when $a_{01} + 3 < 2a_{11}$. In the language of Section 4, when $p = 0$ the SPD is constrained in the former case, and expansive in the latter case. According to the general (though unproved) principle that expansiveness is robust under small random permutations, we conjecture that cooperation emerges when $a_{01} + 3 < 2a_{11}$, to the extent that

$$\rho_e = \lim_{p\to 0} \liminf_t P(\eta_t(0) = 1) > 0. \tag{6.2}$$

The right frame in Figure 7 gives a snapshot of these dynamics at $p = 0.01$ at the time substantial cooperation (black) has started to emerge. In our simulations we have chosen $a_{01} = 4.2$ and $a_{11} = 4$.

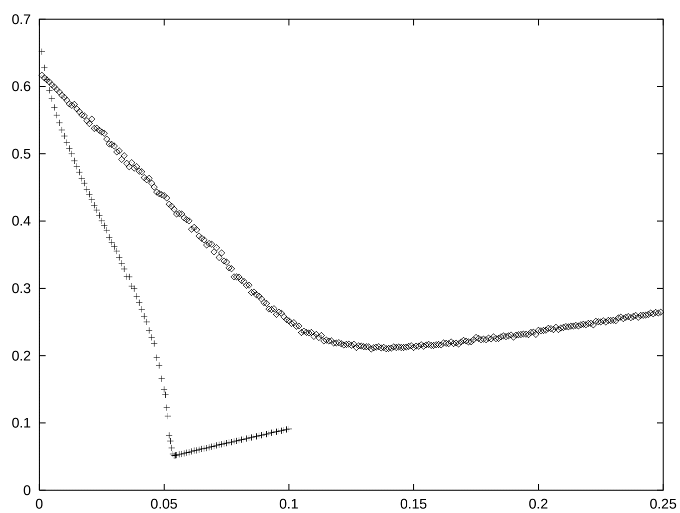

Figure 8. Equilibrium density vs. p in two SPD models.

As an interesting aside, Figure 8 provides a plot of the equilibrium density $\lim_{t\to\infty} P(\eta_t(0) = 1)$ (assuming it exists) versus p in our case of SPD (diamonds) and the range 1 Box case with $a_{01} = 10$ and $a_{11} = 7.5$ (pluses). To estimate the densities, we made every initial state contain a 30×30 square of 1's (surrounded by 0's) to speed up emergence of cooperation. Simulations were run on squares of various sizes and up to various times, depending on the speed of convergence. As a result, we estimate $\rho_e \approx 0.62$. Moreover, as in many artificial life models (and also, presumably, in the real–life counterparts), a high level of mutation makes coherent self–organization impossible, so beyond a critical $p \approx 0.14$ the equilibrium density is driven purely by noise. Note also that the Box neighborhood example suggests a second–order phase transition exactly at the minimal density, which is rather mysterious and merits further study.

In view of (6.2), it is natural to ask how long it takes for the cooperating region to reach a typical point if p is small. This is the statistic which measures *nucleation* of the SPD. To be more precise, call T the first time the box $B_{p^{-1/3}}$ (of $(2p^{-1/3}+1)^2$ sites) around the origin contains $2\rho_e p^{-2/3}$ 1's. The choice of $p^{-1/3}$ reflects the trivial lower bound on the order of T obtained by assuming that a single 1 generates growth which spreads with the speed of light (cf. the derivation of (6.3) to see how this would yield $p^{-1/3}$).

Nucleation questions are connected to studying the smallest seeds which grow, as those are likely to be the first which affect the origin. Again, assume for a moment that $p = 0$. Then a single 1 dies, and so does any pair of 1's. Three 1's may form blinkers:

$$\begin{array}{cccccccccccc} & & & 1 & & & 1 & & & & & \\ 1 & 1 & \longleftrightarrow & 1 & 1 & \qquad\text{or}\qquad & 1 & \longleftrightarrow & 1 & 1 & 1 & . \\ & 1 & & & & & 1 & & & & & \end{array}$$

A 2×2 square of 1's expands linearly (the left frame in Figure 7 is a snapshot of growth from this initial state), and this property apparently persists for small enough $p > 0$. (In fact, positive p seem to make the resulting shape convex.)

To estimate the order of T for small p, we now need to make a few estimates. In the discussion of the next paragraph, all times and probabilities are to be interpreted within the order of the quantity given.

Imagine space–time as embedded in $\mathbf{R}^3$, with the xy plane containing $\mathbf{Z}^2$ and time being the positive z–axis. Form a cone $\mathcal{C}_t$ with apex at $(0,0,t)$, height t, and circular base of radius t. Then cooperation is likely to reach the origin at time t if $\mathcal{C}_t$ contains at least one 2×2 square of 1's. At any fixed time, such a square appears by itself with probability p^4. However, the first type of blinker above appears with a much higher probability p^3 and creates a square by time $1/p$ with a probability which is bounded below (by 0.2, say). Since there are t^3 sites in $\mathcal{C}_t$, T

must satisfy

$$T^3 \cdot p^3 \approx 1. \tag{6.3}$$

We can therefore, with some confidence, conjecture that T is on the order of p^{-1} as $p \to 0$.

Quite formidable obstacles would need to be overcome before our last conjecture could be proved. In fact, there are no rigorous results whatsoever on nucleation in non–monotone dynamics. On the other hand, quite a lot is known about this aspect of monotone CA ([AL], [DS], [GG1], [GG2], [GG5], [Sch2]). As one illustration, consider the BVA (introduced in Section 5) and assume for simplicity that randomness is confined to initial states with density p of 1's. In such cases, a rather general nucleation theory is possible, leading sometimes to power laws in p, and sometimes to exponential metastability. To explain the latter, we consider a specific example with range 2 Box neighborhood and $\beta_1 = 11$, $\delta_1 = 3$. In this case, T can be simply the first time the origin becomes occupied, although the definition given above also works. Then, it can be proved ([Sch1],[GG6]) that every site eventually becomes permanently occupied. Thus the limiting equilibrium density ρ_e of (6.2) equals 1. On the other hand, there exist constants C_1 and C_2 such that

$$P(C_1 p^{-3} \leq \log T \leq C_2 p^{-3}) \to 1$$

as $p \to 0$, so it takes a long while to be occupied when p is small. Simulations with very low p are therefore not feasible. Nevertheless, Figure 9 depicts a 400×400 system with $p = 0.13$ at an intermediate time $t = 117$; occupied sites at different times are periodically shaded to give a basic impression of nucleation and growth in this CA.

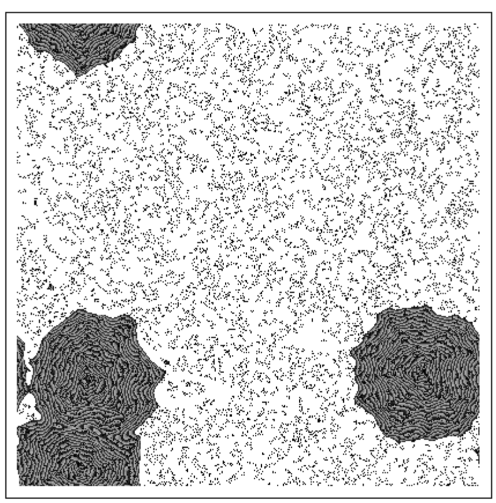

Figure 9. Nucleation in BVA.

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