SCALING LAWS FOR A CLASS OF CRITICAL CELLULAR AUTOMATON GROWTH RULES

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Abstract. Assume that a cellular automaton (CA) rule enlarges subsets of $\mathbf{Z}^2$, and does so in such a way that a larger set results in a larger outcome. Such models are called *monotone solidification CA*. In the *critical* case, these dynamics cannot cover the lattice starting from any finite set, but are able to do so from any set with finite complement. We assume that the initial set is a product measure with small density p, and address various scaling laws for the first passage time to the origin, emergence of shapes, and the ability of the dynamics to overcome pollution of space.

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Janko Gravner, David Griffeath

1. Introduction.

A natural and fairly general class of cellular automata (CA) growth rules consists of for which an occupied site always stays occupied, and any addition of occupied cells can only enhance further growth. It turns out that the former condition is not essential for the results presented here, but it does eliminate a number of complications. However, absence of the latter property fundamentally changes the game, adding major obstacles to the development of rigorous theory ([GG5]). Furthermore, the literature on spatial growth is largely limited to two–dimensional systems, since necessary geometric techniques are much harder to develop in higher dimensions, as are also illuminating simulations and statistical analyses ([Adl], [Sch2]).

Accordingly, we assume that growth is given by a transformation $\mathcal{T} : 2^{\mathbf{Z}^2} \to 2^{\mathbf{Z}^2}$ which satisfies the assumptions (A1–5) below. (A sixth axiom will be introduced at the end of Section 2).

(A1) $\mathcal{T}$ is *translation–invariant*: $\mathcal{T}(x + A) = x + \mathcal{T}(A)$.

(A2) $\mathcal{T}$ is *local*: there exists a finite *neighborhood* $\mathcal{N} \subset \mathbf{Z}^2$, i.e., a set for which $x \in \mathcal{T}(A)$ iff $x \in \mathcal{T}(A \cap (x + \mathcal{N}))$. To avoid trivialities, we assume that $\mathcal{N}$ is *minimal* (no proper subset is a neighborhood) and that it is not included in a line.

(A3) $\mathcal{T}$ is *monotone* (or *attractive*): $A \subset B$ implies $\mathcal{T}(A) \subset \mathcal{T}(B)$.

(A4) $\mathcal{T}$ *solidifies*: $A \subset \mathcal{T}(A)$.

Dynamics with inherent drift, such as various oriented models, are often of interest ([Mou1], [Sch3]), but seem to elude a general theory along the lines presented in this paper. We eliminate them by imposing a further restriction:

(A5) $\mathcal{T}$ is *symmetric*: $\mathcal{T}(-A) = -\mathcal{T}(A)$.

Given a (deterministic or random) initial set $A_0 \subset \mathbf{Z}^2$, $\mathcal{T}^n(A_0), n = 0, 1, \ldots$ is interpreted as the set of occupied points at time n, while $\mathcal{T}^\infty(A_0) = \cup_n \mathcal{T}^n(A_0)$ is the set of eventually occupied points. Sometimes, we represent occupied and unoccupied sites as 1's and 0's, respectively.

We say the dynamics are

- *supercritical* if $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$ for a *finite* initial set A_0,
- *subcritical* if $\mathcal{T}^\infty(A_0) \neq \mathbf{Z}^2$ for an initial set A_0 with finite *complement*,
- *critical* if they are neither supercritical nor subcritical, and
- *convex confined* if $\mathcal{T}^\infty(A_0)$ is finite for every finite A_0.

Our main objective in this paper is to analyze critical dynamics starting from the random set $A_0 = \Pi(p)$, to which each lattice site belongs independently with probability p. Various scaling laws for small p shed light on the size, dynamic nucleation, and propagation of viable growing droplets ([Adl], [AL], [GG2–4], [Mou1–2], [Sch1–3]), and so are illuminating prototypes for such phenomena in crystal growth and adsorption processes. Much of the rigorous theory reviewed here provides a rigorous foundation for the following insights from an early paper by G. Vichniac ([Vic]):

> "[In] convex confined [critical] rules [...], local high–density fluctuations in the initial distribution initiate the growth of clusters of 1's. The growth consists in filling concavities, and halts once the convex shapes are reached. For small p, the clusters stop growing before they can meet. They remain separated by a sea of 0's. To be sure, on an infinite lattice, this 0–phase is metastable: an exceptional fluctuation can create a very large cluster that will grow forever, feeding on isolated small clusters."

Let us now introduce several concrete, illustrative examples. Most fit within the context of *threshold growth* (TG) dynamics ([GG1–5]). Such CA rules have two parameters: a finite *neighborhood* $\mathcal{N} \subset \mathbf{Z}^2$ (with $0 \in \mathcal{N}$), and a positive integer *threshold* θ; the growth transformation $\mathcal{T}$ is then given by

$$\mathcal{T}(A) = \mathcal{T}_{TG}(\mathcal{N}, \theta)(A) = A \cup \{x : |A \cap (x + \mathcal{N})| \geq \theta\}. \tag{1.1}$$

With $\theta = 1$ this is an instance of *additive* dynamics, which are always supercritical (by virtue of (A2)). On the other hand, it is instructive to check that the following four examples constitute critical TG dynamics, and that Example 2 is the only one which is not convex confined.

Example 1.

$$\mathcal{N} = \begin{array}{ccc} & \bullet & \\ \bullet & 0 & \bullet \\ & \bullet & \end{array}, \qquad \theta = 2.$$

Example 2.

$$\mathcal{N} = \begin{array}{ccccc} & & \bullet & & \\ \bullet & \bullet & 0 & \bullet & \bullet \\ & & \bullet & & \end{array}, \qquad \theta = 2.$$

Example 3.

$$\mathcal{N} = \begin{array}{ccccc} & & \bullet & & \\ \bullet & \bullet & 0 & \bullet & \bullet \\ & & \bullet & & \end{array}, \qquad \theta = 3.$$

Example 4.

$$\mathcal{N} = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet \end{array}, \qquad \theta = 4.$$

Example 1, probably the best known instance of critical dynamics, has its own name: *bootstrap percolation* ([Adl], [AL]). A fifth example of critical dynamics, from [TM], is not an instance of TG.

Example 5. In the *Soil Erosion* CA, $\mathcal{T}(A)$ comprises sites in A along with those x which see one of the following local configurations (0's represent soil, 1's eroded areas):

$$\begin{array}{cc} 1 & \\ 1 & x \\ 1 & \end{array} \qquad \begin{array}{ccc} & x & \\ & & \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{cc} & 1 \\ x & 1 \\ & 1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ & x & \\ & & \end{array}$$

We illustrate Examples 2-5 with snapshots of 200×200 systems with periodic boundary conditions. In each case, the initial density p is chosen close to critical, i.e., so that only a few growing clusters emerge, and the dynamics are shown after these clusters have reached a substantial size. Sites added at successive iterates are shaded with a periodic palette, to provide a glimpse of the temporal evolution. Evidently sites of the same shade align with characteristic slopes (± 1 in Examples 2 and 3, $\pm 2^{\pm 1}$ in Example 4, 0 and ∞ in Example 5). Can the reader explain this? A nice illustration of Example 1 is the front page graphic of the *Primordial Soup Kitchen* [Gri]. *WinCA*, a Windows-based program for CA simulation, written by R. Fisch and D. Griffeath, is also available at that Web site.

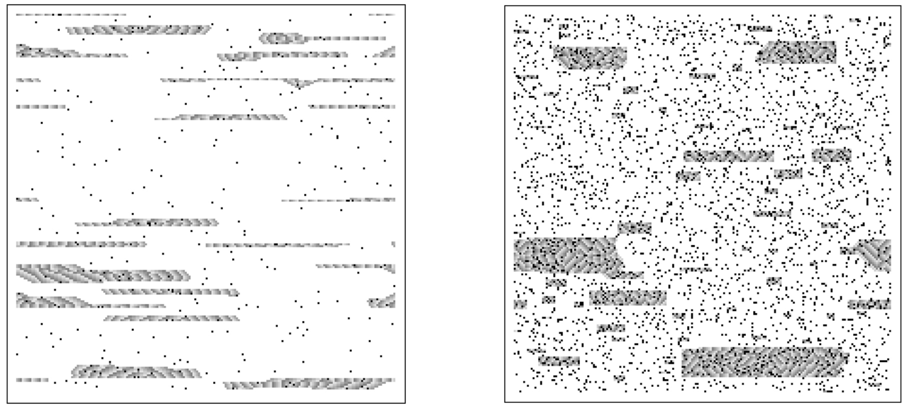

Figure 1. Example 2 with $p = 0.008$, and Example 3 with $p = 0.08$.

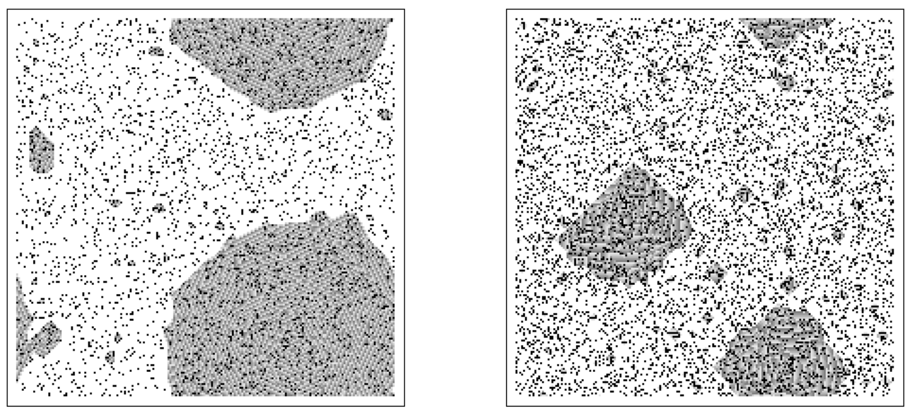

Figure 2. Example 4 with $p = 0.08$, and Example 5 with $p = 0.17$.

Most past and current research on critical dynamics is focused on the four fundamental aspects introduced in the remainder of this Introduction. We will formulate several rigorous results in the remainder of the paper, and give rough sketches of some of the proofs, deferring further details to [GG7]. The emerging message is that this class of CA rules gives rise to a variety of scaling laws in various regimes, most of which can be described in terms of a few combinatorial quantities called *nucleation parameters*.

The natural statistic to quantify the metastable behavior described above by Vichniac is the *first passage time* to the origin:

$$T = \inf\{n : 0 \in \mathcal{T}^n(\Pi(p))\}.$$

For any $p > 0$, the infinite lattice contains every possible finite configuration of occupied sites, so $P(T < \infty) = 1$ if $\mathcal{T}$ is supercritical. Equally clearly, $P(T < \infty) < 1$ in any subcritical case. Which of these two alternatives holds for critical dynamics? In fact, every site is eventually occupied with probability 1 ([GG2]). (This is one property which fails in the absence of (A5); see [Sch3] and [GG2] for examples.) Once it is known that T is finite, our next order of business is to address the questions posed in Problems 1–4 below.

Problem 1. *What is the asymptotic behavior of T as $p \to 0$?*

The magnitude of T for small p is closely related to the size of the smallest L for which the dynamics on an $L \times L$ box (with either free or periodic boundary) eventually reaches total occupancy with substantial probability ([AL]).

As follows from results in Section 3 below, if p is small and the dynamics are convex confined, then nucleation positions of viable growing clusters are widely separated, so that these clusters continue growing for a long time in a sparse helpful environment (which is, in fact, a slightly perturbed product measure).

Problem 2. *Does the appropriately rescaled shape of a typical growing cluster approach a deterministic limit set as $p \to 0$?*

For example, Figure 2 vaguely suggests that said deterministic set might be an octagon in Example 4 and a diamond in Example 5. While the latter shape can be rigorously established, we are currently unable to rule out the possibility that the former shape is a square.

Our next topic deals with the final stage of the dynamics, long after the expanding clusters from Problem 2 have collided, when the last, rare remaining holes are being filled. The ultimate, and very ambitious, goal would be to describe the shape of these holes, a problem currently solved only for additive models ([GG6]). To be precise, let $\mathcal{C}_n(0)$ denote the connected cluster, in the usual site percolation sense, of unoccupied sites containing the origin. Thus, $\{\mathcal{C}_n(0) = \emptyset\} = \{T \le n\}$. Measure the distance between compact subsets of $\mathbf{R}^2$ in the Hausdorff metric, and use the associated Prohorov metric for distance between random compact sets. For a fixed $\delta > 0$, let $\mu_{p,\delta,n}$ be the measure induced on sets $\mathcal{C}_{T-\delta n}(0)/\delta n$ under the conditional measure $P(\,\cdot\,|T > n)$. We call a random set H_p, with associated distribution μ_{H_p}, the *last hole* if

$$\lim_{\delta\to 0} \limsup_{n\to\infty} d(\mu_{p,\delta,n}, \mu_{H_p}) = 0.$$

For additive models, H_p is a deterministic set, which turns out to equal the forward shape L (see Section 4) for every p. It all likelihood, H_p is truly random in most other cases. For instance, we conjecture that for Example 1, H_p equals, or at least converges as $p \to 0$ to, $[-1,1] \times \{0\}$ and $\{0\} \times [-1,1]$, each with probability 1/2.

Since the last holes story involves conditioning on $\{T > n\}$, a fairly detailed understanding of this event seems necessary; our next problem identifies a first step in that direction.

Problem 3. *What is the asymptotic behavior of $P(T > n)$ as $n \to \infty$, when p is small?*

This problem was studied for some special cases in [Sch3], [And], and [AMS]. The last paper contains a surprisingly precise result for a fixed p in the case of *modified bootstrap percolation*, in which $\mathcal{T}(A) = \mathcal{T}_{TG}(\mathcal{N}_1, 1)(A) \cap \mathcal{T}_{TG}(\mathcal{N}_2, 1)(A)$, where $\mathcal{N}_1 = \{(0,0), (\pm 1, 0)\}$ and $\mathcal{N}_2 = \{(0,0), (0, \pm 1)\}$.

Our final topic concerns the ability of critical dynamics to overcome a polluted environment. More precisely, assume that sites are independently *removed* from $\mathbf{Z}^2$ with a small probability $q > 0$, occupied with probability p, and vacant with probability $1-p-q$. Then run the dynamics as before on the non–removed sites, with free boundary conditions.

Problem 4. *What relationship between p and q ensures that $P(T < \infty) \to 1$ as $p \to 0$ (so pollution "does not matter")? When is the pollution significant to the extent that $P(T < \infty) \to 0$ as $p \to 0$?*

In the context of Example 5, Problem 4 asks for the minimal density of forested patches which provides sufficient protection to prevent substantial erosion.

In conclusion, let us briefly mention two areas of growth dynamics which remain largely unexplored. The first concerns random perturbations of the update rule: while shape theory from finite sets can be developed for supercritical rules in some generality ([BG]), basic problems such as existence of interface speeds remain open even in simplest cases ([GG5]). Secondly, there are many interesting questions, but very few answers, concerning multi–color critical models such as the cyclic CA ([FGG]), or various voter models ([Gri]).

2. Half–spaces, nucleation parameters, and voracity.

Unfortunately, our results require quite a lot of notation, and many definitions, to which this section is devoted. The reader may want to skim it for now, and refer back later as necessary.

We start by defining a transformation $\bar{\mathcal{T}}$ on subsets of $\mathbf{R}^2$ which is conjugate to $\mathcal{T}$:

$$\bar{\mathcal{T}}(B) \cap \mathbf{Z}^2 = \mathcal{T}(B \cap \mathbf{Z}^2),$$

for every $B \subset \mathbf{R}^2$. This mapping decides whether to occupy $x \in \mathbf{R}^2$ by basing the lattice at x, intersecting the lattice with B, and then applying the discrete rule:

$$\bar{\mathcal{T}}(B) = \{x \in \mathbf{R}^2 : 0 \in \mathcal{T}(((x + \mathbf{Z}^2) \cap B) - x)\}.$$

One of the main reasons for introducing $\bar{\mathcal{T}}$ is that it translates half–spaces. For a unit vector $u \in S^1$, let $H_u^- = \{x \in \mathbf{R}^2 : \langle x, u\rangle \leq 0\}$. By (A1), there exists a $w(u) \geq 0$ such that

$$\bar{\mathcal{T}}(H_u^-) = H_u^- + w(u)u.$$

Note that $w(u) = w(-u)$, by virtue of (A5).

Next, we state a basic characterization result, which can be proved by approximating bounded sets of small curvature with half spaces (see [GG1] and Section 2 of [GG2]).

Proposition 2.1. *Growth dynamics are supercritical iff $w(u) > 0$ for every u, and subcritical iff $w(u) = 0$ for every u. Moreover, the dynamics are convex confined iff $w(u_1) = w(u_2) = 0$ for two linearly independent u_1 and u_2.*

To apply the above proposition in the threshold growth case, define $\iota_1 = \iota_1(\mathcal{N})$ and $\iota_2 = \iota_2(\mathcal{N})$ to be, respectively, the size of the largest and the second largest intersection of $\mathcal{N}$ with a line $\ell(u) = \{x \in \mathbf{R}^2 : \langle x, u\rangle = 0\}$:

$$\begin{aligned}\iota(u) &= |\mathcal{N} \cap \ell(u)|,\\ \iota_1 &= \iota(u_1) = \max\{\iota(u) : u \in S^1\},\\ \iota_2 &= \max\{\iota(u) : u \neq \pm u_1\}.\end{aligned}$$

Then $w(u) = 0$ iff $\theta > |\text{interior}(H_u^-) \cap \mathcal{N}| = \frac{1}{2}(|\mathcal{N}| - \iota(u))$. Hence $\mathcal{T}$ is critical iff $\theta \in [\frac{1}{2}(|\mathcal{N}| - \iota_1) + 1, \frac{1}{2}(|\mathcal{N}| - 1)]$, and convex confined iff $\theta \geq \frac{1}{2}(|\mathcal{N}| - \iota_2) + 1$.

We now define *nucleation parameters* γ_0, γ_1, and γ_2. These quantify the smallest instabilities which have long–range effects for the evolution, and thus play a crucial role in the scaling laws. In definitions below, we adopt the usual convention that $\inf \emptyset = \infty$.

Say A_0 *generates persistent growth* if $\mathcal{T}^{n+1}(A_0) \neq \mathcal{T}^n(A_0)$ for every $n \geq 0$. Set

$$\gamma_0 = \inf\{|A_0| : A_0 \text{ generates persistent growth}\}.$$

Then, for any $u \in S^1$, let $\gamma(u)$ be the least number of sites one needs to add to the half–space H_u^- to ensure growth, i.e.,

$$\gamma(u) = \inf\{|A_0| : A_0 \cup (H_u^- \cap \mathbf{Z}^2) \text{ generates persistent growth}\}.$$

The next two parameters are the maximal and second largest $\gamma(u)$:

$$\gamma_1 = \gamma(u_1) = \sup\{\gamma(u) : u \in S^1\},$$
$$\gamma_2 = \sup\{\gamma(u) : u \neq \pm u_1\}.$$

The proposition below is now little more than a restatement of the previous one.

Proposition 2.2. *The dynamics are critical iff $0 < \gamma_1 < \infty$. Moreover the dynamics are convex confined iff $\gamma_2 > 0$, or equivalently, iff $\gamma_0 = \infty$. Finally, $P(T < \infty) = 1$ for all critical dynamics.*

Convex confined dynamics are called *balanced* if $\gamma_1 = \gamma_2$.

It is not hard to check that for convex confined TG dynamics, $\gamma_i = \theta - (|\mathcal{N}| - \iota_i)/2$, $i = 1, 2$. Hence a TG model is balanced iff $\iota_1 = \iota_2$. One now readily verifies that the dynamics in Examples 1, 4 and 5 are all balanced, with $\gamma_1 = \gamma_2 = 1$. In Example 3, however, $\gamma_1 = 2$ and $\gamma_2 = 1$.

Almost all our results to follow require a regularity condition ensuring that "if the dynamics do not fill the space they supposed to, then they stop altogether." We call dynamics which satisfy this condition *voracious*. Voracity is easiest to define in the supercritical case ([GG2]): $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$ for every A_0 with $|A_0| = \gamma_0$ which generates persistent growth. In critical cases, we instead have two similar conditions and a third which states that smaller than minimal sets have no effect:

(V1) If $\gamma_0 < \infty$, then every A_0 with $|A_0| = \gamma_0$ which generates persistent growth fills in a strip, i.e., there exist numbers a, b so that $\mathcal{T}^\infty(A_0)$ is the union of $(bu_1 + H^-_{u_1}) \cap (au_1 + H^-_{-u_1}) \cap \mathbf{Z}^2$ and a finite set.

(V2) For $i = 1, 2$, every u with $\gamma(u) = \gamma_i$, and every A_0 with $|A_0| = \gamma_i$ such that $A_0 \cup (H^-_u \cap \mathbf{Z}^2)$ generates persistent growth, there exists a number $a > 0$ such that $\mathcal{T}^\infty(A_0 \cup (H^-_u \cap \mathbf{Z}^2))$ is the union of $(au + H^-_u) \cap \mathbf{Z}^2$ and a finite set. (Interpret the $i = 2$ part of this condition as vacuous if $\gamma_2 = 0$.)

(V3) For $i = 1, 2$, for every u with $\gamma(u) = \gamma_i$, and every A_0 with $|A_0| < \gamma_i$, $\mathcal{T}$ adds no site at all to $A_0 \cup (H^-_u \cap \mathbf{Z}^2)$.

Properties (V1–2) in effect eliminate the possibility of growing checkerboard patterns, parallel strips, etc. Note, for example, that if $\gamma_0 < \infty$, then (V1) implies that $\mathcal{N}$ cannot be a subset of a proper sublattice of $\mathbf{Z}^2$. These conditions involve only minimal growing sets because those are the only ones that matter on relevant scales. For small neighborhoods one can therefore check voracity by hand or by computer. As an illustration, we sketch how one deals with Example 2, the remaining four examples being even more straightforward. First, one finds that $u_1 = e_2$, and there are 8 different two–point sets A_0 which generate persistent growth. (Assume that these sets have their leftmost lowest point at the origin, say, to avoid double–counting.) By examining the first few iterations, one verifies that all of these fill in a strip, thereby establishing (V1). Next, take all A_0 as in (V2) translated so that their leftmost point is on the y–axis. There are two such: the singletons $\{(0,1)\}$ and $\{(0,2)\}$, making $H_{e_2}^-$ advance by 1 and 2 units respectively. Therefore, (V2) holds as well. As for (V3), one easily checks that it holds for any TG model.

Such examples suggest the possibility of a general theory to the effect that voracity should be automatic for "nice" $\mathcal{N}$. Such a theory is far from easy to develop; at present the only result in this direction, due to Bohman ([Boh]) and proved by a complex combinatorial argument, applies to supercritical TG dynamics on box neighborhoods.

To avoid repeatedly stating the voracity hypothesis, we officially add it to the axiom list

(A6) $\mathcal{T}$ generates voracious dynamics,

and assume, from now on, that $\mathcal{T}$ satisfies (A1–6).

3. Scaling laws for first passage times.

To provide the proper context, we start with a theorem for supercritical rules.

Theorem 3.1. *Assume that $\mathcal{T}$ generates supercritical dynamics. Then*

$$p^{\gamma_0/2} \cdot T \xrightarrow{d} \tau,$$

as $p \to 0$, where τ is a non–degenerate random variable.

The power of p is not at all surprising: a viable nucleus occurs at a fixed site with probability on the order p^{γ_0}, so there should be a Poisson number of them among $p^{-\gamma_0}$ sites at distance $p^{-\gamma_0/2}$. However, the proof of Theorem 3.1 (see [GG2]) is complicated by the fact that the resulting growing clusters interact in a non–trivial fashion. Although the scenario is entirely different in critical cases, T may still satisfy a power law.

Theorem 3.2. *Assume that $\mathcal{T}$ is critical, but not convex confined. Then*

$$p^{(\gamma_0+\gamma_1)/2} \cdot T \xrightarrow{d} \tau,$$

for a non–trivial random variable τ.

Example 2 is probably the simplest instance of this theorem, hence the best setting to explain its proof and the exponent $(\gamma_0 + \gamma_1)/2$. Recall that $\gamma_0 = 2$ and $\gamma_1 = 1$ in this example. Start with a *horizontal* scaling of $\mathbf{Z}^2$ by p, thus transforming $\mathbf{Z}^2$ into horizontal copies of $\mathbf{R}$ at distance 1 apart, each equipped with a unit–intensity Poisson point location. Assume that every point in the random field is equipped with a *percolation gadget* of the variety shown on the left side of Figure 3. Moreover, make each location point a *nucleus* with probability $\approx 8p$.

Speed up time by a factor of $1/p$. Starting from a nucleus, the growth progresses with speed 1 in the horizontal direction, and jumps instantaneously in the direction of an arrow whenever it passes through its tail. This is an instance of additive growth dynamics, so the origin will be reached by the time it makes the origin to reach the first nucleus in the *dual* growth model, obtained by reversing the gadget arrows (as on the right side of Figure 3).

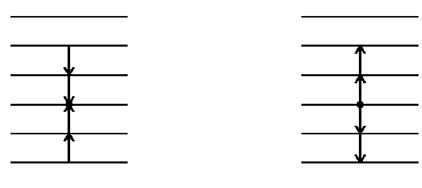

Figure 3. Percolation gadgets at the Poisson point locations.

One can invoke standard subadditivity arguments to show that, starting with the origin occupied, this growth model then has an asymptotic shape $\tilde{L}$:

$$\frac{\{\text{points reached by time } t\}}{t} \to \tilde{L}$$

as $t \to \infty$. Therefore,

$$\begin{aligned}
&P(T \cdot p^{3/2} \geq \lambda) \\
&\approx P(\text{no nucleus reached by time } \lambda \cdot p^{-1/2}) \\
&\approx P(\text{no nuclei in } \lambda p^{-1/2} \tilde{L}) \\
&\approx e^{-8\lambda^2 \text{area}(\tilde{L})}.
\end{aligned}$$

Despite the simplicity of additive growth, $\tilde{L}$ is apparently not computable (although its intersection with the x–axis is obviously $[-1,1]$). Figure 4 depicts a simulation of the rescaled growth and gives a rough idea of what $\tilde{L}$ looks like.

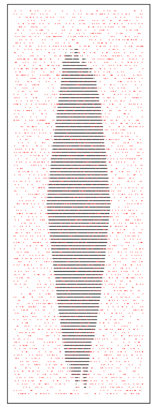

Figure 4. The rescaled growth.

For the general case, one starts with a horizontal scaling by p^{γ_1}, which yields a variety of gadgets at the resulting Poisson point locations. (One appeals to sophisticated Poisson convergence theory here; see [AGG] and [BHJ]). The growing set in the dual model then must proceed until it reaches the first nucleus of γ_0 sites, which happens when its radius is order $p^{(\gamma_1-\gamma_2)/2}$. The product of these two scalings gives the power of $(\gamma_0+\gamma_1)/2$.

By contrast, in convex–confined cases the growing droplet cannot expand in any direction without the help of its sparsely occupied surroundings. This causes T to be much larger. In fact, $\ln T$ behaves like a power of p, possibly with logarithmic corrections.

Theorem 3.3. *Suppose $\mathcal{T}$ is balanced, convex confined, and critical. Then there exist constants $c_1, c_2 > 0$ so that*

$$P\left(c_1 \cdot \frac{1}{p^{\gamma_1}} \leq \ln T \leq c_2 \cdot \frac{1}{p^{\gamma_1}}\right) \to 1 \text{ as } p \to 0.$$

Theorem 3.4. *If $\mathcal{T}$ is convex confined and critical, but not balanced, then there exist constants $c_1, c_2 > 0$ so that*

$$P\left(c_1 \cdot \frac{1}{p^{\gamma_2}} \ln\frac{1}{p} \leq \ln T \leq c_2 \cdot \frac{1}{p^{\gamma_2}} \ln\frac{1}{p}\right) \to 1 \text{ as } p \to 0.$$

To get the rough feeling for these two results, consider first the dynamics of Example 1. Call site x a *nucleus* if it is occupied, and the square annulus $B_\infty(x, n+1) \setminus B_\infty(x, n)$ contains at least one occupied site within each of its four sides. If x is a nucleus and the dynamics cover $B_\infty(x, n)$ at some time, then $B_\infty(x, n+1)$ is occupied in at most $2n$ additional steps. Moreover,

$$\lim_{p \to 0} p \cdot \ln P(x \text{ is a nucleus}) = 4 \int_0^\infty \ln(1 - e^{-2u})\, du = -\frac{\pi^2}{3}.$$

Assuming the nuclei occur at different sites nearly independently, the nearest one to the origin occurs with high probability at ℓ^∞–distance at most $e^{C/p}$, for some $C > 0$. Then the origin is reached no later than at time $2e^{2C/p}$. This heuristic is easily adapted to provide a rigorous upper bound on T of the form given in Theorem 3.3. The lower bound requires a different argument; [AL] and [GG2] contain complete proofs for Examples 1 and 4, respectively.

By contrast, in Example 3, a typical non–balanced case, advance in the vertical direction is very costly compared to horizontal spread. The square annuli above should therefore be replaced by long thin rectangular ones, from which it follows that the "hard" direction only contributes a logarithmic factor. See [GG2] for more details.

In growth processes, random seeds are often sown not only at the beginning, but *continuously* in time. To model this situation, define the rule $\mathcal{T}_c$, in which $\mathcal{T}_c(A)$ consists of points in $\mathcal{T}(A)$, and any other point independently with probability p. By convention, $\mathcal{T}_c$ is started from the vacant lattice. Let T_c be the first passage time to 0. Estimates on how fast T_c grows as $p \to 0$ can be obtained as simple applications of results in this section. The first step is to view $T = T(p)$ and $T_c = T_c(p)$ as random functions of p. By monotonicity, $\mathcal{T}_c^n(\emptyset)$ is bounded above by the result of n random seeding steps followed by n iterates of $\mathcal{T}$. Even more obviously, a lower bound on $\mathcal{T}_c^n(\emptyset)$ is obtained from $n/2$ seeding steps followed by an equal number of iterates of $\mathcal{T}$. Under the coupling which demonstrates the two comparisons,

$$\{T(1-(1-p)^{n/2}) \le n/2\} \subset \{T_c(p) \le n\} \subset \{T(1-(1-p)^n) \le n\},$$

and a simple computation leads to the following result.

Corollary 3.5. *Let $\mathcal{T}$ be critical, convex confined and balanced. Then there exist constants c_1 and c_2 such that*

$$P(c_1 \cdot \frac{1}{p \cdot (\ln(1/p))^{1/\gamma_1}} \le T_c \le c_2 \cdot \frac{1}{p \cdot (\ln(1/p))^{1/\gamma_1}}) \to 1 \text{ as } p \to 0.$$

Non–balanced convex confined cases lead to even more intriguing asymptotics: $T_c \approx p^{-1} \cdot (\ln(1/p))^{-1/\gamma_2} \cdot (\ln\ln(1/p))^{1/\gamma_2}$. Finally, if T is on the order of a power $p^{-\alpha}$, then T_c is of order $p^{-\alpha/(\alpha+2)}$. See [DS] for a similar model.

4. Characteristic shapes of growing droplets.

In supercritical cases, every finite set A_0 for which $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$ generates the *forward shape* L :

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{T}^n(A_0) \to L.$$

In fact L can be characterized as the polar transform of $K_{1/w} = \cup_{u\in S}[0, 1/w(u)]u$. That is, $L = K^*_{1/w} = \{y \in \mathbf{R}^2 : \langle x, y\rangle \le 1 \text{ for every } x \in K_{1/w}\}$ (see [GG2] for a proof and examples). To capture shapes observed when the dynamics are started from randomness, let $\mathcal{C}_n(1)$ be the connected site percolation cluster of 1's at time n which contains the origin. Condition on the event that the origin is part of an occupied set of size γ_0 which generates persistent growth,

Proposition 4.1. *With respect to the above conditional probability,*

$$\frac{1}{n}\mathcal{C}_n(1) \to L, \qquad \text{as } p \to 0,$$

in probability, provided that $n = n(p) \to \infty$ in such a way that $n \cdot p^{\gamma_0/2} \to 0$.

In words, then, the shapes we see are small perturbations of forward shapes. The time n of course needs to be restricted, since even the normalized shape tends to $\mathbf{R}^2$ as $n \cdot p^{\gamma_0/2} \to \infty$. A proof of Proposition 4.1, based on large deviation bounds for probabilities of linear effects, appears in [GG2].

The corresponding story in critical cases is necessarily more complicated, since $\mathcal{C}_n(1)/n$ must be scaled by a power of p to obtain a non–trivial limit. Suppose $\mathcal{T}$ is balanced and critical. Consider the event that $B_\infty(0, 2p^{-4\gamma_1})$ contains an occupied cluster of diameter $p^{-2\gamma_1}$ at time $p^{-4\gamma_1}$. The lattice is thus centered near a large occupied set at a time after the nucleation phase, but far before the growing droplets start to interact.

Theorem 4.2. *Conditional on the above event, there is a convex polygon L, and a $\delta > 0$, so that for any any $n = n(p) \in [p^{-5\gamma_1}, \exp(\delta p^{-\gamma_1})]$,*

$$p^{-\gamma_1}\left(\frac{\mathcal{C}_n(1)}{n}\right) \to L$$

as $p \to 0$ in probability.

The number of sides of L is bounded by the number of directions u for which $\gamma(u) = \gamma_1$. The reason is that growth in all other directions is much faster, and so is not manifest in the asymptotic shape. Hence L must be a square in Example 1, a diamond in Example 5, and at most an octagon in Example 4. Let us concentrate on this last example to explain the main step of the proof.

Essentially, the shape is determined by the speed of horizontal and diagonal half–planes of occupied sites in a random environment which is approximately a density p product measure. We consider $H_{e_3}^-$ first. After horizontal rescaling by p, one gets exactly the same growth model as in Section 2 (defined by Figure 3), started from a Poisson point location on the x–axis). This being an additive model, the half–plane can be shown to have a coherent vertical speed. A similar argument for the diagonal interface H_u^- with $u = (e_1+e_2)/\sqrt{2}$ generates similar dynamics except that the lines corresponding to Figure 3 are now $1/\sqrt{2}$ apart, the Poisson locations have intensity $1/\sqrt{2}$, and the gadgets have range 3 instead of 2. This last fact implies that the shape cannot be a diamond, as the diagonal interfaces are too fast. To eliminate the square as a possible limit shape, one would have to prove that, when locations and distance between lines are the same, the speed with range 3 gadgets is strictly smaller than twice the speed with range 2 gadgets. This seems likely, but a rigorous argument has eluded us so far.

Theorem 4.2 describes the growth of a critical model in a slightly helpful random environment. At first glance, this may look very similar to the following random dynamics, which can be thought of as a small supercritical perturbation of a critical rule. Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ define, respectively, convex confined dynamics and supercritical dynamics. Now define the rule $\mathcal{T}_r$ thus: given a set A, include in $\mathcal{T}_r(A)$ every point in $\mathcal{T}_1(A)$, while every point of $\mathcal{T}_2(A)$ not in $\mathcal{T}_1(A)$ is adjoined independently with probability q. Start from a large deterministic set A_0 and assume that somehow the existence of an asymptotic shape

$$L_q = \lim_{n\to\infty} \frac{1}{n} \mathcal{T}_r^n(A_0)$$

can be established (see [BG] for a general method). How does L_q scale with small q? Perhaps surprisingly, the answer does not depend on nucleation properties of either $\mathcal{T}_1$ or $\mathcal{T}_2$. Rather, the diameter of L_q is always on the order of $\sqrt{q}$. See [KS] and [Gra] for specific examples.

5. Decay rates for first passage times.

A completely satisfactory solution to Problem 3 of the Introduction is challenging even in supercritical cases. At least this much is clear: for any $\epsilon > 0$, a necessary condition for $T > n$ is that there are no minimal sets which generate persistent growth in $(1-\epsilon)n \cdot L$, while a sufficient condition is that, for a large enough R, the event $G_x = \{B_\infty(x, R) \text{ contains } \gamma_0\ 1's\}$ happens for no $x \in (1+\epsilon)n \cdot \text{co}(\mathcal{N})$. (Here L is the forward shape and $\text{co}(\mathcal{N})$ is the convex hull of $\mathcal{N}$.) The fact that the events G_x are positively correlated then implies the following result.

Proposition 5.1. *If $\mathcal{T}$ is supercritical, then there are constants $c_1, c_2 > 0$ such that for every $p < 1$ and sufficiently large n,*

$$c_1 p^{\gamma_0} n^2 \leq -\ln P(T > n) \leq c_2 p^{\gamma_0} n^2.$$

It is natural to conjecture that $-p^{-\gamma_0} n^{-2} \ln P(T > n)$ converges as $n \to \infty$ and then $p \to 0$, but this is known only in the *quasi–additive* cases, i.e., when $K_{1/w}$ is convex, in which case it can be proved ([GG6]) that the limit equals the area of L.

As expected, decay is much slower in critical cases.

Theorem 5.2. *Assume $\mathcal{T}$ is critical and balanced. Then, for every $p \in (0,1)$, $\Gamma_1 = \Gamma_1(p) = -\liminf n^{-1} P(T > n)$ and $\Gamma_2 = \Gamma_2(p) = -\limsup n^{-1} P(T > n)$ are positive and finite.*

The proof of the above result is a relatively straightforward adaptation of the methods in [Sch3] or [And]. The lower bound is easy, since a sufficient condition for $T > n$ is that a long thin rectangle perpendicular to u_1, with length on the order of n and width larger than the diameter of $\mathcal{N}$, is completely unoccupied. The techniques in [And] and [AMS], can be adapted (with some work) to show that, at least for balanced critical dynamics, $\ln \Gamma_i / \ln p \to \gamma_1$ as $p \to 0$, $i = 1, 2$. However more precise results would be welcome, perhaps along the lines of [Mou2] and [AMS], which show for a modified bootstrap rule that $\Gamma_1 = \Gamma_2 = -2\log(1-p)$.

6. Scaling relationships in the presence of pollution.

Supercritical dynamics have no problem circumventing finite obstacles, as evidenced by the following result from [GG3].

Proposition 6.1. *For any supercritical $\mathcal{T}$,*

$$\lim_{q \to 0} \liminf_{p \to 0} P(T < \infty) = 1.$$

On the other hand, pollution may have a dramatic effect on critical dynamics. Call $\nu > 0$ a *lower pollution power (l.p.p.)* if, for every $\epsilon > 0$, $q > p^{\nu - \epsilon}$ implies that $P(T < \infty) \to 0$ as $p \to 0$. Say $\nu < \infty$ is an *upper pollution power (u.p.p.)* if, for every $\epsilon > 0$, $q < p^{\nu + \epsilon}$ implies that $P(T < \infty) \to 1$ as $p \to 0$. Naturally, a *pollution power* is both a u.p.p. and an l.p.p. The letter ν will denote such a pollution power from now on. Note that our terminology makes sense since the larger ν, the more pronounced the effect of site removal on $\mathcal{T}$.

Theorem 5.2. *For any critical $\mathcal{T}$ there exists a u.p.p. and an l.p.p.*

To illuminate this last theorem, we sketch the proof that $\nu = 1$ in Example 2. The main step in general is essentially an appropriately rescaled modification.

First let $q = p^{1+\epsilon}$. Divide the lattice into $1 \times p^{-1-\epsilon/2}$ line segments, arranged in vertical stacks, so that each segment has a neighboring segment directly above, below, to the left, and to the right. Call such a segment *open* if it contains no removed sites and at least one occupied site. Then the probability that a segment is open converges to 1 as $p \to 0$. Hence, with probability tending to 1, the segment containing the origin is connected (by a neighbor–to–neighbor path) to infinitely many segments. Almost surely, one of these is completely occupied and, since occupation is able to spread between neighboring open segments, the origin must eventually be occupied.

Now assume that $q = Cp$. Define a *blocking path* to be an oriented nearest neighbor path on $\mathbf{Z}^2$ which makes only right and up moves, contains no occupied site within ℓ^1–distance 2, and only moves up to a removed site. The reason for this name is that the dynamics cannot penetrate a blocking path from below. For small p, the probability of the existence of an infinite blocking path from the origin approaches the survival probability of the one–sided contact process. Hence such a path exists with very high probability if C is sufficiently large. At this point, standard oriented percolation arguments enable one to encircle large sets with paths that cannot be penetrated from the outside. These sets can be much smaller in size than p^{-2}, thereby ensuring that, with high probability, the encircling paths are safe from the inside as well. The vast majority of sites therefore remains unoccupied forever.

In other critical cases, pollution powers depend on the ability of removed sites to block progress of concavities in the occupied set, as well as the now familiar issue of advancement of occupied half–spaces. However, a general result is quite cumbersome to state, so we will merely mention answers for the four remaining examples, and then conclude with two still frames from computer experiments.

It is proved in [GM] that $\nu = 2$ for Example 1. The fundamental building blocks in the argument are squares of size on the order $(1/p) \times (1/p)$ (in contrast to the horizontal line segments of size $(1/p) \times 1$ in Example 2). Some inessential modifications of the arguments in Sections 1 and 3 of GM establish that $\nu = 2$ in Examples 4 and 5 as well. Example 3 needs a more substantial modification, since the building blocks are $(1/p^2) \times (1/p)$ rectangles in this case, but the ultimate conclusion is that $\nu = 3$.

Figure 5 shows snapshots of fixated states for Examples 3 and 5. The dynamics were run on a 130×130 array with 1–boundary conditions (to eliminate tricky nucleation issues). This time the removed sites are black, and the occupied sites have gray shades. Note the blocking paths which outline the boundaries of the two "frames" of 1's. In both cases, $q = 0.01$ was fixed, then p was increased until the frame of 1's was able to make substantial inroads into the square. The resulting "critical" p's, 0.04 in Example 1 and 0.09 in Example 5, should not suggest that Example 5 has a higher pollution power. Instead, they merely point out that reliable simulations of critical dynamics typically require very large arrays, a (sometimes unpleasant) lesson that has been encountered by researchers several times over the past 15 years.

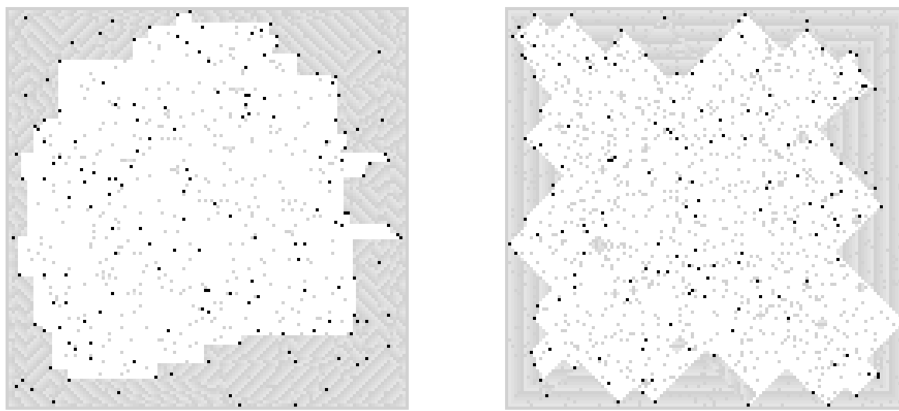

Figure 5. Examples 3 and 5 in polluted environments.

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