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TWO-DIMENSIONAL SUPERCRITICAL GROWTH DYNAMICS WITH ONE-DIMENSIONAL NUCLEATION

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We introduce a class of cellular automata growth models on the two-dimensional integer lattice with finite cross neighborhoods. These dynamics are determined by a Young diagram $\mathcal{Z}$ and the radius ρ of the neighborhood, which we assume to be sufficiently large, so that the resulting dynamics are supercritical. A point becomes occupied if the pair of counts of currently occupied points in the horizontal and vertical parts of the cross neighborhood lies outside $\mathcal{Z}$. Starting with a small density p of occupied points, we focus on the first time T at which the origin is occupied. We show that T scales as a power of $1/p$, and identify that power, when $\mathcal{Z}$ is a triangle, a rectangle, and the union of a finite rectangle with an infinite strip. The case of a triangle corresponds to the classical bootstrap percolation model. We give partial results when $\mathcal{Z}$ is a union of two finite rectangles. The distinguishing feature of these dynamics is nucleation of lines that grow to significant length before most of the space is covered.

1. Introduction. To quote Maris [Mar], "Nucleation means a change in a physical or chemical system that begins within a small region." Nucleation is a key factor in determining self-organization properties of physical systems [JD]. Of particular interest for probabilists is homogeneous nucleation, which happens due to random fluctuations in a statistically homogeneous environment without preferred nucleation sites, such as impurities. Starting in the 1970s, simple models were devised to study such processes [CLR, AL]. In these models, a "small region" is one with a diameter much smaller than the time scale on which a new equilibrium is reached; in this sense, the nucleation is strictly local.

In this paper, we study models in which a nucleation phase is non-local. Instead, the "small region" is now of a lower dimension than the growth environment. As we restrict to two-dimensional environments, this translates to growth models that generate extended one-dimensional tentacles long before the growth covers most of the space. While evidence of such growth in the physical literature is scarce, there is a recent discovery of efficient nucleation of lines in the assembly of two-dimensional molecular arrays [Che], which may motivate further investigation into mathematical models of this type.

1.1. *Background and definitions.* Our model is a local version of the one we introduced in [GSS], a class of rules that can accommodate fairly general interaction between two possible directions. To ensure monotonicity, the key parameter in such rules is a Young diagram. The other parameter is a finite range, which ensures locality, and which we assume to be large enough, but otherwise plays a limited role at our level of precision.

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Cellular automata growth dynamics with cross neighborhoods were introduced in [HLR], and indeed the model considered in that paper fits our definition. It is, however, a critical dynamics [BSU, BDMS], by contrast with the supercritical ones we consider here. Consequently, the scaling of first passage time T is exponential in a power of $1/p$ and the central problem is the calculation of the precise exponential rate, which requires completely different methods. More closely related is the setting in [GSS], where *unbounded* cross neighborhoods enable results that are more general and precise than those obtained here (see Open Problem 6 in Section 10). In another direction, the recent papers [Bla1, Bla2, Bla3] study critical dynamics with d-dimensional versions of cross neighborhoods, for $d \geq 3$.

To proceed with precise definitions, we call a set $\mathcal{Z} \subseteq \mathbb{Z}_+^2$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, a *zero-set* if $(u, v) \in \mathcal{Z}$ implies $[0, u] \times [0, v] \subseteq \mathcal{Z}$, where we use $[a, b]$ to denote the set of integers between a and b. If $\mathcal{Z}$ is finite, then it is equivalent to a Young diagram in the French notation; however, infinite zero-sets are also of interest and will still be referred to as Young diagrams. Sometimes we specify $\mathcal{Z}$ by the *minimal counts*, which are those pairs $(u_0, v_0) \in \mathbb{Z}_+^2 \setminus \mathcal{Z}$ for which both $(u_0 - 1, v_0)$ and $(u_0, v_0 - 1)$ are in $\mathcal{Z} \cup (\mathbb{Z}_+^2)^c$. The reason for this terminology will be clear in the next paragraph. See Figure 1 for an example.

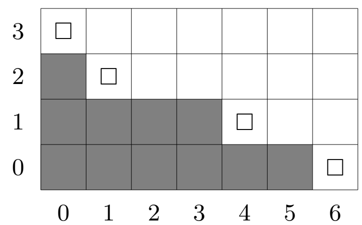


FIG 1. *A example of a zero-set $\mathcal{Z}$. It has width $w(\mathcal{Z}) = 6$, height $h(\mathcal{Z}) = 3$, and its minimal counts $(0,3)$, $(1,2)$, $(4,1)$ and $(6,0)$ are labeled by $\square$.*

Define the *cross neighborhood with range* $\rho \geq 0$ as follows. Let $\mathcal{N}^h = [-\rho, \rho] \times \{0\}$ and $\mathcal{N}^v = \{0\} \times [-\rho, \rho]$ be the horizontal and vertical range ρ line neighborhoods of 0. For $x \in \mathbb{Z}^2$, let $\mathcal{N}_x^h = x + \mathcal{N}^h$ and $\mathcal{N}_x^v = x + \mathcal{N}^v$ and then declare the neighborhood of x to be

$$\mathcal{N}_x = \mathcal{N}_x^h \cup \mathcal{N}_x^v.$$

For each $t \in \mathbb{Z}_+$, we let $\xi_t \subseteq \mathbb{Z}^2$ denote the collection of *occupied* (or *active*) vertices at time t. Given ξ_0, we define ξ_t recursively by

$$\xi_{t+1} = \xi_t \cup \{x \in \mathbb{Z}^2 : (|\mathcal{N}_x^h \cap \xi_t|, |\mathcal{N}_x^v \cap \xi_t|) \notin \mathcal{Z}\}. \tag{1.1}$$

That is, for each unoccupied point $x \in \mathbb{Z}^2$ we compute the pair $(|\mathcal{N}_x^h \cap \xi_t|, |\mathcal{N}_x^v \cap \xi_t|)$ of counts of currently occupied points in the horizontal and vertical parts of its neighborhood, and then we add x to the occupied set if this pair lies outside of $\mathcal{Z}$; equivalently, x becomes occupied when

$$(|\mathcal{N}_x^h \cap \xi_t|, |\mathcal{N}_x^h \cap \xi_t|) \geq (u_0, v_0)$$

for some minimal counts (u_0, v_0) of $\mathcal{Z}$, using the standard partial order on $\mathbb{Z}^2$. It is convenient to specify the dynamics with minimal counts when their number is small, e.g., in Theorems 1.4 and 1.5.

As the occupied sites are never removed, we have *solidification*: $\xi_t \subseteq \xi_{t+1}$. The definition of a zero-set also ensures *monotonicity*: enlarging ξ_0 can only enlarge any ξ_t, $t \geq 0$. We will also consider such dynamics on a finite set S, with 0-*boundary* conditions, whereby we assume $\xi_0 \subseteq S$ and only allow $x \in S$ to become occupied at all times. The most useful such set is $B_n = [0, n-1]^2$. We also often impose periodic boundary conditions on B_n (see Section 2). We set $\xi_\infty = \cup_{t \ge 0} \xi_t$, and call a set $A \subseteq \mathbb{Z}^2$ *inert* if $\xi_0 = A$ implies $\xi_1 = A$.

Let $h(\mathcal{Z}) = |\mathcal{Z} \cap (\{0\} \times \mathbb{Z}_+)|$ and $w(\mathcal{Z}) = |\mathcal{Z} \cap (\mathbb{Z}_+ \times \{0\})|$ be the height and width of the zero-set $\mathcal{Z}$. Without loss of generality, we assume that $h(\mathcal{Z}) \le w(\mathcal{Z})$. We will also assume that $h(\mathcal{Z}) < \infty$ and that $\rho \ge h(\mathcal{Z})$. When $w(\mathcal{Z}) < \infty$, we also assume that $\rho \ge w(\mathcal{Z})$. (In fact, this is not a restriction: if $w(\mathcal{Z}) > \rho$, we can obtain the same process by replacing $\mathcal{Z}$ with the new zero-set $\mathcal{Z}'$ that agrees with $\mathcal{Z}$ except that the rows of $\mathcal{Z}$ that exceed ρ are made infinite in $\mathcal{Z}'$.) These constraints make our dynamics supercritical [BSU, BBMS1]. Such dynamics are *voracious* [GG] if every starting set of minimal cardinality selected from

$$\mathcal{A} = \{A \text{ finite} : \xi_0 = A \text{ leads to } \xi_\infty \text{ with } |\xi_\infty| = \infty\}$$

results in $\xi_\infty = \mathbb{Z}^2$. Nucleation properties of voracious dynamics are relatively transparent as they are determined by the minimal sets of $\mathcal{A}$ [GG].

However, unless $\mathcal{Z}$ consists of a single point, our dynamics are *not* voracious. To see this, assume that $w(\mathcal{Z}) \ge \max(h(\mathcal{Z}), 2)$. If $|\xi_\infty| = \infty$, then $|\xi_0| \ge h(\mathcal{Z})$ (as a set of $h(\mathcal{Z}) - 1$ horizontal or vertical parallel lines is inert), but a vertical interval of $h(\mathcal{Z})$ sites generates an occupied vertical line, which is inert. Therefore, the results of [GG] do not apply. Instead, nucleation is governed by the most efficient configurations of occupied sites that generate lines that grow in different directions and interact to finally produce a configuration that expands in all directions; see Figure 2.

To study the nucleation properties, we assume that each $x \in \mathbb{Z}^2$ is included in ξ_0 independently with probability $p \in [0, 1]$, and investigate the scaling of

$$T = \inf\{t \ge 0 : (0, 0) \in \xi_t\},$$

the first time that the origin is occupied, as $p \to 0$. Besides T, another natural quantity is the *critical length* L_c for spanning [Mor], defined as follows. We say $B_n = [0, n-1]^2$ is *spanned* if the dynamics on B_n with 0-boundary eventually occupies every point of B_n. Then

$$L_c := \inf\{n \ge 0 : \mathbb{P}(B_n \text{ is spanned}) \ge 1/2\}.$$

The quantity L_c naturally reflects nucleation density when the Young diagram $\mathcal{Z}$ is symmetric (with respect to switching the roles of x and y), but imposes an artificial constraint by restricting the dynamics to a square when $\mathcal{Z}$ is not symmetric. The advantage of the statistic T is that it is not constrained in this way, and thus allows the dynamics to find its own most efficient domain for nucleation.

The most studied special case of growth dynamics is known as *bootstrap percolation* [CLR] or *threshold growth* [GG]. In our context, it is given by an integer *threshold* $r \ge 1$, and the triangular zero-set $\mathcal{Z} = \{(u, v) \in \mathbb{Z}_+^2 : u + v \le r - 1\}$. Therefore, a site x joins the occupied set whenever the number of currently occupied sites in $\mathcal{N}_x$ is at least r. Bootstrap percolation was introduced on trees in [CLR] and has been since extensively studied on various graphs, with many deep and surprising results, beginning with early papers [vE, AL]. Particularly impressive are results on $\mathbb{Z}^d$; for some of the highlights, see [Hol, HLR, BBDM, HMo, BDMS], recent papers [BBMS1, BBMS2], and survey [Mor], which contains a wealth of further references. Analysis of bootstrap percolation on graphs with longer range connectivity, related to the present setup, is more recent. It was introduced in [GHPS] and further explored in [Sli, GSS, GS1, GS2, GPS, GKMR].

Another special case is *line growth*, with finite rectangular zero-set $\mathcal{Z} = [0, r-1] \times [0, s-1]$, where $1 \le s \le r$. Thus, a site x becomes occupied when either the number of currently occupied horizontal neighbors is at least r, or the the number of currently occupied vertical neighbors is at least s. These dynamics were introduced as *line percolation* [BBLN, GSS]

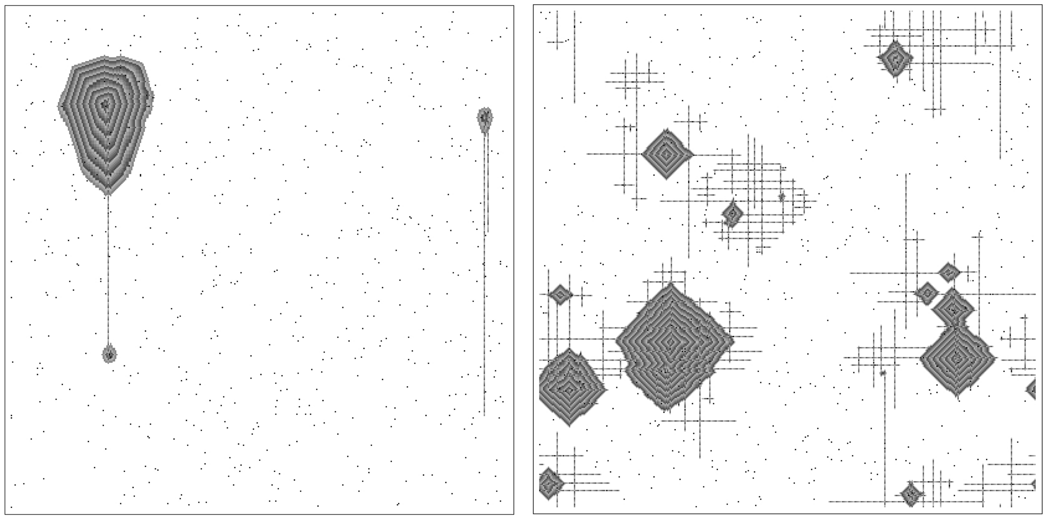


FIG 2. *Simulations of bootstrap percolation with threshold $r = 3$ and with $\rho = 3$ at $p = 0.003$ (left), and of symmetric line growth with parameters $r = s = 2$ and with $\rho = 2$ at $p = 0.002$ (right). The still-frames are taken at the time when large occupied sets are about to quickly take over the space. Sites occupied at time $t > 0$ are colored periodically with shades of grey.*

on Hamming graphs (which have $\rho = \infty$, i.e., the neighborhood is an infinite cross in both directions). On Hamming graphs, these dynamics have the property that any point gets occupied together with an entire line through it, which is of great help in the analysis. It turns out that this property approximately holds in the local version of the present paper and yields our main results. On the other hand, we suspect, and are able to prove in one case, logarithmic corrections to the power laws when $r = s > 1$, which have no counterpart on the Hamming plane and are somewhat surprising for symmetric supercritical rules. See Figure 2 for simulations of bootstrap percolation and line growth dynamics.

1.2. *Statements of main results.* Assume that a is a deterministic sequence depending on p and X is a sequence of nonnegative random variables depending on p. We write:

- $X \lesssim a$ if $\lim_{\lambda\to\infty} \limsup_{p\to 0} \mathbb{P}(X \geq \lambda a) = 0$;
- $X \gtrsim a$ if $\lim_{\lambda\to 0} \limsup_{p\to 0} \mathbb{P}(X \leq \lambda a) = 0$; and
- $X \asymp a$ if $X \lesssim a$ and $X \gtrsim a$.

In this sense, $X \asymp a$ means that $X = \Theta(a)$ in probability.

We call $\gamma > 0$ a *lower power* (resp., an *upper power*) for $\mathcal{Z}$ if, for every $\epsilon > 0$, $T \gtrsim p^{-(\gamma-\epsilon)}$ (resp., $T \lesssim p^{-(\gamma+\epsilon)}$). Then $\gamma_c = \gamma_c(\mathcal{Z})$ is the *critical power* for the zero-set $\mathcal{Z}$ if it is both an upper and a lower power.

Existence of a critical power still allows for, say, logarithmic corrections in the scaling of T, so it is meaningful to ask whether such corrections are absent. We call the critical power γ_c *pure* if $T \asymp p^{-\gamma_c}$.

It follows from [BSU] (see also [BBMS1]) that finite lower and upper powers always exist under our assumptions on $\mathcal{Z}$ and ρ that guarantee supercriticality. However, there seems to be no general method that would prove that γ_c always exists, and even good inequalities may be very difficult if not impossible to obtain in general; see [BBMS2, HMe] for discussion on related computational problems. Our methods demonstrate the existence of γ_c only when it is possible to compute it exactly, which includes many small zero-sets; see Section 3.

THEOREM 1.1. *Assume $r \geq 1$ and $\mathcal{Z} = \{(u,v) \in \mathbb{Z}_+^2 : u+v \leq r-1\}$, so that the process $(\xi_t : t \geq 0)$ is the threshold-r bootstrap percolation with a cross neighborhood. Let*

$$\widehat{m} = \left\lceil \frac{\sqrt{9+8r}-5}{2} \right\rceil.$$

Then the critical power for $\mathcal{Z}$ is

$$\gamma_c = \frac{(\widehat{m}+1)(2r-\widehat{m})}{2(\widehat{m}+2)}.$$

Moreover, this critical power is pure.

The upper bounds implied by Theorem 1.1 play a role in the determination of the critical length for three dimensional critical bootstrap percolation processes (see Proposition 1.2 in [Bla2]), so our main contributions are the matching lower bounds.

THEOREM 1.2. *Assume $2 \leq s \leq r$ and $\mathcal{Z} = [0, r-1] \times [0, s-1]$, so that the process $(\xi_t : t \geq 0)$ is the line growth with parameters r and s. Then the critical power for $\mathcal{Z}$ is*

$$\gamma_c = \frac{(r-1)s}{r}.$$

Line growth with $1 = s \leq r$ is different: the critical power is $r/(r+1)$ and is pure (see Proposition 3.1). In general, we are not able to determine purity for line growth with $s \geq 2$. We can, however, demonstrate that the critical power is not pure in one case. Also, to our knowledge, this is the only symmetric growth rule for which the scalings of T and L_c provably differ. The reason for this discrepancy is as follows. The occupied sites needed for occupation of the origin are likely to be found within distance $1/p$, which determines the scaling for L_c. However, their influence is spread through an embedded tree with root at the origin on which the geodesics are longer by a logarithmic factor, which increases the occupation time by the same factor.

THEOREM 1.3. *Assume $\mathcal{Z} = [0,1] \times [0,1]$, so that the process $(\xi_t : t \geq 0)$ is the line growth with parameters $r = s = 2$. Then*

$$T \asymp p^{-1} \log p^{-1},$$

and $L_c \asymp p^{-1}$.

Next, we consider L-shaped Young diagrams, for which we can only determine γ_c in special cases, and we instead give power bounds in general.

THEOREM 1.4. *Assume $\mathcal{Z}$ is given by the minimal counts $(0,r)$, (s_1, s_2), and $(r,0)$, and assume $1 \leq s_1, s_2 < r$. Thus, $\mathcal{Z}$ is the union of two bounded rectangles,*

$$\mathcal{Z} = ([0, r-1] \times [0, s_2 - 1]) \cup ([0, s_1 - 1] \times [0, r-1]).$$

1. *If $s_1 = 1$ and $s_2 = s$ with $s \leq r/2$, the pure critical power is $\gamma_c = r/2$.*
2. *If $s_1 = s_2 = 2$ and $r \geq 6$, the pure critical power is $\gamma_c = 2r/3$.*
3. *If $s_1 = s_2 = s$, a lower and an upper power are, respectively,*

$$\gamma_\ell = \frac{\widehat{m}(r - \widehat{m} + 1)}{1 + \widehat{m}}, \text{ where } \widehat{m} = \min(\lfloor(-1+\sqrt{4r+9})/2\rfloor, \lfloor s/2 \rfloor),$$

and

$$\gamma_u = \frac{rs}{s+1}.$$

4. *Assume* $s_1 = 1$ *and* $s_2 = s$. *For* $s > r/2$, *we have lower and upper power* $s - \frac{s}{r}$ *and* $s + 1 - \frac{2s+1}{r+1}$, *which differ by at most* $1/2$.

Throughout this article we use the asymptotic notation $\ll$, $\gg$, o, $\mathcal{O}$, and Θ in the standard fashion. Assume the zero set $\mathcal{Z}$ has height and width r, and r is large. If $\mathcal{Z}$ is between bootstrap percolation and symmetric line growth with the same threshold r, Theorems 1.1 and 1.2 give lower and upper powers of the form $r - \mathcal{O}(\sqrt{r})$. On the other hand, concave zero-sets may no longer have powers of the form $r - o(r)$. For example, the powers are $r/2$ for the zero-set of part 1 of Theorem 1.4, but if both arms have thickness s they change to $2r/3$ when $s = 2$ (part 2) and become $r - o(r)$ when $s \to \infty$ and $r - \Theta(\sqrt{r})$ when $s = \Theta(\sqrt{r})$ (part 3). By contrast, if one arm of the zero-set has thickness 1 and the other has thickness s, the powers do not change at all up to $s = \lfloor r/2 \rfloor$ and become $r - o(r)$ only when $s = r - o(r)$ (parts 1 and 4).

Finally, we address a family of infinite zero-sets, for which the critical power can again be established.

THEOREM 1.5. *Assume the infinite zero set is given by the minimal counts* $(0, r)$ *and* (s_1, s_2), *where* $1 \le s_2 < r$ *and* $1 \le s_1$. *Thus,* $\mathcal{Z}$ *is the union of a bounded and an unbounded rectangle,*

$$\mathcal{Z} = ([0, s_1 - 1] \times [0, r - 1]) \cup ([0, \infty) \times [0, s_2 - 1]).$$

Then the pure critical power for $\mathcal{Z}$ *is*

$$\gamma_c = \frac{r s_1 + s_2}{1 + s_1}.$$

The rest of the paper is organized as follows. In Section 2 we give definitions and simple preliminary results that we routinely use throughout the paper. In Section 3 we address the 13 zero-sets that fit into a 3×3 box, assuming the results from subsequent sections. Section 4 is devoted to bootstrap percolation and the proof of Theorem 1.1. Section 5 gives the upper bound for the symmetric line growth for arbitrary $r \ge 2$, which is more precise than stated in Theorem 1.2, and is needed for Theorem 1.3, whose proof is completed in Section 6. The most substantial and technical portion of the paper is Section 7, which completes the proof of Theorem 1.2. At heart, that proof is an optimization argument, establishing which nucleation scenarios are optimal. The next two sections address L-shaped Young diagrams, finite ones in Section 8 (proving Theorem 1.4) and infinite ones in Section 9 (proving Theorem 1.5). We conclude with a selection of open problems in Section 10.

2. Preliminaries. We start by introducing a convenient notation for tracking the powers of p. For a sequence X of nonnegative random variables and a real number α, we write $X \rhd \alpha$ if $X p^\alpha \to \infty$ in probability, and $X \lhd \alpha$ if $X p^\alpha \to 0$ in probability. If $\alpha \le 0$ and X is an integer-valued random variable, the meaning of $X \lhd \alpha$ is simply that $X \to 0$ in probability. We artificially declare $X \rhd \alpha$ whenever $\alpha < 0$.

The next lemma, whose proof is a simple exercise, provides useful reformulations of the asymptotic notion defined at the start of Section 1.2.

LEMMA 2.1. *Let* a *be a deterministic sequence depending on* p *and* X *a sequence of nonnegative random variables depending on* p. *Then* $X \lesssim a$ *if and only if* $\mathbb{P}(X \le t) \to 1$ *for every sequence* t *such that* $t \gg a$, *and* $X \gtrsim a$ *if and only if* $\mathbb{P}(X \le t) \to 0$ *for every sequence* t *such that* $t \ll a$. *Moreover,* $X \lhd \alpha$ *and* $X \rhd \alpha$ *can be respectively characterized as* $\mathbb{P}(X \le t) \to 1$ *for any* $t \gtrsim p^{-\alpha}$ *and* $\mathbb{P}(X \le t) \to 0$ *for any* $t \lesssim p^{-\alpha}$.

We next state two simple lemmas, which we routinely use throughout. As mentioned in Section 1, we commonly consider dynamics on boxes B_n with the default 0-boundary. However, it is also useful to consider *periodic boundary* on B_n, whereby a site $(x_1+i, x_2+j) \in \mathcal{N}_{(x_1,x_2)}$ is understood to be $((x_1+i) \mod n, (x_2+j) \mod n)$, giving B_n the topology of the discrete torus, the Cartesian product of two cycles with n vertices. In either case, we call B_n *spanned at time* t if $\xi_t = B_n$; and *spanned* if this holds at $t=\infty$. We say a sequence of events E (depending on p) occurs *with high probability* if $\mathbb{P}(E) \to 1$ as $p \to 0$.

LEMMA 2.2. *Assume that there exists a constant C, and a sequence $n = n(p)$ such that B_n (with the default 0-boundary) is spanned at time Cn with high probability. Then, with high probability, $T \le Cn$.*

PROOF. This is a simple consequence of monotonicity. □

LEMMA 2.3. *Assume that a sequence $n = n(p)$ has the property that, for the dynamics ξ_t on B_n with periodic boundary, $|\xi_\infty|/|B_n| \to 0$ in probability, as $p \to 0$. Then, for the dynamics on $\mathbb{Z}^2$, $\mathbb{P}(T \ge n/(2\rho)) \to 1$, as $p \to 0$.*

PROOF. In this proof, we temporarily use the notation ξ_t^P to denote the dynamics on B_n with periodic boundary conditions, and ξ_t^E for the dynamics on $\mathbb{Z}^2$. We assume that $\xi_0^P = \xi_0^E$ on B_n. Let $w = (\lfloor (n-1)/2 \rfloor, \lfloor (n-1)/2 \rfloor) \in B_n$ be a central point of B_n. Let T_w be the first time the dynamics ξ_t^E on $\mathbb{Z}^2$ occupies w. As T_w and T have the same distribution, it suffices to show that $T_w \ge n/(2\rho)$ with high probability. By translation invariance of ξ_∞^P,

$$\mathbb{P}\left(w \in \xi_\infty^P\right) = \mathbb{E}\left(\frac{|\xi_\infty^P|}{|B_n|}\right) \to 0,$$

by dominated convergence. If $w \notin \xi_\infty^P$, then $w \notin \xi_{\lfloor n/2\rho \rfloor}^E$, as boundary effects cannot travel faster than the "speed of light," that is, ℓ^1-distance ρ per update. Therefore,

$$\mathbb{P}(T_w \ge n/(2\rho)) \ge \mathbb{P}\left(w \notin \xi_\infty^P\right) \to 1,$$

ending the proof. □

For upper bounds, we frequently use the following two inequalities.

LEMMA 2.4. *If p is small enough, $\mathbb{P}(\text{Binomial}(n,p) \le np/2) \le e^{-np/7}$. If np is small enough, $\mathbb{P}(\text{Binomial}(n,p) > 0) \ge np/2$.*

On the other hand, we use the following consequence of the Markov inequality for lower bounds.

LEMMA 2.5. *If X is a sequence of random variables that depends on p, and a is a deterministic such sequence, then $\mathbb{E}X \ll a$ implies $X/a \to 0$ in probability. In particular, for $\gamma \in \mathbb{R}$, $\mathbb{E}X \lhd \gamma$ implies $X \lhd \gamma$.*

We often use color-coding to distinguished between sites that get occupied at different times in our various updating schemes. In all schemes, we consistently refer to initially occupied sites as *black*, and nonoccupied vertices as *white*.

3. Examples: all zero-sets with height and width at most 3. Up to reflection symmetry, there are 13 zero-sets that fit into the 3×3 box. Our methods provide powers in all 13 cases, and the results are below. Three cases are not completely covered by our results elsewhere in the paper, so we provide separate arguments, which also serve as simple illustrations of the methods we use.

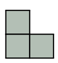
$\mathcal{Z} =$  This is the only voracious case, which is easily seen to have pure critical power $\gamma_c = 1/2$.

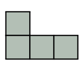
$\mathcal{Z} =$  The pure critical power is $\gamma_c = 2/3$ (Proposition 3.1).

$\mathcal{Z} =$  The pure critical power is $\gamma_c = 3/4$ (Proposition 3.1).

PROPOSITION 3.1. *The line growth with $1 = s \leq r$ has pure critical power $\gamma_c = r/(r+1)$.*

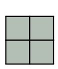
PROOF. If $n \gg p^{-r/(r+1)}$, that is, $n(np)^r \gg 1$, then with high probability the box B_n contains r initially nonempty neighboring vertical lines. Such a configuration spans B_n by time $2n$, so Lemmas 2.1 and 2.2 finish the proof of the upper bound on T. To prove the matching lower bound, consider the dynamics on B_n with periodic boundary. If no $(2\rho + 1) \times n$ vertical strip contains r initially nonempty vertical lines, no site in any initially empty vertical line gets occupied. This happens with high probability if $n \ll p^{-r/(r+1)}$, in which case also the expected number of sites on initially nonempty vertical lines is at most a constant times $n^2 \cdot np \ll n^2$. Lemma 2.3 then implies the lower bound. □

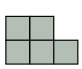
$\mathcal{Z} =$  This is bootstrap percolation with $r = 2$ and pure critical power $\gamma_c = 1$ (Theorem 1.1).

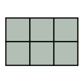
$\mathcal{Z} =$  The pure critical power is $\gamma_c = 1$ (Proposition 3.2).

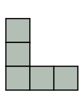
PROPOSITION 3.2. *The zero-set with minimal counts $(0,2)$, $(1,1)$ and $(r,0)$, with $r \geq 2$, has pure critical power $\gamma_c = 1$.*

PROOF. The lower bound on T follows by comparison with bootstrap percolation with $r = 2$, whose proof can also be adapted to get the upper bound. Indeed, if $n \gg p^{-1}$, the probability that a fixed $r \times n$ strip in B_n has a vertical line with two neighboring black (initially occupied) sites and the other $r - 1$ vertical lines each have at least one black site is $\gg p$. Therefore, the number of such strips in B_n is large with high probability, and one such strip will occupy the entire box by time $4n$, so the upper bound follows by Lemmas 2.1 and 2.2. □

$\mathcal{Z} =$  This case still has critical power $\gamma_c = 1$, but is no longer pure. From Theorem 1.3, we get that $T \asymp p^{-1} \log p^{-1}$.

$\mathcal{Z} =$  The critical power is $\gamma_c = 4/3$, and purity is unresolved (Proposition 7.6).

$\mathcal{Z} =$  This is line percolation with $r = 3$ and $s = 2$, so it has the same critical power $\gamma_c = 4/3$, with unresolved purity (Theorem 1.2).

$\mathcal{Z} =$  The pure critical power is $\gamma_c = 3/2$ (Theorem 1.4).

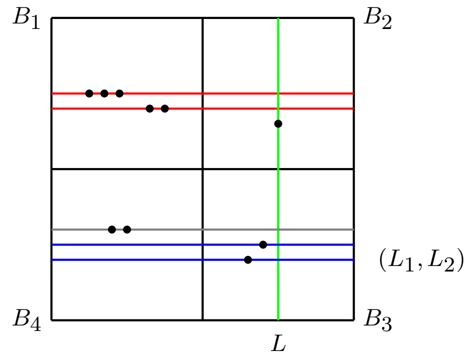
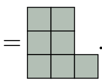
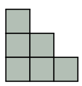
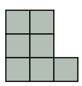


FIG 3. *Occupying 3 neighboring parallel horizontal lines for the zero set* $\mathcal{Z} =$ . *The red lines get occupied first, then the green line* L*, then the two blue lines* (L_1, L_2)*, and finally the grey line.*

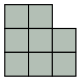
$\mathcal{Z} =$  This is bootstrap percolation with $r = 3$ and pure critical power $\gamma_c = 5/3$ (Theorem 1.1).

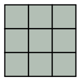
$\mathcal{Z} =$  The critical power is still $\gamma_c = 5/3$ and is still pure (Proposition 3.3).

PROPOSITION 3.3. *For the above zero-set,* $\gamma_c = 5/3$ *is the pure critical power.*

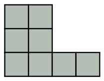
PROOF. The lower bound on T follows from bootstrap percolation with $r = 3$ (Theorem 1.1), so we only need to prove the upper bound.

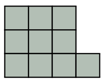
Assume $n \gg p^{-5/3}$. Divide the box B_{2n} into 4 congruent $n \times n$ boxes B_1 (top left), B_2 (top right), B_3 (bottom right), and B_4 (bottom left); see Figure 3. The probability of the event that there is a horizontal line with 3 contiguous black sites next to a horizontal line with 2 neighboring black sites, both in B_1, approaches 1. Conditioned on this event, there is at least one pair of fully occupied neighboring horizontal lines in B_2 (by time $4n$). Let V_1 be the number of vertical lines that contain at least one black site in the neighborhood of the pair. As $np \gg p^{-2/3}$, $V_1 \rhd 2/3$. We may also assume that these lines are at distance at least 2 from one another. All V_1 lines become occupied within B_{2n} by additional time $2n$. Let H_2 be the number of neighboring pairs of horizontal lines (L_1, L_2), such that there exists a line L among the V_1 eventually occupied vertical lines, and two non-collinear black sites in B_3 and in the neighborhood of L, one on L_1 and one on L_2. Then L_1 and L_2 become occupied within B_{2n} by additional time $2n$. We also assume that these pairs are separated by distance at least 1. As $p^{-2/3}np^2 \gg p^{-1/3}$, $H_2 \rhd 1/3$. Now consider the event that there are two neighboring black sites on a horizontal line in B_4 and in the neighborhood of one of the H_2 horizontal pairs. As $p^{-1/3}np^2 \gg 1$, this event happens with high probability, and results in 3 occupied neighboring horizontal lines by additional time $2n$. Then, by additional time $2n$, the entire B_{2n} is finally occupied. Thus, with high probability, B_{2n} is occupied by time $12n$, and Lemmas 2.1 and 2.2 finish the argument. □

$\mathcal{Z} =$  The critical power is $\gamma_c = 2$, and purity is unresolved (Proposition 7.6).

$\mathcal{Z} =$  This is symmetric line growth with $r = 3$ and critical power $\gamma_c = 2$ (Theorem 1.2), which we suspect is not pure due to Theorem 1.3, though purity is unresolved.

Two small examples for which our methods leave critical powers unresolved are the following.

$\mathcal{Z} =$  The lower and upper powers are $5/3$ and 2, which follow from Theorem 1.1 and part 1 of Theorem 1.4, respectively.

$\mathcal{Z} =$  The lower and upper powers are 2 and $9/4$, which follow from Theorem 1.2.

4. Bootstrap percolation.

4.1. *Upper bounds.* For an integer $k \geq 1$, call a row of the box B_n *k-filled* if there is a contiguous interval of length k consisting entirely of black sites. Assume $n \geq k$. If $np^k < 1$, the probability that a fixed row is k-filled is for small p between $\frac{1}{4k} \cdot np^k$ and np^k. If $np^k \geq 1$, this probability is at least $\frac{1}{4k}$, and it goes to 1 if $np^k \gg 1$.

A strip of r consecutive rows in B_n is *packed* if it has the following property: starting from the bottom of the strip, the first row is r-filled, the next row is $(r-1)$-filled, ..., and the top row is 1-filled. If a strip is packed, then it is easy to see that it is spanned in at most rn time steps and then the entire B_n is spanned in at most additional n time steps.

Assume $r \geq 2$, p is small, and $n \sim Cp^{-\gamma}$ for some large constant C. The probability that a fixed strip is packed is at least a constant times

$$np^r \cdot np^{r-1} \cdots np^{r-\overline{m}},$$

where $\overline{m} = \overline{m}(\gamma) \in [1, r]$ is the largest integer m such that $np^{r-m} < 1$, that is, such that $-\gamma + r - m > 0$. There are $\lfloor n/r \rfloor$ independent candidates for a packed strip in B_n. Therefore, with this choice of $\overline{m}$, the probability that there exists a packed strip is close to 1 as soon as

$$n \cdot np^r \cdot np^{r-1} \cdots np^{r-\overline{m}} = n^{\overline{m}+2} p^{(\overline{m}+1)(2r-\overline{m})/2} \tag{4.1}$$

is large. Let $\widehat{m}$ be the largest m for which

$$(r-m) - \frac{(m+1)(2r-m)}{2(m+2)} > 0,$$

that is,

$$-m^2 - 3m + 2r > 0.$$

Then we pick γ so that the power of p in (4.1) vanishes if $\overline{m}$ is replaced by $\widehat{m}$ (so that (4.1) is a large constant). These computations imply the following result.

LEMMA 4.1. *For $r \geq 2$ let*

$$\widehat{m} = \left\lceil \frac{\sqrt{9+8r}-5}{2} \right\rceil$$

and

$$\gamma = \frac{(\widehat{m}+1)(2r-\widehat{m})}{2(\widehat{m}+2)}. \tag{4.2}$$

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
γ_c	$\frac{1}{2}$	1	$\frac{5}{3}$	$\frac{7}{3}$	3	$\frac{15}{4}$	$\frac{9}{2}$	$\frac{21}{4}$	6	$\frac{34}{5}$	$\frac{38}{5}$	$\frac{42}{5}$	$\frac{46}{5}$	10	$\frac{65}{6}$	$\frac{35}{3}$	$\frac{25}{2}$
$\widehat{m}$	0	0	1	1	1	2	2	2	2	3	3	3	3	3	4	4	4

TABLE 1

The critical powers γ_c and the corresponding $\widehat{m}$ for small thresholds.

Then $n \gg p^{-\gamma}$ implies $\mathbb{P}(B_n$ spanned at time $(r+1)n) \to 1$. Consequently, $t \gg p^{-\gamma}$ implies $T \lesssim t$.

PROOF. We need to check that, for the choice of γ in (4.2), $\overline{m}(\gamma) = \widehat{m}$; this is equivalent to $-(\widehat{m}+1)^2 - 3(\widehat{m}+1) + 2r \le 0$, which holds by definition of $\widehat{m}$. The last statement follows from Lemmas 2.1 and 2.2. □

We will prove the matching lower bounds in the next subsection, so in fact γ is the critical power. Its values, together with the corresponding $\widehat{m}$'s, for small r are in Table 1. For large r, $\widehat{m} = \sqrt{2r} + \mathcal{O}(1)$, and consequently

$$\gamma = r - \sqrt{2r} + \mathcal{O}(1).$$

4.2. *Lower bounds.*

LEMMA 4.2. *For $r \ge 2$, let $\widehat{m}$ and γ be as in Lemma 4.1. Then $p^{-(\gamma-1)} \ll n \ll p^{-\gamma}$ implies $\mathbb{P}(B_n$ spanned$) \to 0$. Moreover, if $t \ll p^{-\gamma}$, then $\mathbb{P}(T \le t) \to 0$.*

We use the following growth dynamics as an upper bound for our process on B_n with periodic boundary. As will be the case throughout the paper, we use color-coding to keep track of vertices that get occupied at different times in our comparison dynamics. Recall that initially occupied points are black and non-occupied points are white. We will also use red and blue colors to mark vertices that are not occupied initially, but get occupied at later times. Fix an ordering of the vertices of B_n. At step ℓ, we examine the white vertices in order, and if a vertex has at least a combined number r of black and blue neighbors, then it is marked red, along with all points in the same row and column of B_n. At the end of a step, all red points switch to blue. Note that if a vertex is marked red before it is examined, then it will be skipped.

For an integer $m \ge 1$, we say that a collection of m rows (resp. columns) form an *m-clump* if they are within distance 3ρ of each other and they are fully blue. (The clumps are not necessarily disjoint, so for example every m-clump contains $\binom{m}{k}$ k-clumps for $k < m$.) Let $L_m^{(\ell)}$ denote the number of m-clumps of lines (rows or columns) after round ℓ. We let $L_0^{(\ell)} = n$. Let $\Delta L_m^{(\ell)} = L_m^{(\ell)} - L_m^{(\ell-1)}$ denote the number of new m-clumps added at step ℓ.

Let γ be given by (4.2). For $m \ge 0$, let

$$\beta(m) = \gamma(m+1) - \frac{m(2r-m+1)}{2} = \gamma(m+1) - rm + \frac{m(m-1)}{2}.$$

Recall that $\widehat{m}$ is the largest integer m for which $-\gamma + r - m > 0$. Also, $\beta(\widehat{m}+1) = 0$.

LEMMA 4.3. *For $0 \le m \le \widehat{m}$, $\beta(m) + m$ is a (strictly) decreasing function of m. Consequently, $\beta(m) + m \le \gamma$.*

PROOF. For $m < \widehat{m}$,

$$\beta(m) - \beta(m+1) = -\gamma + r - m = -\gamma + r - (m+1) + 1 > 1,$$

which, together with $\beta(0) = \gamma$, implies the two statements. □

LEMMA 4.4. *Assume $n \ll p^{-\gamma}$. For each $1 \leq \ell \leq \widehat{m} + 1$ and $0 \leq m \leq \widehat{m} + 1$,*

$$L_m^{(\ell)} \lhd \beta(m).$$

Moreover, for each $1 \leq \ell \leq \widehat{m} + 1$ and $1 \leq m \leq \ell$,

$$\Delta L_m^{(\ell)} \lhd \beta(\ell).$$

PROOF. Let $\mathcal{L}$ be a fixed (deterministic) set of red and blue (horizontal and vertical) lines in B_n. Condition on the event F that exactly they are colored at a certain point by the described comparison occupation dynamics. By FKG, under this conditioning, the probability of any increasing event E depending on the configuration outside $\mathcal{L}$ is smaller than the non-conditional probability of E. This is because the event F is decreasing as a function of the configuration on $B_n \setminus \mathcal{L}$.

We proceed by induction on ℓ. For $\ell = 1$, the expected number of m-clumps is at most a constant times $n \cdot (np^r)^m$ and so

$$\Delta L_m^{(1)} = L_m^{(1)} \lhd (m+1)\gamma - rm \leq \beta(m).$$

We now proceed to proving the $\ell \to \ell+1$ induction step, for $\ell \leq \widehat{m}$. An m-clump is created by adding $m - a$ parallel lines to some a-clump, for some $a \in [0, m-1]$. Each of those lines may receive help from a perpendicular b_i-clump, where $0 \leq b_i \leq r - a$, $i = 1, \ldots m - a$.

We now have two possibilities. The first is that the a-clump was created at step ℓ. By the induction hypothesis, for $a \leq \ell$, $\Delta L_a^{(\ell)} \leq \beta(\ell)$. As the ℓ-clumps offer the most help for expansion, we may assume that $a \geq \ell$ and so $m \geq \ell + 1$. For fixed a and b_i, we use our observation on negative correlations due to the FKG inequality and conditioning on all $L_k^{(\ell')}$ and $\Delta L_k^{(\ell')}$ for $\ell' \leq \ell$, to get that, with probability tending to 1 as $C \to \infty$,

$$\Delta L_m^{(\ell+1)} \leq C \Delta L_a^{(\ell)} \prod_i L_{b_i}^{(\ell)} p^{r-a-b_i}.$$

It follows from the induction hypothesis that

$$\begin{aligned}
\Delta L_m^{(\ell+1)} &\lhd \max_{a,b_i} \left[\beta(a) + \sum_i (a - r + b_i + \beta(b_i))\right] \\
&= \max_{a,b_i} \left[\beta(a) + (m-a)(a-r) + \sum_i (b_i + \beta(b_i))\right] \\
&= \max_a \left[\beta(a) + (m-a)(a - r + \gamma)\right] \\
&= \max_a \left[\beta(m) - \frac{1}{2}(m-a)(m-a-1)\right] \\
&= \beta(m).
\end{aligned}$$

Here, we used Lemma 4.3 on the third line, and the observation that the maximum on the penultimate line is achieved at $a = m - 1$.

The second possibility is that *every* b_i-clump was created at step ℓ. In this case, the inequality that holds with probability tending to 1 as $C \to \infty$ is

$$\Delta L_m^{(\ell+1)} \le C L_a^{(\ell)} \prod_i \Delta L_{b_i}^{(\ell)} p^{r-a-b_i}.$$

We may assume that $a \le \widehat{m}$, as it follows from the induction hypothesis that there is no $(\widehat{m}+1)$-clump at step ℓ. When $m \ge \ell$,

$$\begin{aligned}
\Delta L_m^{(\ell+1)} &\lhd \max_{a,b_i} \left[\beta(a) + \sum_i (a - r + b_i + \beta(\max(b_i,\ell)))\right] \\
&= \max_{a,b_i \ge \ell} \left[\beta(a) + \sum_i (a - r + b_i + \beta(b_i))\right] \\
&= \max_{a,b_i \ge \ell} \left[\beta(a) + (m-a)(a-r) + \sum_i (b_i + \beta(b_i))\right] \\
&= \max_a \left[\beta(a) + (m-a)(a - r + \ell + \beta(\ell))\right] \\
&\le \max_a \left[\beta(a) + (m-a)(a - r + \gamma)\right] \\
&= \beta(m),
\end{aligned}$$

as before. One the other hand, when $m \le \ell$, we join the above computation on the fourth line, and use $\beta(a) \le \gamma - a$, to get

$$\Delta L_m^{(\ell+1)} \lhd \max_a \left[\gamma - a + (m-a)(a - r + \ell + \beta(\ell))\right].$$

Differentiating the expression inside the maximum with respect to a, we obtain

$$\begin{aligned}
m + r - 2a - \beta(\ell) - \ell - 1 &\ge m + r - 2(m-1) - \gamma - 1 \\
&= r + 1 - m - \gamma \\
&\ge r - \widehat{m} - \gamma > 0,
\end{aligned}$$

which implies that the maximum is obtained at $a = m - 1$, and then

$$\begin{aligned}
\Delta L_m^{(\ell+1)} &\lhd \gamma - m + 1 + m - 1 - r + \ell + \beta(\ell) \\
&= \gamma - r + \ell + \beta(\ell) \\
&= \beta(\ell + 1).
\end{aligned}$$

This completes verification of the induction step and ends the proof. □

PROOF OF LEMMA 4.2. By Lemma 4.4, $\Delta L_1^{(\widehat{m}+1)} = 0$ with probability tending to 1 as $p \to 0$, which implies that our comparison dynamics stops with high probability at step $\widehat{m}$, which is independent of p. By Lemma 4.4 and Lemma 4.3, $L_1^{(\widehat{m})} \lhd \gamma - 1$ and so $L_1^{(\widehat{m})} \ll p^{-(\gamma-1)} \ll n$ with high probability. Therefore, after the step $\widehat{m}$, the number of blue points is $o(n^2)$ with high probability. The number of black points has expectation pn^2 and is therefore also $o(n^2)$ with high probability. This proves the first statement and Lemma 2.3 proves the second one. □

PROOF OF THEOREM 1.1. The conclusion follows immediately from Lemmas 4.1 and 4.2. □

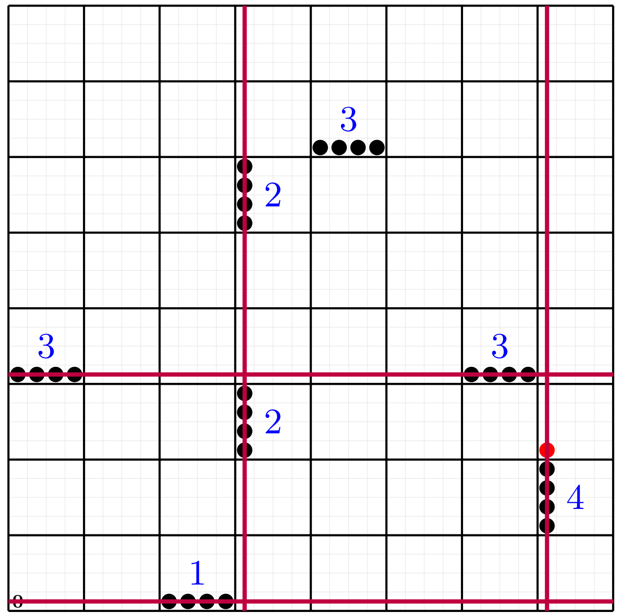


FIG 4. *An example of successful occupation of* **0** *in our branching construction for* $r = 5$ *and* $n = 8$*. The black circles are occupied in the basic configuration, as detected by the branching construction; the blue numbers indicate the generation at which a lot is marked. The one red circle is occupied in the sprinkled configuration. The purple lines are successively occupied in the order* $4 - 3 - 2 - 1$.

5. Symmetric line growth: general upper bound.

LEMMA 5.1. *Assume* $r \geq 2$*. If* $t \gg p^{-(r-1)} \log \frac{1}{p}$*, then* $T \lesssim t$*. Moreover, if* $n \gg p^{-(r-1)}$*, then* B_n *is spanned with high probability.*

PROOF. Let $n = A/p^{r-1}$. Divide the $n(r-1) \times n(r-1)$ box S with **0** at its lower left corner into $(r-1) \times (r-1)$ subboxes, which we call *lots*. A lot is *bottom-filled* (resp., *left-filled*) if its bottommost row (resp., leftmost column) is all black. A bottom-filled (resp., left-filled) lot is a *nucleus* if the bottommost row (resp., leftmost column) is part of a horizontal (resp., vertical) interval of r black sites. Within this proof, when we refer to columns comprised of lots in S simply as columns, and analogously for rows. Thus S has n rows and columns.

In the remainder of the proof, we consider the density of black sites to be $2p$ instead of p, which does not affect our claim. The black sites thus stochastically dominate the union of two independent configurations, each with density p. We call the two the *basic* and the *sprinkled* configuration. Until further notice, all events will be with respect to the basic configuration.

We will create a set of marked lots by the following sequential procedure. At each step, in addition to newly marking lots, we enlarge a set of rows and columns that are removed from future consideration and are called *stamped*. Initially, no lots are marked and we stamp the rightmost column and the topmost row.

In the first step, we inspect the bottom row. If, excluding the lot in the stamped column, it contains no bottom-filled lots, the process stops and the set of marked lots is empty. Otherwise, we mark all bottom-filled lots in this row. We then stamp the bottom row. If the process has not stopped, we proceed to the second step, when we consider exactly the columns immediately to the right of the newly marked lots. Inspect the columns from left to right. In any column under inspection, mark the left-filled lots that are not in any stamped row and also not in the same row with any already marked lots. If no such left-filled lots exist, leave that column without marked lots. In any case, stamp all the inspected columns.

In the third step, consider rows which are just above the lots marked on the second step, from bottom to top. For each of these rows, do the analogous procedure as on the previous step: mark all bottom-filled lots within it that are not in any stamped columns and not in the same column as any previously marked lot, and stamp it.

Continue with this procedure, alternating rows and columns, until one of two contingencies. If in some step no new lots are marked, we stop after that step. On the other hand, if at some point the number of marked lots exceeds $n/5$, we stop immediately, even if this happens within a step that is not yet complete. Observe that, by our construction, the column immediately to the right of any newly marked lot in a row is not stamped before being inspected, and the analogous property holds for the rows.

If, at some point in the procedure, the number of marked lots is m, and we proceed to inspect a row or a column, we check at least $n-2m$ lots and each of them is marked independently with probability p^{r-1}. If $m \le n/5$, the expected number of marked lots is at least $A/2$. Therefore, if A is large enough, the marking process dominates a supercritical branching process up to stopping. It follows that, for any ϵ, we can choose a large enough A so that the procedure reaches $n/5$ marked lots in $\log n$ steps with probability at least $1-\epsilon$. We now condition on this event.

The key observation is that marked lots transfer the occupation to the origin, provided at least one of them is "triggered" by an additional black site; see Figure 4. To be more precise, if any of the marked lots is a nucleus, the origin becomes occupied by time $rn \log n$.

The final step in our proof is a standard sprinkling argument. Under our conditioning, the union of the basic and sprinkled black sites generate a nucleus with probability at least $1-(1-p)^{n/5} > 1-\epsilon$, when A is large enough.

To establish the property of T, for a large enough constant A and small enough p, the probability that the origin is occupied by time $rn\log n \le 2Ar^2 p^{-(r-1)} \log \frac{1}{p}$ is at least $1-\epsilon$.

By another sprinkling argument, using the same construction, we can make r adjacent lines in B_n occupied with probability at least $(1-\epsilon)^r$, which results in B_n being spanned. □

6. Lower bound for the symmetric line growth with $r = 2$.

In this case, we have a lower bound that asymptotically matches the bound from Lemma 5.1. For $r > 2$, we prove a weaker bound of the form $p^{-(r-1)+o(1)}$ in Section 7.3.

LEMMA 6.1. *Assume $r = 2$. If $t \ll p^{-1} \log \frac{1}{p}$, then $T \gtrsim t$.*

PROOF. A *neighborhood path* is a sequence $x_0, x_1, \dots, x_n \in \mathbb{Z}^2$ of length n, starting at x_0 which makes neighborhood steps, that is, $x_i - x_{i-1} \in \mathcal{N}_0$, $1 \le i \le n$. We say that the path makes a *turn* at i, $1 \le i \le n-1$, if the vectors $x_i - x_{i-1}$ and $x_{i+1} - x_i$ are perpendicular. We call such a path *transmissive* if:

(1) x_0 has two black sites in its neighborhood;
(2) at every turn i, there is a black site on a horizontal or vertical line through x_i within distance 2ρ;
(3) all black sites in (1) and (2) are distinct; and
(4) all steps are of size 1, and between any two turns the steps are all in the same coordinate direction and orientation.

At any time, call a line *saturated* within a box at a given time if it contains two or more occupied sites within a neighborhood of one of its sites inside of the box. When a point becomes occupied at some time $t > 0$, we call it *red*. Thus, every occupied point at time t is either red or black, but not both.

Step 1. Assume that there are no parallel saturated lines within distance 5ρ of each other at time t for the dynamics restricted to $[-2\rho t, 2\rho t]^2$, with 0-boundary, by time t. Assume also that $\mathbf{0}$ is occupied by time t. Then we claim that there exists a transmissive path ending at 0 of length at most ρt.

Assume that x gets occupied at time $t > 0$. We will first prove by induction that there is a neighborhood path of length at most t, ending at x, that satisfies properties (1) and (2), but not necessarily (3) and (4), and is contained entirely in lines that are saturated within $[-2\rho t, 2\rho t]^2$ by time $t - 1$.

When $t = 1$, the path consists of only the site x. We now proceed to proving the $t - 1 \to t$ induction step, for $t \geq 2$.

At time $t - 1$, x has two occupied neighbors, say y_1 and y_2, so that y_1, y_2, x are all on the same line, and so that y_1 and y_2 are not both black. If, say, y_1 is black and y_2 is red, then we use the induction hypothesis on y_2, take the resulting neighborhood path that ends at y_2 and append the step from y_2 to x. It is possible that the so constructed path makes a turn at y_2, but then (2) holds.

Now assume both y_1 and y_2 are red. By the induction hypothesis, there exist neighborhood paths that satisfy (1) and (2) and end at y_1 and y_2, with penultimate sites z_1 and z_2. Here, we add the proviso that, if y_1 is occupied at time 1, z_1 is one of the requisite black sites, and similarly for y_2. If, say, $x - y_1$ is parallel to the last step $z_1 - y_1$ of the path to y_1, we may complete the induction step by appending x to the path to y_1. We proceed analogously if $x - y_2$ is parallel to the last step $z_2 - y_2$. In all cases so far, the saturation claim holds for the new path to x. The last possibility is that both $z_1 - y_1$ and $z_2 - y_2$ are perpendicular to $x - y_1$, and thus parallel. By induction, they lie on parallel saturated lines, and are within distance 2ρ, in contradiction with our assumption.

We thus constructed a path that satisfies (1) and (2), and we proceed to modify it to satisfy the other two requirements. We satisfy (4) by simply replacing all steps between two turns by steps of length 1 on the same line and with the same orientation. Now, if our path happens to make a turn at i, and $x_i = x_j$ for some $i < j$, we eliminate the loop $x_{i+1}, \dots, x_j$ from the path. Similarly, we may eliminate a loop if $x_i = x_0$ for some $i > 0$. (However, we cannot make the path self-avoiding, as we cannot eliminate loops if two sites coincide at non-turns.)

Finally, suppose that the path makes turns at $i < j$, and x_i and x_j are at distance $5k$ or less. Then we have two parallel saturated lines within this distance, which contradicts our assumption. The same holds if there is a turn within distance 5ρ of x_0. It follows that (3) is also satisfied, and the claim in Step 1 is proved.

For future use, we consider two distinct red sites x and y that are occupied by time t. We have two transmissive paths (x_i) and (y_j), one to x and the other to y. If the path to y makes a turn at some j so that $y_j = x_i$ is on the path to x, then we replace to $x_0, \dots, x_i$ with the path to $y_0, \dots, y_j$. Analogously if we switch the roles of the two paths or if x_0 or y_0 are on the other path. After this, we say the two paths are *m-merged* if $x_i = y_i$ for $i \leq m$. In addition, we also declare them *0-merged* if x_0 and y_0 are within distance $5k\rho$.

The result is that now any two turns that occur at different sites use distinct black sites, and any turns use different sites than the initial sites. If the paths are not 0-merged, the initial sites use different black sites as well.

Take time $t = c(1/p)\log(1/p)$. We will choose the constant $c = c(\rho)$ to be small enough. Let F_n, $n \geq 0$, be the event that there is a transmissive path of length n ending at $\mathbf{0}$.

Step 2. If c is small enough, $\mathbb{P}\left(\cup_{n \leq \rho t} F_n\right) \to 0$ as $p \to 0$.

Dividing according to the number i of turns this path makes, we get that, for some constant $C = C(\rho)$,

$$\mathbb{P}(F_n) \le C \sum_{i=0}^{n-1} \binom{n}{i} (Cp)^i p^2 \le C(1+Cp)^n p^2 \le Ce^{Cnp} p^2.$$

Therefore,

$$\mathbb{P}(\cup_{n \le \rho t} F_n) \le Cte^{Ctp} p^2 \le Cp^{1-cC} \log(1/p),$$

which establishes the claim in Step 2.

Next we address the probability that there are two parallel saturated lines within distance 5ρ for the dynamics restricted to $[-2\rho t, 2\rho t]^2$, with 0-boundary, by time t. Consider the first such pair of lines. Each of these two lines must have, at the saturation time two occupied sites, at least one of which is black, within the neighborhood of a site. Therefore, there are two transmissive paths, one for each line, that each end at a site on the respective line, which has an additional black site within distance 2ρ. Let G_1 be the event that the two paths do not merge for this first pair of parallel lines.

Step 3. For c small enough, $\mathbb{P}(G_1) \to 0$.

As black sites on all turns of both paths are distinct,

$$\mathbb{P}(G_1) \le Ct(tp)^2 (te^{Ctp} p^2)^2 = Cp(tp)^5 e^{2Ctp},$$

reflecting the selection of the pair of lines, choosing the position of the pair of occupied sites on each line, one of them being black, and then the choice of non-merged transmissive paths. The claim in step 3 follows.

Finally, we consider the event G_2 that the two paths do merge for the described first pair of parallel lines.

Step 4. For c small enough, $\mathbb{P}(G_2) \to 0$.

Fix two sites y and z, which do not lie on the same line, but are otherwise arbitrary. The key is to find an upper bound for the probability that there is a neighborhood path that satisfies (2)–(4) of length exactly n connecting y and z. Consider first the case when $n > ||y - z||_1$. Then the path makes $i \ge 2$ turns, and there are some two successive turns at which the path makes a U-turn. If the locations of all other $i - 2$ turns are fixed, there are at most 2 (symmetric) possibilities for locations of these two turns; all other locations change the length. So the upper bound on the probability is

$$\sum_{i=2}^{n} \binom{n}{i-2} i(Cp)^i \le Cp^2(np)e^{Cnp}, \tag{6.1}$$

reflecting the choice of the number of turns, position of the first U-turn, and positions of the black sites at each turn.

Now consider the case $n = ||y - z||_1$. Now the number of turns is $i \ge 1$, and we merely observe that after the choice of first $i - 1$ turns, there is at most one possibility for the last turn, which gives the upper bound

$$\sum_{i=1}^{n} \binom{n}{i-1} (Cp)^i \le Cpe^{Cnp}, \tag{6.2}$$

as we can now only choose the positions of $i - 1$ turns.

If the first transmissive path makes i turns, the second transmissive path can join it at i locations. By choosing which path is the second one (i.e., the one that joins) we can also make sure that the initial point of the second path and the joining location are not on the same line. Putting all this together, we sequentially make the choices of: the pair of close parallel saturated lines; the pair of occupied sites, with one black, on each line; length of the first path; number i of turns of the first path; positions of the i turns on the first path; positions of black sites that make the first path transmissive; and locations where the second path joins. For the probability bound that the second path contributes, we use (6.1) and (6.2). This gives the following upper bound:

$$\begin{aligned}\mathbb{P}(G_2) &\leq Ct(t^2p^2)\cdot t\sum_{i=0}^{\rho t}\binom{\rho t}{i}p^ip^2i\cdot(tp^2(tp)e^{Ctp}+pe^{Ctp})\\ &\leq Ct(t^2p^2)\cdot t(pt)e^{Cpt}p^2\cdot(tp^2(tp)e^{Ctp}+pe^{Ctp})\\ &= C(tp)^5\cdot e^{2Cpt}\cdot((tp)^2+1)\cdot p.\end{aligned}$$

This again goes to 0 if c is small enough.

From Steps 2–4, for small enough c,

$$\mathbb{P}(\mathbf{0}\text{ is occupied at time }t)\leq\mathbb{P}((\cup_nF_n)\cup G_1\cup G_2)\to 0,$$

as desired. □

PROOF OF THEOREM 1.3. The scaling of T follows from Lemmas 5.1 and 6.1.

We need to prove the conclusion for the critical length, L_c. Lemma 5.1 gives the upper bound $L_c=\mathcal{O}(p^{-1})$. On the other hand, if $n\ll p^{-1}$, the initial configuration in B_n with 0-boundary is inert with high probability. It follows that $L_c=\Theta(1/p)$. □

7. General upper and lower bounds for the line growth. In this section we establish the possibly non-pure critical power for the line growth dynamics in which a site becomes newly occupied by having either r horizontal or s vertical occupied neighbors, with $2\leq s\leq r$. We denote

$$\gamma=\frac{(r-1)s}{r},$$

the claimed critical power.

7.1. *Upper bound in the asymmetric case.* In this subsection we assume that $2\leq s<r$.

LEMMA 7.1. *Assume that $n\gg p^{-(\gamma+\epsilon)}$ for some $\epsilon>0$. Then there exists a large enough constant $C>0$ such that the dynamics spans B_n by time Cn with high probability.*

We consider the following comparison process, evolving in steps $\ell=1,2\ldots$. Divide the $n\times n$ box into vertical strips of width r and horizontal strips of width s. Call a vertical (resp. horizontal) line *saturated* if it contains s (resp. r) occupied sites, all in the same horizontal (resp. vertical) strip. Any step ℓ consists of two half-steps. When $\ell=1$, the first half step occupies, in every vertical strip, the maximal contiguous interval of saturated vertical lines that includes the leftmost line. In the first half step for $\ell\geq 2$, in each vertical strip, consider for occupation only the leftmost vertical line that is not already occupied: if that line is saturated, make it occupied; otherwise, that strip remains unchanged. In the second half-step, for any $\ell\geq 1$, we only consider the bottom-most unoccupied line in each horizontal strip: if that line is saturated, we occupy that line; otherwise, there are no changes in that strip. After the ℓth step, let $V_m^{(\ell)}$ be the number of vertical strips with at least m occupied lines. Also, let $H^{(\ell)}$ be the number of occupied horizontal lines after the ℓth step.

We stop the process after completion of step $\hat{\ell}$ where

$$\hat{\ell} := \min\{\ell : (\ell r + 1)\epsilon \geq s/r\}.$$

LEMMA 7.2. *For a small enough $\epsilon > 0$ and for $\ell \leq \hat{\ell}$,*

$$V_{r-2}^{(\ell)} \rhd \frac{s}{r} + (r-1)\epsilon$$

$$V_{r-1}^{(\ell)} \rhd \ell r\epsilon$$

$$H^{(\ell)} \rhd \ell r\epsilon + \epsilon + s - \frac{s}{r} - 1.$$

PROOF. We prove that the lower power bound on $V_{r-2}^{(\ell)}$ holds at $\ell = 1$, and the other two bounds are proved by induction on ℓ. As we perform only a bounded number of steps, we may (by a standard sprinkling argument) assume that the configuration of black sites is, at every half step, an independent product measure off already occupied lines. For $\ell = 1$, $V_m^{(1)}$ is with high probability larger than a constant times

$$n(np^s)^m = n^{m+1}p^{sm} \gg p^{-((\gamma+\epsilon)(m+1)-sm)},$$

whenever the power of p is non-positive, and so

$$V_m^{(1)} \rhd s - \frac{s}{r} + \epsilon + m\left(-\frac{s}{r} + \epsilon\right).$$

In particular, we get the desired two lower power bounds on $V_{r-2}^{(1)}$ and $V_{r-1}^{(1)}$. For $H^{(\ell)}$, we will consider help from occupied vertical strips with at least $(r-1)$ occupied lines through the first half step of step ℓ. Thus, we need a single black site next to the occupied $(r-1)$ vertical lines within a vertical strip to saturate a horizontal line. In particular, $H^{(1)}$ is with high probability at least a constant times

$$p^{-\epsilon r}np \gg p^{-(\gamma+\epsilon+\epsilon r-1)},$$

and so

$$H^{(1)} \rhd r\epsilon + \epsilon + s - \frac{s}{r} - 1.$$

For the $\ell \to \ell + 1$ inductive step, we need to bound from below the probability that a fixed vertical strip that contains $r-2$ occupied vertical lines occupies an additional line, with the help of occupied horizontal lines. Using the lower bound on $H^{(\ell)}$ from the induction hypothesis and $s-1$ additional black points in a horizontal strip, we get a lower bound on this probability of at least a constant times

$$p^{-(\ell r\epsilon+\epsilon+s-\frac{s}{r}-1)+s-1}.$$

The stopping rule $\ell \leq \hat{\ell}$ ensures that the power of p above is positive, so our lower bound for this probability is small. This gives

$$V_{r-1}^{(\ell+1)} \rhd \frac{s}{r} + (r-1)\epsilon + \ell r\epsilon + \epsilon + s - \frac{s}{r} - 1 - (s-1),$$

which gives the desired lower power bound for $V_{r-1}^{(\ell+1)}$. We use this bound in the subsequent half step to bound $H^{(\ell+1)}$. Observe that the stopping rule $\ell \leq \hat{\ell}$ guarantees that

$$(\ell+1)r\epsilon - 1 < \frac{s}{r} - 1 + (r-1)\epsilon < 0,$$

so our lower bound on the probability that a fixed horizontal line gets occupied is small. This gives

$$H^{(\ell+1)} \triangleright (\ell+1)r\epsilon + \gamma + \epsilon - 1,$$

again confirming the induction step. □

PROOF OF LEMMA 7.1. After our process stops, the probability of occupation of a vertical line in our comparison process is bounded from below by a strictly positive constant. It takes the original dynamics at most Cn steps to occupy all vertices in the lines occupied by the comparison dynamics through step $\hat{\ell}$. The original dynamics, then, completely occupies a vertical strip in at most n additional steps with high probability. Then the entire box B_n is occupied in at most n additional steps. □

7.2. *Lower bound in the asymmetric case.* In this subsection we again assume that $2 \le s < r$ and $\gamma = (r-1)s/r$.

LEMMA 7.3. *Assume that $p^{-\gamma+\delta} \ll n \ll p^{-\gamma}$. For small enough $\delta > 0$, the dynamics on B_n with periodic boundary has $|\xi_\infty| = o(n^2)$ with probability converging to* 1.

We now call a vertical (resp. horizontal) line *saturated* if it contains s (resp. r) occupied points within a neighborhood. We now, at every step ℓ, perform the following two half steps. In the first half-step, we occupy all saturated vertical lines and in the second half-step, all horizontal saturated lines. Recall the definition of an m-clump from Section 4.2. After the ℓth step is completed, we let $V_m^{(\ell)}$ and $H_m^{(\ell)}$ be the numbers of vertical and horizontal clumps of size at least m, with $V_m^{(0)} = H_m^{(0)} = 0$, for $m > 0$ and $V_0^{(0)} = H_0^{(0)} = n$. Also let $\Delta V_m^{(\ell)} = V_m^{(\ell)} - V_m^{(\ell-1)}$ and $\Delta H_m^{(\ell)} = H_m^{(\ell)} - H_m^{(\ell-1)}$.

LEMMA 7.4. *Assume that $n \ll p^{-\gamma}$. For all $\ell \ge 1$,*

$$\begin{aligned}
V_m^{(\ell)} &\triangleleft m\left(-\frac{s}{r}\right) + s - \frac{s}{r} \\
H_m^{(\ell)} &\triangleleft m\left(\frac{s}{r} - 2\right) + s - \frac{s}{r} \\
\Delta V_m^{(\ell)} &\triangleleft m\left(-\frac{s}{r}\right) + s - \frac{s}{r} + (\ell-1)\left(\frac{s}{r} - 1\right) \\
\Delta H_m^{(\ell)} &\triangleleft m\left(\frac{s}{r} - 2\right) + s - \frac{s}{r} + (\ell-1)\left(\frac{s}{r} - 1\right).
\end{aligned}$$

Moreover, $V_{r-1}^{(\ell)} = 0$ with high probability.

PROOF. We proceed by induction. Observe that, as $s/r < 1$, the upper power bounds for $\Delta V_m^{(\ell)}$ and $\Delta H_m^{(\ell)}$ are less than or equal to the respective bounds for $V_m^{(\ell)}$ and $H_m^{(\ell)}$. For $\ell = 1$, the number of vertical m-clumps is bounded by a constant times

$$n(np^s)^m \ll p^{-\gamma(m+1)+sm}$$

and therefore

$$V_m^{(1)} \triangleleft m(\gamma - s) + \gamma =: \beta(m).$$

To bound the number of horizontal m-clumps, we employ an optimization scheme described below.

Similarly as in Section 4.2, we use FKG in the following fashion. Given a fixed (deterministic) set $\mathcal{H}$ of, say, horizontal lines, condition on exactly them being occupied by the procedure so far. By FKG, under this conditioning, the probability of any increasing event E depending on the configuration outside $\mathcal{H}$ is smaller than the non-conditional probability of E.

Any horizontal m-clump consists of m lines enumerated, say, from the lowest line in the clump. The line i may be saturated by some vertical a_i-clump together with $r - a_i$ nearby black points on this line; here, $0 \le a_i \le r-2$, since $V^{(1)}_{r-1} = 0$ with high probability. Thus the number of horizontal m-clumps with helping numbers $a_1, \dots, a_m$ is with high probability bounded above by a constant times

$$n \cdot p^{-\beta(a_1)} p^{r-a_1} \cdots p^{-\beta(a_m)} p^{r-a_m}.$$

Let $a = a_1 + \cdots + a_m$. Then we get

$$H^{(1)}_m \lhd \max_a (\gamma - mr + \gamma m + (\gamma - s + 1)a).$$

As $\gamma - s + 1 > 0$, the maximum is achieved at $a = m(r-2)$ (that is, all $a_i = r-2$). It follows that

$$H^{(1)}_m \lhd \gamma - mr + \gamma m + (\gamma - s + 1)m(r-2) = m\left(\frac{s}{r} - 2\right) + \gamma =: \alpha(m).$$

For the $\ell \to \ell + 1$ induction step, let $\beta'(m)$ and $\alpha'(m)$ be the right-hand side of the upper power bounds for $\Delta V^{(\ell)}_m$ and $\Delta H^{(\ell)}_m$.

To get a new vertical m-clump at step $\ell + 1$, pick a b and add $m - b$ vertical lines to an existing vertical b-clump, $0 \le b \le m-1$. Let $a_1, \dots, a_{m-b}$ be the sizes of helping horizontal clumps for lines to be saturated $1, \dots, m-b$, counted from the left. The number of such new m-clumps is with high probability at most constant times

$$p^{-\beta(b)} \cdot p^{-\alpha'(a_1)} p^{s-a_1} \cdots p^{-\alpha'(a_{m-b})} p^{s-a_{m-b}}.$$

Let $a = \sum_i a_i$. Observe that $a_i \ge 1$ for all i since $\ell \ge 1$, so $a \ge m - b$. We get

$$\begin{aligned}
\Delta V^{(\ell+1)}_m \lhd \max_{a,b}\ & b(\gamma - s) + \gamma - s(m-b) + a + a\left(\frac{s}{r} - 2\right) \\
& + (m-b)\left(\gamma + (\ell - 1)\left(\frac{s}{r} - 1\right)\right) \\
&= \max_{a,b}\ a\left(\frac{s}{r} - 1\right) + (m-b)(\ell-1)\left(\frac{s}{r} - 1\right) - m\frac{s}{r} + \gamma \\
&= \max_{b}\ (m-b)\left(\frac{s}{r} - 1\right) + (m-b)(\ell-1)\left(\frac{s}{r} - 1\right) - m\frac{s}{r} + \gamma \\
&= \max_{b}\ (m-b)\ell\left(\frac{s}{r} - 1\right) - m\frac{s}{r} + \gamma \\
&= \ell\left(\frac{s}{r} - 1\right) - m\frac{s}{r} + \gamma,
\end{aligned}$$

as claimed.

Proceeding to formation of horizontal clumps, we similarly get, with $0 \le b \le m-1$ and $1 \le a_i \le r-2$, $i = 1, \dots, m-b$, $a = \sum_i a_i$, and noting that we must use $\beta'(a_i)$ with ℓ

replaced by $\ell+1$,

$$\begin{aligned}\Delta H_m^{(\ell+1)} &\lhd \max_{a_i,b}\ \alpha(b)+\sum_i \beta'(a_i)+\sum_i a_i - r(m-b)\\ &= \max_{a,b}\ a\left(1-\frac{s}{r}\right)+(m-b)\left(\gamma+\ell\left(\frac{s}{r}-1\right)-r-\frac{s}{r}+2\right)+m\left(\frac{s}{r}-2\right)+\gamma\\ &= \max_{b}\,(m-b)(r-2)\left(1-\frac{s}{r}\right)+(m-b)\left(\gamma+\ell\left(\frac{s}{r}-1\right)-r-\frac{s}{r}+2\right)\\ &\qquad\qquad +m\left(\frac{s}{r}-2\right)+\gamma\\ &= \max_{b}\,(m-b)\ell\left(\frac{s}{r}-1\right)+m\left(\frac{s}{r}-2\right)+\gamma\\ &= \ell\left(\frac{s}{r}-1\right)+m\left(\frac{s}{r}-2\right)+\gamma,\end{aligned}$$

again verifying the inductive claim. □

PROOF OF LEMMA 7.3. It follows from Lemma 7.4 that, after a bounded number ℓ of steps, $\Delta V_1^{(\ell)} \lhd 0$ and $\Delta H_1^{(\ell)} \lhd 0$. Therefore the configuration after ℓ steps is with high probability inert. As $V_1^{(\ell)} \lhd \gamma - s/r < \gamma$, for small enough $\delta > 0$, the final configuration occupies $o(n^2)$ sites, and Lemma 2.3 applies. □

7.3. *The general lower bound for the symmetric case..* In this section, we assume $s = r \geq 2$, so that $\gamma = r-1$.

LEMMA 7.5. *Fix an $\epsilon \in (0,1/2)$. Assume that $p^{-(r-3/2-\epsilon)} \ll n \ll p^{-(r-1-\epsilon)}$. Then the dynamics on B_n with periodic boundary has $|\xi_\infty| = o(n^2)$ with probability converging to 1.*

PROOF. Let $\gamma' = r-1-\epsilon$. We now at each step occupy all r-saturated horizontal and vertical lines simultaneously. Then we have, as in the asymmetric case,

$$H_m^{(1)}, V_m^{(1)} \lhd m(\gamma'-r)+\gamma' =: \alpha(m)$$

so that both numbers vanish with high probability for $m \geq r-1$. Now we claim that

$$\Delta H_m^{(\ell)}, \Delta V_m^{(\ell)} \lhd m(\gamma'-r)+\gamma'-(\ell-1)\epsilon =: \alpha'(m)$$

for all $\ell \geq 1$.

To prove the claim by induction, we again consider ways to get a new vertical m-clump at step $\ell+1$. We pick a b and add $m-b$ vertical lines to an existing vertical b-clump, $0 \leq b \leq m-1$, and let $a_1,\dots,a_{m-b}$ be the helping horizontal clumps for line $1,\dots,m-b$, as before. The number of such new m-clumps is with high probability at most constant times

$$p^{-\alpha(b)}\cdot p^{-\alpha'(a_1)}p^{r-a_1}\cdots p^{-\alpha'(a_{m-b})}p^{r-a_{m-b}}.$$

Let $a = \sum_i a_i$. Observe that $1 \leq a_i \leq r-2$ for all i. We get

$$\begin{aligned}\Delta V_m^{(\ell+1)} &\lhd \max_{a,b}\ b(\gamma'-r)+\gamma'-r(m-b)+a+a\left(\gamma'-r\right)+(m-b)\left(\gamma'-(\ell-1)\epsilon\right)\\ &= \max_{a,b}\ -a\epsilon-(m-b)(\ell-1)\epsilon+m(\gamma'-r)+\gamma'\end{aligned}$$

$$= \max_b -(m-b)\epsilon - (m-b)(\ell-1)\epsilon + m(\gamma' - r) + \gamma'$$

$$= \max_b -(m-b)\ell\epsilon + m(\gamma' - r) + \gamma'$$

$$= -\ell\epsilon + m(\gamma' - r) + \gamma',$$

as claimed. The proof is concluded as that of Lemma 7.3. □

PROOF OF THEOREM 1.2. The conclusion follows from Lemmas 7.1 and 7.3 for the case $s < r$ and from Lemmas 5.1, and 7.5 for the case $s = r$. □

7.4. *The perturbed line growth.*

PROPOSITION 7.6. *Assume $2 \le s \le r$ and remove the $(r-1, s-1)$ square from the line-growth Young diagram of Theorem 1.2 to get the zero-set with minimal counts $(0,s)$, $(r,0)$ and $(r-1,s-1)$. Then $\gamma_c = (r-1)s/r$ remains the same as in Theorem 1.2.*

PROOF. We will address the asymmetric case $s < r$ first, proving that the lower bound in Lemma 7.3 still holds. We join the proof of Lemma 7.4, with the corresponding definitions of α and β, and claim that the final configuration is still with high probability inert, which clearly suffices to verify this claim.

Fix $k \ge 1$ and let N be the number of sites in the final configuration in the proof of Lemma 7.4 that have at least k occupied neighbors. The neighborhood of such a site must intersect some a occupied horizontal lines and some b occupied vertical lines and contain additional $k-a-b$ black sites. Therefore, N is with high probability much smaller than

$$p^{-\alpha(a)} p^{-\beta(b)} p^{k-a-b},$$

and then

$$N \lhd \max_{0 \le a, 0 \le b \le r-2} a\left(\frac{s}{r} - 1\right) + b\left(1 - \frac{s}{r}\right) + 2\gamma - k$$

$$= (r-2)\left(1 - \frac{s}{r}\right) + 2\gamma - k$$

$$= r + s - 2 - k,$$

and so $N = 0$ with high probability as soon as $k \ge r - s - 2$, which covers exactly our case.

To address the symmetric case, we prove that the lower bound in Lemma 7.5 still holds, again joining its proof, with the corresponding definition of α. We observe that the procedure in the proof produces the final configuration in which there are no $(r-1)$-clumps. We claim that this configuration is still with high probability inert. To prove this claim, let $k \ge 1$ and let N be the number of sites in this configuration with k occupied neighbors. The neighborhood of such a site must intersect some a occupied horizontal lines and some b vertical lines and contain additional $k-a-b$ black sites. Therefore, N is with high probability much smaller than

$$p^{-\alpha(a)} p^{-\alpha(b)} p^{k-a-b},$$

and then

$$N \lhd \max_{0 \le a, b \le r-2} \left[-\epsilon(a+b) + 2\gamma' - k\right] = 2\gamma' - k,$$

and so $N = 0$ with high probability as soon as $k \ge 2\gamma'$, which holds when $k = 2(r-1)$. □

8. Finite L-shaped Young diagrams. It follows from the results of the Theorems 1.1 and 1.2 that any zero-set that lies between the threshold-r bootstrap percolation triangle and the $r \times r$ box has the the upper and lower powers that are both $r - \mathcal{O}(\sqrt{r})$, for large r. In this section, we consider some examples of zero-sets that do not satisfy this restriction. Our main focus will be symmetric L-shaped Young diagrams, but we will also briefly address the asymmetric ones. We begin with a lemma that illustrates our methods for the simplest L-shaped Young diagrams, and provides a lower bound used in Lemma 8.4.

LEMMA 8.1. *Assume $\mathcal{Z}$ has minimal counts $(0,r)$, $(r,0)$ and $(1,1)$ for $r \ge 2$. Then $\gamma_c = r/2$.*

PROOF. As r diagonally adjacent black sites occupy an $r \times r$ square in r steps. If $n \gg p^{-r/2}$, B_n contains such an $r \times r$ box with high probability, so $\gamma_c \le r/2$.

To prove the lower bound, consider, as usual, the dynamics on the box B_n with periodic boundary. Assume $n \ll p^{-r/2}$. Choose some sequence M such that $1 \ll M^r \ll 1/(p^{r/2}n)$.

Call a line *saturated* if it contains r occupied sites within an interval of length M. Let τ be the first time B_n contains a saturated line. If a site x is occupied for the first time at time $t \in [1,\tau]$, then both $\mathcal{N}_x^h$ and $\mathcal{N}_x^v$ must contain an occupied site at time $t-1$. Each of these occupied sites is either black or becomes occupied at some previous time $u \in [1, t-2]$. Continuing backwards in time, we find that on each line through x there is a black site within distance $(r-1)\rho$, as we cannot continue for more than $r-1$ steps, or else a line through x would be saturated before time τ. If L is a line that becomes saturated at time τ, and $x_1, \dots, x_r$ are the occupied sites on L at time τ within distance M, then, for each i, either x_i is black or there is a black site within distance $(r-1)\rho$ on the line through x_i, perpendicular to L. Therefore, if B_n ever contains a saturated line, there is an $M \times M$ box within B_n that contains r black sites, and this happens with probability at most $M^{2r}n^2p^r \to 0$. If the box never contains a saturated line,

$$|\xi_\infty| \le n \cdot (n/M + 1)(r-1) \ll |B_n|,$$

and Lemma 2.3 implies that $\gamma_c \ge r/2$. □

LEMMA 8.2. *Assume $\mathcal{Z}$ has minimal counts $(0,r)$, $(r,0)$ and $(2,2)$ for $r \ge 6$. Then $\gamma_c = 2r/3$.*

PROOF. To prove the upper bound assume first that $n \gg p^{-2r/3}$ and consider the dynamics on B_n. This box will be spanned by time Cn if the following event happens: there are two neighboring horizontal lines in B_n that each contain an interval of r black sites, and next to this pair of lines there is a strip of $r-2$ additional horizontal lines each of which contains 2 neighboring black sites. This happens with high probability as $n(np^r)^2 \gg 1$ and $np^2 \gg p^{-(2r/3-2)} \gg 1$. It follows that $\gamma_c \le 2r/3$.

To prove the lower bound, assume $n \ll p^{-2r/3}$. We will consider the slow version of the dynamics on B_n with periodic boundary, in which we occupy at every time step a single site that can be occupied, chosen arbitrarily, provided it exists.

We will often refer to sites and lines as being *near* each other or *nearby*. Within the context of this proof, this means that that the distance between objects in question is at most $Cr\rho$, for some suitable large constant C. We will choose M so that $1 \ll M \ll 1/(p^{2r/3}n)^\epsilon$, for any $\epsilon > 0$. For a line L and a set of sites S, we call S *M-near* L if: all sites of S are near L; and the projection of S to L fits into an interval of length M.

Again, a *saturated* line contains r occupied sites within an interval of length M, which we call the *saturation sites* of this line. (Note that they do not need to be near each other as M

is much larger than $Cr\rho$.) Upon saturation, we declare the line completely occupied. If a line is saturated initially it may have more than one choice of r saturation sites in which case we just make an arbitrary selection of them. If x is a saturation site on a line L, any black site near x on a line perpendicular to L through x is *associated* to x.

Our strategy is to prove that saturated parallel lines likely remain at distance of order M. To this end, let time τ_2 be the first time when there are two parallel saturated lines within distance $3M$. Assume that $\tau_2 < \infty$. Denote the resulting lines by L_1 and L_2, and assume they are horizontal and that L_1 was saturated first, at time $\tau_1 < \tau_2$. The main problem that we face in the rest of the argument is that there may be isolated vertical or horizontal lines that get saturated before time τ_2 and help in saturation of L_1 and L_2 (see Figure 5). We claim that one of the following must hold:

(1) there are at least $2r-2$ black sites in some $5M \times 5M$ box inside B_n; or
(2) the following occur disjointly:
- either there are r black sites M-near L_1, or there are $r-1$ black sites M-near L_1 *and* a line near them perpendicular to L_1, which disjointly has $r-1$ M-near black sites; and
- the same is true for L_1 replaced by L_2.

If a site x on L_1 gets occupied at time $t \in [1, \tau_1]$, and it does not lie on a saturated vertical line at time t, then it must have two previously occupied sites on the vertical part of its neighborhood; if at least one of them is non-black, we can find two previously occupied sites in the vertical part of its neighborhood, and so on. Therefore, if such an x is a saturation site for L_1, it is either black or there are two black sites associated to x. Similarly, every saturation site y on L_2 that is occupied at time $t \le \tau_2$ and does not lie on a saturated vertical line at time t is either black or has one black site off L_1 associated to it.

Assume first that a saturation site on L_1 and a saturation site on L_2 lie on nearby vertical lines. We claim that in this case (1) holds. Note that then all saturation sites on L_1 and L_2 are within horizontal distance $3M$ and therefore intersect at most one vertical line that is saturated by time τ_2; this follows by minimality of τ_2. This vertical line contains at most two saturation sites for L_1 and L_2, one on each of these two horizontal lines. Now consider all vertical lines through saturation sites on L_1 and L_2 that are *not* saturated at time τ_2. Assume that $a \in [0, r]$ of these lines contain two saturation sites, one on L_1 and one on L_2. Note that each of these a lines must contain two black sites near L_1. If $a = r$, then (1) clearly holds. If $a \le r-1$, there are a least $2(r-1) - 2a$ remaining lines, each of which contains at least one black site near L_1 or L_2, and consequently these black sites are within vertical distance $3M$ of each other. This produces $2(r-1)$ black sites that satisfy (1).

Now assume that no vertical line through a saturation site on L_1 is near a vertical line through a saturation site on L_2. Then each of the two lines L_1 and L_2 may have zero or one saturation site included in a previously saturated vertical line. If neither have such a vertical line, then clearly (2) occurs.

If a vertical line L_3 covers a saturation site of L_1 and is saturated before time τ_1, then there are $r-1$ saturation sites for L_1 not on L_3; they are either all black or else there are at least r associated black sites for L_1 not on L_3. In the former case, we have two additional possibilities. The first possibility is that a saturation site for L_3 is near a saturation site for L_1, in which case we use the fact that at least $r-1$ saturation sites, and therefore at least $r-1$ associated black sites, for L_3 are not on L_1 and (1) occurs. The second possibility is that no saturation sites for L_1 and L_3 are nearby in which case L_3 has at least $r-1$ associated black sites, disjoint from the $r-1$ black sites on L_1.

Now we look at L_2, whose saturation sites are now horizontally far away from those for L_1. If there is no vertical saturated line at time τ_2 that covers one of the saturation sites for L_2, we have r black sites M-near L_2, disjoint from those for L_1 and L_3 (if it exists) in the

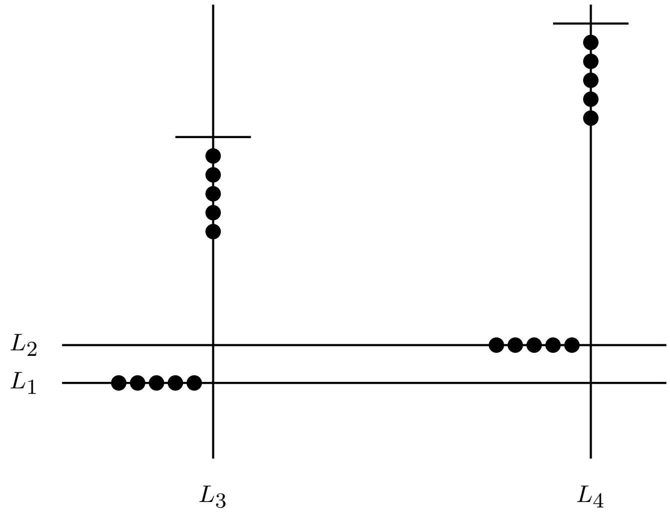


FIG 5. *An illustration of one case from the proof of Lemma 8.2, with $r = 6$ of the configuration of black sites leading to the last term on the first line of (8.1). One of the saturation sites on L_1 is generated by the previously saturated line L_3, which, in turn, may have one of its saturation sites generated by a previously saturated line. Analogously for L_2.*

previous paragraph. Otherwise, let L_4 be such a line. Then L_2 has $r-1$ associated black sites not on $L_1 \cup L_4$. If a saturation site for L_4 is within distance $3M$ of L_2, then it is possible that one saturation site for L_4 is on L_1 and we either have: r associated black sites for L_4 (which are all within distance $4M$ of L_2); or $r-1$ black saturation sites on L_4 — in either case (1) occurs. The last possibility (see Figure 5) is that all saturation sites for L_4 are at least distance $3M$ from L_2, in which case it is possible that another previously saturated line (not L_1) covers one of them, which means that we have at least $r-1$ associated black sites, disjoint from all others found so far. It follows that (2) holds and the claim is established.

From the claim, we get that

$$\begin{aligned} \mathbb{P}(\tau_2 < \infty) &\le CM^{4r}\left[n^2p^{2r-2} + n(np^r)^2 + n(np^{r-1})^2np^r + n(np^{r-1})^4\right] \\ &= CM^{4r}\left[n^2p^{2r-2} + n^3p^{2r} + n^4p^{3r-2} + n^5p^{4r-4}\right] \\ &\to 0, \end{aligned} \tag{8.1}$$

provided that $2r/3 \le r-1$, $8r/3 \le 3r-2$, and $10r/3 \le 4r-4$, which all hold for $r \ge 6$.

If $\tau_2 = \infty$, then we have at most n/M saturated lines in ξ_∞, so that

$$|\xi_\infty| \le (n/M)\cdot n + n\cdot(n/M+1)(r-1) \ll |B_n|,$$

thus Lemma 2.3 implies that $\gamma_c \ge 2r/3$. □

The restriction $r \ge 6$ is likely not necessary. Indeed, for $r = 3$ we have a separate proof that $\gamma_c = 2$ (Proposition 7.6), while for $r = 4, 5$ the argument for the lower bound can possibly be extended (starting with the two previously saturated lines that contribute to saturation sites on L_3 and L_4 in Figure 5), which we however did not pursue.

LEMMA 8.3. *Assume $\mathcal{Z}$ is given by the minimal counts $(0,r)$, (s,s), and $(r,0)$, with $1 \le s < r$. Then a lower and an upper power are, respectively,*

$$\gamma_\ell = \frac{\widehat{m}(r-\widehat{m}+1)}{1+\widehat{m}}, \text{ where } \widehat{m} = \min(\lfloor(-1+\sqrt{4r+9})/2\rfloor, \lfloor s/2\rfloor),$$

and

$$\gamma_u = \frac{rs}{s+1}.$$

PROOF. For the upper power, consider the dynamics on B_n with $n \gg p^{-\gamma_u} \geq p^{-s}$. Every fixed line is then likely to contain an interval of s black sites. Moreover, a horizontal strip of width s with each of its horizontal lines containing an interval of r black sites is also likely to exist within B_n, as $n(np^r)^s = n^{s+1}p^{rs} \gg 1$. With high probability, then, there is such a strip with $r-s$ neighboring horizontal lines, such that each contains an interval of s black sites. This configuration causes complete occupation of B_n by time $3n$.

For the lower power, assume $n \ll p^{-\gamma_\ell}$. adopt the same slow version of the dynamics as in the previous proof, with M such that $1 \ll M \ll 1/(np^{-\gamma_\ell})^\epsilon$ for every $\epsilon > 0$, and the same definitions of saturated lines and nearby points. Pick an integer $m \leq s/2$.

Let τ now be the first time the dynamics produces m saturated parallel lines within distance $3rM$. We claim that $\mathbb{P}(\tau < \infty) \to 0$, for a judicious choice of m. Assume that $\tau < \infty$ and that the m lines produced at time τ are horizontal. At time τ, each occupied site on one of these lines either: is on a previously saturated vertical line; is a black site; or has $s-m+1$ black sites nearby on the vertical line through it.

Divide the m horizontal lines into k groups of $a_1, \dots, a_k$ lines so that, for two lines L_1 and L_2 in different groups, no vertical line through an occupied site on L_1 is near a vertical line through an occupied site on L_2. Here $a_1 + \ldots + a_k = m$ and $1 \leq a_i \leq m$. We now proceed to finding an upper bound for the probability for a fixed m and a fixed division.

Once we fix the vertical locations of the m lines, the first group, say, yields the probability bound a power of M times $np^{a_1(r-m+1)}$, as: every occupied site on these lines that does not lie on a vertical saturated line is either black or has $s - a_1 \geq a_1$ nearby black sites on its vertical line; and there are at most $m-1$ saturated vertical lines that can help, by minimality of τ. This results in the bound a power of M times

$$n \cdot n^k p^{(a_1+a_2+\cdots+a_k)(r-m+1)} = n^{1+k}p^{m(r-m+1)} \leq n^{1+m}p^{m(r-m+1)}.$$

The proof of the lower bound now reduces to the routine verification that the function

$$\frac{m(r-m+1)}{1+m} = r+3-\left(1+m+\frac{r+2}{1+m}\right),$$

achieves its maximum for $m \in [1, \lfloor s/2 \rfloor]$ at $m = \widehat{m}$. □

LEMMA 8.4. *Assume $\mathcal{Z}$ is given by the minimal counts $(0,r)$, $(s,1)$, and $(r,0)$, with $1 \leq s < r$. For $s \leq r/2$ we have pure critical power $\gamma_c = r/2$. For $s > r/2$, we have lower and upper powers $s - \frac{s}{r}$ and $s + 1 - \frac{2s+1}{r+1}$.*

PROOF. We only need to provide the upper bounds, as lower bounds follow from Lemmas 8.1 and 7.3. As usual, we consider the dynamics on B_n. Observe that a horizontal strip of width r, which contains a horizontal line with an interval of r black sites and $r-1$ lines, all with an interval of s black sites, will occupy B_n by time $(r+1)n$.

When $s \leq r/2$, and $n \gg p^{-r/2}$, such a strip exists with high probability, as $n(np^r) \gg 1$ and $np^s \gg 1$. When $s > r/2$ and $n \gg p^{-(s+1-(2s+1)/(r+1))}$, the same holds as

$$n(np^r)(np^s)^{r-1} = n^{r+1}p^{r+s(r-1)} \gg 1.$$

□

PROOF OF THEOREM 1.4. The four parts are established by the four lemmas in this section. □

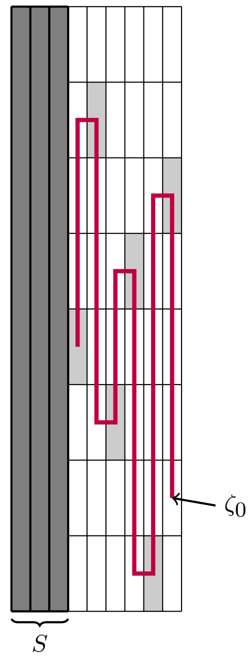


FIG 6. *Transmission of occupation to the cell ζ_0. The dark grey strip S has a vertical interval of r black sites in each of its s_1 (which is set to 3 for the purpose of this illustration) columns, which all get occupied. Light grey columns have thickness 1 and are cells that each include a vertical interval of s_2 black sites. All cells on the purple path get fully occupied, starting with the one next to S.*

9. Infinite L-shaped Young diagrams. In this section, we consider the infinite Young diagrams with two minimal counts (s_1, s_2) and $(0, r)$, with $r \geq 2$, $1 \leq s_2 \leq r-1$, and $s_1 \geq 1$. We let

$$\gamma = \frac{rs_1 + s_2}{1 + s_1},$$

so that our plan is to prove $\gamma_c = \gamma$. We will make use of the dynamics on a rectangular box B of dimensions $A \times Ap^{-s_2}$, for a suitably chosen A that increases as a power of p.

LEMMA 9.1. *The upper bound $T \lesssim p^{-\gamma}$ holds. Hence $\gamma_c \leq \gamma$.*

PROOF. Assume that $A \gg p^{-(r-s_2)s_1/(1+s_1)} = p^{-(\gamma - s_2)}$. Let G_1 be the event that B contains s_1 neighboring vertical lines, each of which contains an interval of r black sites. The event G_1 occurs with high probability because

$$A(Ap^{-s_2}p^r)^{s_1} \gg 1.$$

On G_1, we get an occupied vertical strip of width s_1 in Ap^{-s_2} time steps.

Now divide B into vertical strips of height $h = Cp^{-s_2}$ and width 1, which we call *cells*. We call such a cell *good* if it contains a vertical subinterval of length s_2 full of black sites. The probability P_g that a fixed cell is good can be made arbitrarily close to 1 if C is chosen to be sufficiently large and then p small enough. Connect every good cell with the two horizontally adjacent cells (good or not) and connect *any* two vertically adjacent cells (again, good or not). Assume that G_1 happens, resulting in the occupied strip S by time T_1. Then, any cell to the right (resp., left) of S that is connected through a path of connected cells of length k to the left (resp., right) edge of B is completely occupied by additional time hk (see Figure 6).

Let G_2 be the event that the top and the bottom halves of every column both contain a good cell. As

$$\mathbb{P}(G_2^c) \le A(1-P_g)^{A/(2C)} \le A\exp(-P_g A/(2C)) \to 0,$$

G_2 happens with high probability.

Fix a cell ζ_0. If ζ_0 is below (resp. above) the middle line of B, find the closest good cell ζ_1 above (resp. below) the midline in the column of ζ_0; for the purposes of this part of the argument, we allow cells outside B, but this will not happen on the event G_2. Then find the first good cell ζ_2 below (resp. above) the midline in the column to the left of ζ_1, then a good cell ζ_3 in the column to the left of ζ_2, with the same above-below convention, etc., until reaching a cell in the vertical line that contains the leftmost column of B. This constructs a path of connected cells whose length is bounded by A plus twice the sum of at most A independent Geometric(P_g) random variables, where the additional A accounts for the initial portion of the path from ζ_0 to the midline. The probability that this length exceeds $5A$ for any fixed cell ζ_0 is at most $\exp(-cA)$ for some constant $c > 0$. Let G_3 be the event that the length of the path is at most $5A$ for *every* cell ζ_0. As the number of cells within B is bounded by A^2,

$$\mathbb{P}(G_3^c) \le A^2 \exp(-cA).$$

The event G_4 is the mirror image of the event G_3: every "left" is replaced by "right" and vice versa, so $\mathbb{P}(G_4) = \mathbb{P}(G_3)$. If $G_1 \cap G_2 \cap G_3 \cap G_4$ happens, then box B gets fully occupied by time a constant times Ap^{-s_2}. Finally, by FKG, $\mathbb{P}(G_1 \cap G_2 \cap G_3 \cap G_4) \ge \mathbb{P}(G_1)\,\mathbb{P}(G_2)\,\mathbb{P}(G_3)\,\mathbb{P}(G_4) \to 1$. □

It remains to prove of the matching lower bound, thus we assume for the rest of this section that $A \ll p^{-(\gamma - s_2)}$. Again, the term *nearby* will mean within ℓ^∞-distance $Cr\rho$ for a suitable large constant C. We start by two straightforward observations. First, with high probability, the event that

(E1) there are at most $s_1 - 1$ nearby vertical lines such that, disjointly, each line has r nearby black sites that are nearby each other

happens, as now $A(Ap^{-s_2}p^r)^{s_1} \ll 1$. Also,

$$|B| = \frac{A^2}{p^{s_2}} \ll p^{-\frac{2rs_1 - s_1 s_2 + s_2}{1+s_1}},$$

and so with high probability the event that

(E2) there exists no set of nearby black sites of size

$$\left\lceil \frac{2rs_1 - s_1 s_2 + s_2}{1+s_1} \right\rceil \le 2r$$

also happens. From now on, we assume that the two events (E1) and (E2) happen.

Starting with the black sites, we perform the following two-stage occupation process, in which we call the sites that get occupied in the first step *red* and those that get occupied in the second step *blue*. Declare a vertical line L to be *saturated* if it contains r nearby sites, each of which is either black or is covered by a horizontal line that contains two nearby black sites that are also near L. We call such r sites on L *saturating* sites. In the first step we occupy, i.e., make red, all non-black points on all saturated vertical lines.

LEMMA 9.2. *No two saturating sites, for two different nearby vertical lines, can be on the same horizontal line. Therefore, two different vertical lines cannot share black sites that make them saturated.*

PROOF. Assume that L_1 and L_2 are two nearby saturated lines, and that $a \geq 1$ is the number of horizontal lines that contain saturating sites of both L_1 and L_2, which together contain at least $2a$ nearby black sites. Further, the lines through the remaining $2(r-a)$ saturating sites ($r-a$ on each of L_1 and L_2) together contain at least $2(r-a)$ black sites. This results in at least $2a + 2(r-a) = 2r$ nearby black sites, violating (E2). (Here we used $a \geq 1$ only to force these black sites to be nearby.) □

In the second stage, we use the dynamics with zero-set determined by the single minimal count (s_1, s_2), which is a critical dynamics [BBMS1]. (Note that this is exactly the classic modified bootstrap percolation [Hol] when $\rho = 1$ and $s_1 = s_2 = 1$.) We start from the occupied set consisting of red and black sites. We use the slow version of this process, in which we occupy, that is, make blue, exactly one site that can be occupied, chosen arbitrarily, per time step, until we reach an inert configuration. Call a *cluster* a maximal connected set of black and blue sites, where we use the graph induced by our range ρ neighborhood $\mathcal{N}_x$ for connectivity. The *hull* of a cluster is the smallest rectangle that contains it. Note that hulls of different clusters may intersect. We call a black site x *enhanced* if it has s_1 red or black sites (including itself) within a horizontal interval of length $2\rho + 1$ including x.

LEMMA 9.3. *At any step of the second stage dynamics, every blue site has an enhanced black site in the row of its hull, and s_2 black sites within a vertical interval of $2\rho + 1$ sites in the column of its hull.*

PROOF. We argue by induction: this property trivially holds initially, and then is maintained by addition of each blue point. Indeed, assume x gets painted blue at some step. To prove the claim about the row of x, note that x must have s_1 already colored horizontal neighbors, not all of which can be red by (E1) and Lemma 9.2. If at least one is blue, then the claim follows by induction; otherwise, one of them is black and thus is enhanced. To prove the claim about the column of x, note that x must have s_2 already colored vertical neighbors, but now none can be red. If at least one is blue, the claim follows by induction, and if all are black the claim follows immediately. □

LEMMA 9.4. *The second-stage dynamics never produces a hull with either dimension exceeding $b_0 = 2r\rho$.*

PROOF. Assume that b is the largest dimension of a hull at time t, the first time the claim is violated. Such a hull contains at least $\lceil b/\rho \rceil$ black sites, by Lemma 9.3. Therefore, the claim holds initially, before any blue sites are created. Assume now that $t > 0$ and let x be the site that becomes blue at time t. As x is connected to at most 4 clusters at step $t-1$,

$$2r\rho \leq b \leq 4 \cdot 2r\rho + 2\rho + 1 \leq 11r\rho.$$

It follows that we have at least $2r$ black sites in a $11r\rho \times 11r\rho$ box, contrary to (E2). □

It follows from the above two lemmas that, for every blue site x, there are s_2 black sites in the vertical line through x within an interval of size $2\rho r$ containing x. Also, there are at least s_1 red or black sites, at least one of which is black, on the horizontal line through x within an interval of the same size. This will enable us to prove the key property of the colored configuration.

LEMMA 9.5. *The configuration consisting of black, red, and blue sites on B is, with high probability, inert.*

PROOF. Assume that, after the second stage, the configuration of the colored sites is not inert. Then there is a non-red vertical line that contains r black or blue sites within a vertical interval of size $2\rho + 1$. Of these, at least one is blue (or the line would become red in the first stage) and therefore, by Lemma 9.3, there are at least s_2 black sites within a vertical interval of size $5r\rho$. So within such an interval, we can find some $i \in [1, r - s_2]$ blue sites and $r - i$ black sites. Assume also that there are $j \in [0, s_1 - 1]$ red lines near this line. Each of the i blue sites has to have $s_1 - j$ black sites on the horizontal line through it, nearby but not on the red lines; call these black sites, together with the $r - i$ black sites, *assisting*. Observe that we cannot have $j \le s_1 - 2$, as then again the line would be saturated at stage 1. Therefore, $j = s_1 - 1$.

Now we claim that none of the assisting black sites can be on the same horizontal line as a saturating site of one of the j red lines. (Recall that different nearby red lines do not have saturation sites on the same horizontal line by Lemma 9.2.) Indeed, if this were true, we would again have $a \ge 1$ horizontal lines that each cover an assisting and a saturating site. Each of these contains 2 black sites, and additionally we have at least $2(r - a)$ black sites on the horizontal lines through saturation sites and through assisting sites, and they are all nearby. This would again produce $2r$ nearby black sites, demonstrating that the black sites involved in saturating the lines and assisting sites are disjoint.

It follows that we have $s_1 - 1$ nearby red lines that get saturated disjointly and, also disjointly, a nearby set of r nearby black sites. The probability for this is at most a constant times

$$A(Ap^{-s_2}p^r)^{s_1-1}Ap^{-s_2}p^r \ll 1.$$

Consequently, the probability that the configuration of colored sites is not inert converges to 0. □

We have so far dealt with the dynamics on B, and obtained an inert configuration, which has low density, as we will observe in the proof of Lemma 9.8. However, this is not sufficient in this instance. The reason is that the box B only has width A, and the simple "speed of light" argument of Lemma 2.3 only gives a lower bound on the order of A, which is far smaller than the claimed size of T. We need to provide a different argument, a probabilistic one, that demonstrates that the influence from outside B likely spreads horizontally with speed no larger than on the order of p^{s_2}.

To achieve this, we assume we have $A \times (A/p^{s_2})$ box B with origin in its center, and enlarge the initial configuration by replacing it with the occupied set of black, red, and blue sites produced by the two-stage procedure above. For this part, we will only need that every blue site has s_2 black sites within some distance $R = R(\rho)$. We make all sites outside of the box occupied and call them *green*. Now run the original dynamics with the zero-set $\mathcal{Z}$ on the resulting configuration and also call all newly occupied sites green. We claim that there exist an $\epsilon > 0$ so that the origin turns green by time ϵ/p^{s_2} with probability approaching 0. We call a sequence $x_0, x_1, ..., x_n$ a *neighborhood path* if $x_i \in \mathcal{N}_{x_{i-1}}$ for $i = 1, \dots, n$.

LEMMA 9.6. *Assume that some site x in B turns green at time $t \ge 1$. Then there exists a neighborhood path $x_0, x_1, ..., x_n = x$ of length t, so that x_0 is outside B and for every horizontal step from x_i to x_{i+1} there are s_2 black sites within distance $2R$ of x_{i+1}.*

PROOF. We argue by induction. Due to inertness of other colors, x must have a green neighbor at time $t - 1$. If that neighbor is vertical, the induction step is trivial. Otherwise, x must have a horizontal green neighbor, and also s_2 non-red vertical neighbors (because x itself is not red). If all of these vertical neighbors are black, the induction step is completed. If one of them is blue, it has s_2 black sites within distance R, which again completes the induction step. □

The problem with the path in the previous lemma is that the same black sites may be used at many steps, so we need to construct a path that does not do that. For that, we abandon the "neighborhood" assumption. We say that a path $y_0, ..., y_n = x$ makes a *lateral move* from y_i to y_{i+1} if $||y_i - y_{i+1}|| \le 5R$. Thus, a lateral move is allowed to be in any direction within the restricted distance.

LEMMA 9.7. *Let x be as in Lemma 9.6. Then there exists a path $y_0, \dots, y_n = x$ of length $n \le \rho t$ with the following properties:*

- *it is self-avoiding;*
- *it makes vertical neighborhood moves of unit distance; and*
- *if there is a lateral move from y_i to y_{i+1}, then there are s_2 black sites within distance $2R$ of y_{i+1}; and*
- *all black sites at different lateral moves are distinct.*

PROOF. Take the path $x_0, \dots, x_n$ from the previous lemma let the associated set of black sites be Z. For every $z \in Z$, let X_z be the set of sites x_i such that there is a horizontal step to x_i which has z as one of the requisite neighboring black sites. All elements of X_z are within distance $4R$ of each other. The y-path is constructed as follows. For every X_z, let x_ℓ be the the first site on the path in X_z and x_m the last. Now eliminate from the path all sites from x_ℓ up to, but not including, x_m. The new path has steps bounded by $5R$: the step from $x_{\ell-1}$ to x_ℓ is bounded by ρ and the distance from x_ℓ to x_m is bounded by $4R$. Moreover, any vertical segment of the path can be replaced by a sequence of unit moves, all in the same direction, which lengthens the path by at most a factor ρ. □

We now have everything in place for the last step in the proof of Theorem 1.5.

LEMMA 9.8. *The lower bound $T \gtrsim p^{-\gamma}$ holds. Hence $\gamma_c \ge \gamma$.*

PROOF. The probability that there exists a path from Lemma 9.7 of length s that makes j lateral moves is at most

$$\binom{s}{j} C^j p^{s_2 j} \le \left(\frac{Cesp^{s_2}}{j}\right)^j,$$

where $C = C(R)$ is a constant, the number of choices for a move at a lateral step (and those are the only steps at which we have a choice) times the number of possible positions of a black site at a lateral step.

Let D be a large constant, $D > 2Ce\rho$. Assume $s \le \rho t$. The probability that there exists a path from Lemma 9.7 of length s that makes at least Dtp^{s_2} lateral moves then is at most

$$\sum_{j \ge Dtp^{s_2}} \left(\frac{Cesp^{s_2}}{j}\right)^j \le \sum_{j \ge Dtp^{s_2}} \left(\frac{Ce\rho}{D}\right)^j \le 2(0.5)^{Dtp^{s_2}}.$$

Now the probability that there exists a path of length at most ρt that makes at least Dtp^{s_2} lateral moves is at most

$$2\rho t (0.5)^{Dtp^{s_2}}.$$

Now let $t = \epsilon A p^{-s_2}$, where $\epsilon = 1/(20DR)$. A path from Lemma 9.7, with x the origin, would have to make at least $A/(10R) = 2Dtp^{s_2}$ lateral moves. Note that it is impossible to connect to initially green sites through the top or the bottom of the box, due to the speed of light. Therefore, the probability that such a path exists is at most

$$2\rho t (0.5)^{A/(20R)} \to 0.$$

It follows that the origin is not green by time t with high probability.

Finally, we verify that the origin is also unlikely to be any other color. First, we recall that the probability that the origin is black is p. Next, the probability that it is red is $\mathcal{O}(Ap^{r-s_2})$ by definition of a red line. Finally, the probability that it is blue is $o(1)$ by Lemma 9.4. □

PROOF OF THEOREM 1.5. The conclusion follows from Lemmas 9.1 and 9.8. □

10. Open Problems.

1. Does the critical power γ_c exist for all zero-sets $\mathcal{Z}$?
2. For the symmetric line growth of Theorem 1.2 with $r = s \geq 3$, is there a matching lower bound to the upper bound in Lemma 5.1, i.e., is it true that $t \ll p^{-(r-1)} \log \frac{1}{p}$ implies that $\mathbb{P}(T \leq t) \to 0$?
3. For the line growth of Theorem 1.2 with $1 < s < r$, is the critical power pure?
4. For the L-shaped zero-set of part 3 of Theorem 1.4, is $\gamma_c = r - \mathcal{O}(1)$ if $s = \Theta(r)$, for large r?
5. What is the critical power for the zero-set which is the union of the bootstrap percolation zero-set with threshold r, and an infinite strip of width $s < r$? Thus, in this rule, the condition for occupation is that the total number of occupied neighbors is at least r, and the number of vertical occupied neighbors is at least s.
6. As in [GSS], consider the dynamics with zero-set $\mathcal{Z}$ on a $M \times N$ box B, with periodic boundary, and assume that $\log M / \log p \to \alpha$ and $\log N / \log p \to \beta$. Call $I = I(\alpha, \beta, \mathcal{Z})$ the *large deviation rate* if $\log \mathbb{P}(\xi_\infty = B) / \log p \to I$. When can the large deviation rate be shown to exist and when can it be computed? (In [GSS], it was shown that, when $\rho = \infty$, I always exists and can be computed for line growth and for bootstrap percolation when $\alpha = \beta$.)

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