POLLUTED BOOTSTRAP PERCOLATION WITH THRESHOLD TWO IN ALL DIMENSIONS

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Abstract. In the polluted bootstrap percolation model, the vertices of a graph are independently declared initially occupied with probability p or closed with probability q. At subsequent steps, a vertex becomes occupied if it is not closed and it has at least r occupied neighbors. On the cubic lattice $\mathbb{Z}^d$ of dimension $d \geq 3$ with threshold $r = 2$, we prove that the final density of occupied sites converges to 1 as p and q both approach 0, regardless of their relative scaling. Our result partially resolves a conjecture of Morris, and contrasts with the $d = 2$ case, where Gravner and McDonald proved that the critical parameter is q/p^2.

1. Introduction

Bootstrap percolation is a fundamental cellular automaton model for nucleation and growth from sparse random initial seeds. In this article we address how the model is affected by the presence of pollution in the form of sparse random permanent obstacles.

Let $\mathbb{Z}^d$ be the set of d-vectors of integers, which we call **sites**, and let $p, q \in [0, 1]$ be parameters. In the **initial** (time zero) configuration, each site is chosen to have exactly one of three possible states:

$$\begin{cases} \textbf{closed} & \text{with probability } q; \\ \textbf{open} \text{ and } \textbf{initially occupied} & \text{with probability } p; \\ \textbf{open} \text{ but not initially occupied} & \text{with probability } 1 - p - q. \end{cases}$$

Initial states are chosen independently for different sites. Closed sites represent pollution or obstacles, while occupied sites represent a growing agent.

The configuration evolves in discrete time steps $t = 0, 1, 2, \ldots$ as follows. As usual we make $\mathbb{Z}^d$ into a graph by declaring sites $u, v \in \mathbb{Z}^d$ to be neighbors if $\|u - v\|_1 = 1$. The **threshold** r is an integer parameter. An open site x that is unoccupied at time t becomes occupied at time $t + 1$ if and only if

$$\text{at least } r \text{ neighbors of } x \text{ are occupied} \tag{1}$$

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at time t. Closed sites remain closed forever and cannot become occupied. Open sites remain open. Once a site is occupied, it remains occupied. In the main cases of interest, $d \geq r \geq 2$.

Bootstrap percolation without pollution (the case $q = 0$ in our formulation) has a long and rich history with many surprises. For $d \geq r \geq 1$, there is no phase transition in p, in the sense that every site of $\mathbb{Z}^d$ is eventually occupied almost surely for every $p > 0$, as proved in [vE] ($d = 2$) and [Sch] ($d \geq 3$). The metastability properties of the model on finite regions are understood in great depth (see e.g. [AL, Hol, BBDM, GHM]), while a broad range of variant growth rules have also been explored (e.g. [GG, DvE, BDMS]). For further background see the discussion later in the introduction, and the excellent recent survey [Mor].

The polluted bootstrap model (i.e. the case $q > 0$) was introduced by Gravner and McDonald [GM2] in 1997. The principal quantity of interest is the *final density* of occupied sites, i.e. the probability that the origin is eventually occupied, in the regime where p and q are both small. In dimension $d = 2$ with threshold $r = 2$, Gravner and McDonald proved that the final density is strongly dependent on the relative scaling of p and q. Specifically, there exist constants $c, C > 0$ such that, as $p \to 0$ and $q \to 0$ simultaneously,

$$\mathbb{P}\big(\text{the origin is eventually occupied}\big) \to \begin{cases} 1, & \text{if } q < cp^2; \\ 0, & \text{if } q > Cp^2. \end{cases}$$

In this article we give the first rigorous treatment of the polluted bootstrap percolation model in dimensions $d \geq 3$. We take the threshold r to be 2. (The case $d = r = 3$ is addressed in a companion paper [GHS] by the current authors together with Sivakoff, as discussed below). Our main result is that, in contrast with dimension $d = 2$, occupation prevails regardless of the p versus q scaling.

Theorem 1. *Consider polluted bootstrap percolation on $\mathbb{Z}^d$ with $d \geq 3$, threshold $r = 2$, density $p > 0$ of initially occupied sites, and density $q > 0$ of closed sites. We have*

$$\mathbb{P}\big(\text{the origin is eventually occupied}\big) \to 1 \qquad \text{as } (p, q) \to (0, 0).$$

Moreover, the probability that the origin lies in an infinite connected set of eventually occupied sites also tends to 1. *The same statements hold for modified bootstrap percolation.*

In the above statement, a set of sites is called **connected** if it induces a connected subgraph of $\mathbb{Z}^d$. The **modified** bootstrap percolation model is a well-known variant of the standard model, in which the condition (1) for a site to become occupied is replaced with:

for at least r of the directions $i = 1, \ldots, d$, either $x - e_i$ or $x + e_i$ is occupied,

where e_i is the ith coordinate vector. (As before, closed sites cannot become occupied, and occupied sites remain occupied forever).

Theorem 1 resolves Conjecture 4.6 of Morris [Mor] in the case $r = 2$. To be precise, this conjecture may be expressed as: for all $d > r \geq 1$, there exists an infinite connected eventually occupied set with probability at least $1/2$ for (p, q) sufficiently close to $(0, 0)$. Morris states that the conjecture seems to be very difficult.

Defining $\phi(p, q) = \phi_{d,r}(p, q)$ to be the probability that the origin is eventually occupied, it follows from the obvious monotonicities of the model that ϕ is (weakly) increasing in p and decreasing in q. Therefore, the convergence in Theorem 1 is equivalent to $\lim_{q\to 0} \lim_{p\to 0} \phi(p, q) = 1$. This formulation will be reflected in our proof. We will show that for q sufficiently small there is an infinite structure of open sites on which occupation can spread, no matter how small p, and that the density of this structure tends to 1 as $q \to 0$. Our methods are very different from those in previous works on bootstrap percolation, and involve the technology of oriented surfaces introduced recently in [DDG$^+$].

Our result reveals an interesting phase transition. Let $r = 2$ and $d \geq 3$ and consider the decreasing function $\phi^+(q) := \phi(0^+, q) = \lim_{p\to 0^+} \phi(p, q)$. Theorem 1 implies that $\phi^+(q) > 0$ for q sufficiently close to 0. On the other hand, standard arguments imply that $\phi^+(q) = 0$ if q exceeds one minus the critical probability $p_c^{\mathrm{site}}(\mathbb{Z}^d)$ of site percolation. Therefore the critical probability

$$q_c := \inf\{q : \phi^+(q) = 0\}$$

is nontrivial. In fact, we show the following slightly stronger fact involving a strict inequality.

Corollary 2. *Consider the setting of Theorem 1. The critical value q_c defined above satisfies $0 < q_c \leq 1 - p_c^{\mathrm{site}}(\mathbb{Z}^d)$. For $d = 3$, the latter inequality is strict. The function ϕ^+ vanishes on $(q_c, 1]$, is strictly positive on $[0, q_c)$, and converges to 1 as $q \to 0$.*

Our methods do not produce a good lower bound on q_c, and give no information on the behavior of ϕ^+ near q_c.

As mentioned earlier, the companion paper [GHS] treats polluted bootstrap percolation with threshold $r = 3$ in dimension $d = 3$. The strongest result of [GHS] is for the modified bootstrap percolation model with $d = r = 3$. Similarly to the case $d = r = 2$ of [GM2], but in contrast with the $d > r = 2$ case of Theorem 1, the final density here depends on the p versus q scaling, but now with a *cube* law (modulo logarithmic factors). Specifically, as $p, q \to 0$, the final occupied density converges to 1 if $q \ll (p/\log p^{-1})^3$, and to 0 if $q > Cp^3$. Interestingly, the first of these bounds relies crucially on Theorem 1 of the current article (together with a straightforward renormalization argument). The second bound (which is far from straightforward) again uses oriented surfaces, but in a completely different way: to block growth rather than to facilitate it.

We record some simple observations about other choices of the threshold r. For $d = r$, notwithstanding the detailed results of [GM2,GHS], an easy argument rules out the conclusion $\lim_{(p,q)\to(0,0)} \phi(p,q) = 1$ of Theorem 1. Indeed, if $p, q \to 0$ with $p = o(q^{2^d})$ then with high probability there exist $M < 0 < N$ such that no site in the box $\{0,1\}^{d-1} \times [M, N]$ is initially occupied but every site on the two ends $\{0,1\}^{d-1} \times \{M, N\}$ is closed. On this event, the origin cannot become occupied. (For the modified model, the same argument works even for the line $\{0\}^{d-1} \times [M, N]$, giving the same conclusion under the weaker assumption $p = o(q)$. Similar comparisons involving $\{0,1\}^{d-d'} \times \mathbb{Z}^{d'}$ or $\{0\}^{d-d'} \times \mathbb{Z}^{d'}$ for $d > d'$ are available, which, when combined with the results of [GM2] for $d' = 2$ or [GHS] for $d' = 3$, yield further improvements.) On the other hand, the case of threshold $r = 1$ is easily understood via standard site percolation: the final occupied set is simply the union of all open clusters that contain initially occupied sites. (This observation is relevant to Corollary 2.) Finally, thresholds $r > d$ are less interesting to us, since, even with no closed sites, there are finite sets such as $\{0,1\}^d$ that remain unoccupied forever if unoccupied initially, so $\lim_{(p,q)\to(0,0)} \phi(p,q) = 0$.

Background. Bootstrap percolation is an established model for nucleation and metastability, and one of very few cellular automaton models with a well-developed mathematical theory. It has been applied in physics, biology, and social science to various growth phenomena, including crack formation, crystal growth, and spread of information or infection. See [GZH] for a recent example. Bootstrap percolation has been used in the rigorous analysis of other models such as sandpile and Ising models; see e.g. [Mor]. The evolving set method in Markov mixing theory can be viewed as bootstrap percolation with a randomly varying threshold [MP].

Bootstrap percolation was first considered on trees [CLR], but the lattice $\mathbb{Z}^d$ with its physics connections has received the most attention. There has been recent interest in mean-field and power-law graphs, motivated in part by applications to social networks; see e.g. [JŁTV, AFP, KL].

Polluted bootstrap percolation was introduced in [GM2] on the two dimensional lattice. Potential areas of application include the effects of impurities on crystal growth, of immunization on epidemics, or of interventions on spread of rumors. Since [GM2], rigorous progress on growth processes in random environments has been limited, and the case of polluted bootstrap percolation in three and higher dimensions has been entirely open until now. Here are some examples of work on related models. Investigation of asymptotic shapes in models related to polluted bootstrap percolation with $r = 1$ was initiated in [GM1]; a recent paper [JŁTV] studies such processes on a complete graph with excluded edges; and [DEK$^+$] addresses a Glauber dynamics (which can be viewed as a non-monotone version of bootstrap percolation) with "frozen" vertices. Polluted bootstrap percolation and closely related models have been used in empirical studies of complex networks with "damaged" vertices [BDGM1, BDGM2].

A key element in our proof will be the simple but powerful method of random oriented surfaces recently introduced in [DDG$^+$]. This method has been further used and developed in a variety of contexts [DDS, GH1, GH2, GH3, HM, BT], but ours is the first application to cellular automata so far as we are aware. A distinct application to polluted bootstrap percolation will appear in [GHS].

Another useful tool will be the results of [LSS] concerning domination of finitely dependent processes. A random configuration $X = (X_v)_{v\in\mathbb{Z}^d}$ taking values in $\{0,1\}^{\mathbb{Z}^d}$ is called ***m*-dependent** if $(X_v)_{v\in A}$ and $(X_v)_{v\in B}$ are independent of each other whenever the sets A and B are at distance greater than m. The relevant result of [LSS] is that for any $p < 1$ there exists $p' = p'(p, d, m) < 1$ such that, if X is m-dependent and satisfies $\mathbb{E}X_v \geq p'$ for all v, then X stochastically dominates an i.i.d. process with parameter p.

Outline of proof and organization. The modified bootstrap percolation model is "weaker" than the standard model, in the sense that it is more difficult for a site to become occupied, so that for a given initial configuration, the occupied set for the modified model is a subset of that for the standard model at each time t. Therefore, it suffices to prove the conclusions of Theorem 1 for the modified model. Moreover, we may without loss of generality assume that $d = 3$. Indeed, for $d \geq 4$ we may restrict to the 3-dimensional subspace $\mathbb{Z}^3 \times \{0\}^{d-3}$. Any site that becomes occupied in the $d = 3$ model restricted to the subspace also becomes occupied in the full model on $\mathbb{Z}^d$ (where in both cases $r = 2$). Therefore, for the remainder of the paper we consider the modified bootstrap percolation model with $r = 2$ on $\mathbb{Z}^3$ except where explicitly stated otherwise.

In the absence of closed sites, the two-dimensional bootstrap rule fills $\mathbb{Z}^2$ from any positive density p of occupied sites. This suggests the following approach. For q sufficiently small we may attempt to construct an infinite two-dimensional surface that avoids closed sites and behaves like $\mathbb{Z}^2$, in the sense that it also admits growth by the $r = 2$ model for any $p > 0$. In Section 2 we indeed construct an oriented surface, called a *curtain*, with some of the required properties. In particular, starting from an infinite fully occupied half space of $\mathbb{Z}^3$, a curtain will become fully occupied almost surely for any $p > 0$. The construction of the curtain itself does not involve p, and does not depend on the locations of initially occupied sites.

A curtain alone is not sufficient to prove Theorem 1, because a *finite* occupied nucleus does not lead to indefinite growth on a curtain. To address this, we will use a renormalization argument involving curtains with different orientations that intersect each other. This part of the argument *will* involve p, in the determination of a length scale. In Section 3 we construct the unit of our renormalization, which is a curtain restricted to a finite box, with carefully constrained geometry, and scaled to facilitate the required intersections. This modified curtain is called a *sail*. The size of the box is chosen to be a power of p^{-1}, which allows the sail to contain sufficient initially occupied sites for growth similar to that on a curtain. In Section 4 we use comparison methods to show that if two sails intersect appropriately then occupation is transmitted from one to the other. Finally, Section 5 completes the renormalization argument, which involves comparison of an infinite network of sails with supercritical oriented percolation, together with "sprinkling" for the initial nucleation.

We conclude the paper with a list of open problems.

Notation and conventions. As stated earlier, we work with the polluted modified bootstrap percolation model with threshold $r = 2$ on $\mathbb{Z}^3$ unless stated otherwise. The cubic lattice, also denoted $\mathbb{Z}^3$, is the graph with vertex set $\mathbb{Z}^3$ and with an edge between sites u and v whenever $\|u - v\|_1 = 1$. When discussing sets of sites, connectivity and components always refer to this graph.

When describing subsets of $\mathbb{Z}^3$, intervals will be understood to denote their intersections with $\mathbb{Z}$, so $[a, b)$ denotes $[a, b) \cap \mathbb{Z} = \{a, a + 1, \ldots, b - 1\}$, etc. Let $\mathbb{N}$ be the set of nonnegative integers. We will frequently wish to consider 2-dimensional layers of $\mathbb{Z}^3$, which by convention will be taken perpendicular to the 3rd coordinate. Thus, for $k \in \mathbb{Z}$ we define the kth **layer** to be

$$\Lambda_k := \mathbb{Z}^2 \times \{k\} = \big\{x \in \mathbb{Z}^3 : x_3 = k\big\}.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{Z}^3$, and let e_1, e_2, e_3 be the standard coordinate vectors.

We will consider paths of various types, not always with nearest-neighbor steps. In general, a **path** is a finite or infinite sequence of sites $(\ldots,)x_0, x_1, \ldots, x_n(,\ldots)$. Its **steps** are the vectors $(\ldots,)x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}(,\ldots)$. It is a nearest-neighbor path if all steps are of the form $\pm e_i$. It is self-avoiding if all its sites are distinct.

We will frequently use the following elementary properties of the model. The set of eventually occupied sites is an increasing function of the set of initially occupied sites and a decreasing function of the set of initially closed sites. Moreover, if one only cares about eventually occupied sites, then sites may be updated in any order so long as each site is updated infinitely often. In particular, if we know that every site in some set A becomes occupied eventually, then we may modify the initial configuration by making A occupied initially – this will not change the eventually occupied set.

2. Curtains

In this section we introduce the oriented surfaces underlying our construction in their pure form. Later they will be modified by scaling and restricting to finite boxes.

Definition. A **curtain** is a set $D \subset \mathbb{Z}^3$ satisfying the following.

- (C1) For any $k \in \mathbb{Z}$, the intersection $D \cap \Lambda_k$ with layer k is an infinite path comprising steps e_1 and $-e_2$, with no three consecutive steps in the same direction; i.e. no e_1, e_1, e_1 or $-e_2, -e_2, -e_2$.
- (C2) For all $x \in D$, either $x + (0, 0, -1) \in D$ or $x + (1, 1, -1) \in D$.

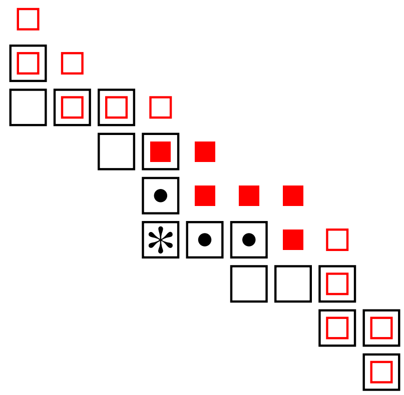

FIGURE 1. An illustration of the proof of Lemma 3. Two consecutive layers of a curtain are shown from above. Large squares belong to the upper layer of the curtain, and small (red) squares to the lower layer. The argument showing that the upper layer site marked with a star becomes occupied is indicated. We consider the portion of the lower layer path shown by filled squares, and deduce that all the upper layer sites marked with discs become occupied.

Figure 2 in the next section shows the intersection of a curtain with a box. The main goal of this section is to construct an infinite open curtain when q is sufficiently small. This will be done adapting the duality technique introduced in [DDG$^+$] for construction of Lipshitz surfaces. The curtain will form the outer boundary of a set reachable by certain paths from a fixed half space. Before giving the construction, we illustrate the relevance of curtains to bootstrap percolation with the following lemma. (Formally, the lemma will not be used in the proof of Theorem 1. Instead we will use a more specialized variant, Lemma 8.)

Lemma 3. *Let D be a curtain. Suppose that for every $x \in D$, the three sites x and $x + (0,0,1)$ and $x + (-1,-1,1)$ are all open. Moreover, suppose that for every $k \in \mathbb{N}$, the set $(D \cap \Lambda_k) + e_3$ contains some initially occupied site. If $D \cap \Lambda_0$ is initially entirely occupied, then $D \cap \bigcup_{k\in\mathbb{N}} \Lambda_k$ becomes entirely occupied in the modified bootstrap model on $\mathbb{Z}^3$.*

Proof. By induction on the layer, it suffices to prove that $D \cap \Lambda_1$ becomes entirely occupied. This verification is given in two steps below, and is illustrated in Figure 1. Let $Y := (D \cap \Lambda_0) + e_3$ be the set above the intersection with the bottom layer.

First we claim that every site in Y eventually becomes occupied. Indeed, Y is connected, open, and contains an occupied site, and every $y \in Y$ has an occupied neighbor $y - e_3 \notin Y$. The claim therefore follows from the bootstrap rule.

We now claim that every site in $Y-(1,1,0)$ also eventually becomes occupied. Indeed, consider such a site $z=y-(1,1,0)$ where $y\in Y$. Since Y is a path with the properties given in (C1), there exist sites $z+ae_1$ and $z+be_2$ in Y, where $a,b\in[0,3]$. Moreover, the intervening sites $z+ie_1$ and $z+je_2$ for $i\in(0,a)$ and $j\in(0,b)$ are open by (C2), and each has a neighbor in Y distinct from $z+ae_1$ and $z+be_2$. Since all sites in Y become occupied, so do all these sites, whence so does z.

The proof is now concluded by observing that $D\cap\Lambda_1\subseteq Y\cup(Y-(1,1,0))$. □

Now we proceed with the construction of a curtain. A **permissible path** is a finite sequence of sites $x_0,\dots,x_n\in\mathbb{Z}^3$ such that every step $x_{i+1}-x_i$ satisfies the following. Either it is a **taxed** step, which is to say that x_{i+1} is *closed*, and $x_{i+1}-x_i$ equals

$$(1,1,0).$$

Otherwise, the step is **free**, that is, $x_{i+1}-x_i$ lies in

$$\Big\{(-1,0,0),(0,-1,0),(0,0,-1),(-2,1,0),(1,-2,0),(-1,-1,1)\Big\}$$

(with no restriction on the states of sites).

Fix any (deterministic) set $H\subset\mathbb{Z}^3$ and let A be the (random) set reachable by permissible paths from H. Then define the following outer boundary:

$$D:=\big\{x\notin A : x-(1,1,0)\in A\big\}. \tag{2}$$

Lemma 4. *For any choice of H, the set D is either empty or an open curtain.*

The lemma is of course only useful when D is nonempty. This will be proved to hold under suitable circumstances in Proposition 5 below.

Proof of Lemma 4. We must prove that if D is nonempty then it is open and has properties (C1) and (C2). Consider any $x\in D$. By translation invariance of the definition, we assume without loss of generality that x is the origin $\mathbf{0}=(0,0,0)$.

Clearly, $\mathbf{0}$ is open, since otherwise the taxed step from $(-1,-1,0)\in A$ would make $\mathbf{0}\in A$.

Turning to property (C1), we have $(-1,-1,0)\in A$ but $\mathbf{0}\notin A$, so using the definition of free steps, $(-1,-2,0)\in A$ but $(1,0,0)\notin A$. We claim that either $(1,0,0)\in D$ or $(0,-1,0)\in D$, but not both. Indeed, if $(0,-1,0)\in A$ then $(1,0,0)\in D$, while if $(0,-1,0)\notin A$ then $(0,-1,0)\in D$. A similar argument shows that either $(-1,0,0)\in D$ or $(0,1,0)\in D$ but not both. This shows that $D\cap\Lambda_0$ is a union of disjoint paths with steps e_1 and $-e_2$. To check the restriction on three consecutive steps, note that $(0,-3,0)\in A$ but $(2,-1,0)\notin A$, which implies $(0,-3,0),(3,0,0)\notin D$. To show that $D\cap\Lambda_0$ comprises only one path, note that $(-1,-1,0)$ is a sum of two free steps, so the diagonal $\{(k,k,0):k\in\mathbb{Z}\}$ is partitioned into an interval belonging to A and an interval belonging to A^C. If D is nonempty then both intervals are nonempty, and so the diagonal contains exactly one site in D.

To prove property (C2), note that $(-1,-1,-1) \in A$ but $(1,1,-1) \notin A$. Consequently, if $(0,0,-1) \notin A$, then $(0,0,-1) \in D$. On the other hand, if $(0,0,-1) \in A$, then $(1,1,-1) \in D$. □

We now choose H to be the half-space

$$H := \{x : x_1 + x_2 + x_3 \leq 0\}, \tag{3}$$

and let A and D by defined as above. Note that, by property (C1), a curtain intersects the ray $\{(t,t,0) : t \in \mathbb{Z}\}$ in exactly one site.

Proposition 5. *There exist positive constants q_0 and c such that the following holds. For $q < q_0$, the set D constructed above is, almost surely, an open curtain. Furthermore, the probability that $(1,1,0) \in D$ tends to 1 as $q \to 0$, while for any $q < q_0$, the probability that D intersects the ray $\{(t,t,0) : t > k\}$ is at most e^{-ck} for all $k > 0$.*

Proof. Let $h = (1,1,1)$. Observe that the scalar product $\langle x, h\rangle$: equals 2 when x is the taxed step; equals -1 when x is a free step; and is nonpositive when $x \in H$.

Fix $k \geq 1$ and suppose $(k,k,0) \in A$. Then there exists a permissible path from H to $(k,k,0)$. By erasing loops, we may assume that the path is self-avoiding. Let n_F and n_T be the number of free and taxed steps of the path, respectively, and let $n = n_F + n_T$ be the total length. As $\langle (k,k,0), h\rangle = 2k$, the above observations about scalar products imply $-n_F + 2n_T \geq 2k$. It follows that $n \geq n_T \geq k$ and $n_T \geq (2k+n)/3$. Therefore,

$$\begin{aligned} \mathbb{P}\big((k,k,0) \in A\big) &\leq \mathbb{P}(\text{there exists a path as above}) \\ &\leq \sum_{n \geq k} 7^n q^{(2k+n)/3} \\ &= (7q)^k \cdot \sum_{n \geq 0} (7q^{1/3})^n \\ &\leq 8 \cdot (7q)^k, \end{aligned} \tag{4}$$

provided $q < q_0 := 8^{-3}$.

Note that $0 \in A$, so that if $(k,k,0) \notin A$ then $\{(t,t,0) : t \in [1,k]\}$ intersects D while $\{(t,t,0) : t > k\}$ does not. Thus, if $q < q_0$ then Lemma 4 implies that D is almost surely nonempty, and thus is an open curtain by Lemma 4. Equation (4) also gives the claimed exponential bound, since $8 \cdot (7q)^k \leq (56q_0)^k$. Taking $k = 1$ in (4), we get $\mathbb{P}((1,1,0) \notin D) \leq 56q$, giving the second claim. □

The results from this section are already strongly suggestive of the conclusions of Theorem 1, although by no means sufficient to prove them. Indeed, consider the initial configuration consisting of the fully occupied half-space $\{x : \langle x, e_3\rangle \leq 0\}$, and elsewhere product measure with densities p and q as usual. Then it is straightforward to deduce from Lemmas 3 and 4 and Proposition 5 that the probability that any fixed site $x \in \mathbb{Z}^3$ is eventually occupied converges to 1 as

$(p, q) \to (0, 0)$. Indeed, with high probability x lies in a curtain that has the properties in Lemma 3: the presence of the appropriately placed open sites can be guaranteed via [LSS], while the presence of an occupied site in each layer of the curtain holds almost surely. The remaining difficulty in the proof of Theorem 1 is the need to replace the occupied half-space with a finite nucleus. In particular, the argument sketched above will be replaced with an appropriate finite-volume version in Proposition 6 below.

3. Sails

A **box** in $\mathbb{Z}^3$ is a Cartesian product of any three integer intervals. Its **dimensions** are the cardinalities of the three intervals (in order). An **oriented box** is a box with a distinguished corner.

Fix an integer length scale L. This scale will be later chosen to be a suitable function of p. A **brick** is an oriented box of dimensions $4L$, $16L$, and $32L$, in any order. Bricks will be the units of our renormalization. We will formulate the required properties of bricks by translating and scaling a smaller box. The **proto-brick** $\widehat{B}$ is the oriented box $[0, 4L) \times [0, 4L) \times [0, 2L)$ with the distinguished corner at the origin.

We now formulate the key definition in our renormalization argument. The idea is that the proto-brick contains a suitably placed portion of a curtain, with properties analogous to those in Lemma 3, but restricted to the proto-brick. See Figure 2 for an illustration.

Definition. The proto-brick $\widehat{B}$ is **good** if there exists a set $\widehat{S} \subseteq \widehat{B}$ with the following properties:

(G1) all sites in the following set are open:
$$\sigma(\widehat{S}) := \big\{x,\ x + (0, 0, 1),\ x + (-1, -1, 1) : x \in \widehat{S}\big\} \cap \widehat{B};$$
(G2) $\widehat{S}$ satisfies (C2) in the definition of a curtain except at the bottom layer: for all $x \in \widehat{S} \setminus \Lambda_0$, either $x + (0, 0, -1) \in \widehat{S}$ or $x + (1, 1, -1) \in \widehat{S}$;
(G3) $\widehat{S} \subseteq \{x : 3L < x_1 + x_2 + x_3 < 4L\}$;
(G4) for each layer $k \in [0, 2L)$, the intersection $\widehat{S} \cap \Lambda_k$ is an oriented path that starts on $\{x : x_1 = 0\}$, ends on $\{x : x_2 = 0\}$ and makes steps $-e_2$ or e_1 with no consecutive three steps of the same type; and
(G5) for each layer except the top, there is an occupied site immediately above its intersection with $\widehat{S}$, i.e. $(\widehat{S} \cap \Lambda_k) + e_3$ contains an occupied site for each $k \in [0, 2L - 1)$.

Next we scale up this definition to a brick, starting with one in a **standard** location and orientation; namely, let B be the brick $[0, 4L) \times [0, 16L) \times [0, 32L)$ with the distinguished corner at the origin. For $x \in \widehat{B}$, define the following subset of B:
$$\texttt{cell}(x) = (x_1, 4x_2, 16x_3) + \{0\} \times [0, 4) \times [0, 16).$$

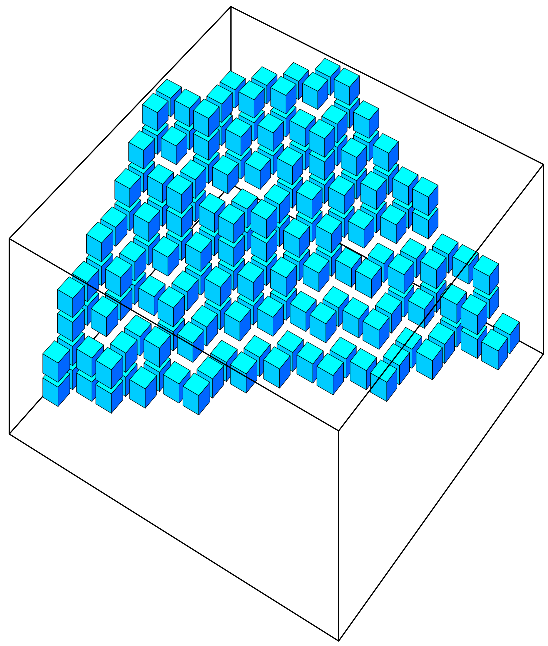

FIGURE 2. A good proto-brick with $L = 4$. The set $\widehat{S}$ comprises the (sites in the centers of) colored cubes. The origin is at the back corner, hidden by the set. The third coordinate axis is vertical.

See Figure 3. For a given configuration on B, we define an **auxiliary configuration** on $\widehat{B}$ by declaring a site $x \in \widehat{B}$ open if all sites in $\mathtt{cell}(x)$ are open; otherwise, we declare x closed. We also call x initially occupied if all sites in $\mathtt{cell}(x)$ are initially occupied. We call B **good** if, in the auxiliary configuration, $\widehat{B}$ is good. See Figure 3.

If B is good and $\widehat{S}$ is any set satisfying the above conditions, then we call

$$S = \bigcup_{x \in \widehat{S}} \mathtt{cell}(x)$$

a **sail** for B. Thus B is good if and only it has a sail.

Define the **tail** and **head** of B to be its lower and upper halves, $[0, 4L) \times [0, 16L) \times [0, 16L)$ and $[0, 4L) \times [0, 16L) \times [16L, 32L)$ respectively. The **tip** of B is the box $[0, 4L) \times [0, 4L) \times [16L, 32L)$, which is a quarter of the head. See Figure 3. The **base** of B is the bottom layer of cells $\bigcup_{x \in \widehat{B} \cap \Lambda_0} \mathtt{cell}(x)$. If B is good and S is a sail for B, then the **head**, **tail**, **base**, and **tip** of S are the intersections of S with the corresponding subsets of B.

If the brick B is good, and S is a sail for B, then we say that S is **activated** by time t if every site in the head of S is occupied at time t.

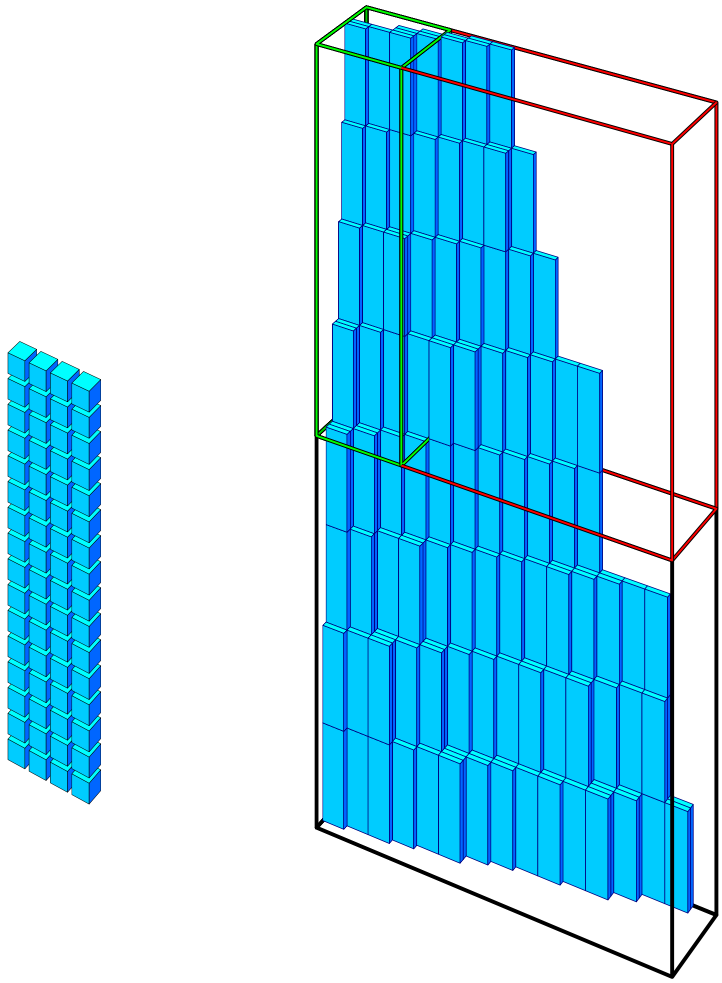

FIGURE 3. *Left:* An example of a cell $\mathtt{cell}(x)$, which is a box of dimensions $(1, 4, 16)$. *Right:* A good brick B and its sail S for $L = 4$. This is obtained by scaling the set $\widehat{S}$ of Figure 2 and replacing each of its sites with a cell. The head of the brick is outlined in red, and its tip in green. Again, the distinguished corner, the origin, is at the back, hidden by the sail.

Now we transfer all the above definitions to an arbitrary brick B' by isometry. More precisely, let η be an isometry of $\mathbb{Z}^3$ that maps B to B', respecting the distinguished corners. The head of B' is the image under η of the head of B. The brick B' is good if applying η^{-1} to the configuration makes B good, in which case a sail for B' is an image under η of a sail for B in that configuration, and so on.

We next show that with high probability a brick is good, and moreover the sail can be chosen to contains a specific site.

Proposition 6. *Let* $L = \lceil p^{-128} \rceil$*. Then the probability that* B *is good and has a sail* S *that contains the site* $x_0 := (L+1, 4L+4, 16L)$ *converges to* 1 *as* $(p,q) \to (0,0)$*.*

We remark that L is chosen to be such a large power of p^{-1} in the interests of simplicity, and since there is no need to optimize it. With suitable modifications to the definitions and proofs (perhaps at the expense of increased complexity), our arguments could likely be modified to work with renormalization units (bricks) of scale $p^{-1} \log p^{-1}$ — the fundamental requirement is that they should be large enough that every layer of a suitable surface spanning a brick contain an intially occupied site with high probability.

Proof of Proposition 6. Fix $\epsilon > 0$. For most of the proof we will consider the relevant event on the proto-brick $\widehat{B}$. Therefore let $\widehat{p} = p^{64}$ and $\widehat{q} = 1 - (1-q)^{64}$, which are the probabilities that a site is, respectively, initially occupied and open in the auxiliary configuration. Fix any $\epsilon > 0$.

Call a site x **swell** if x and $x + (0,0,1)$ and $x + (-1,-1,1)$ are all open in the auxilliary configuration. By the results of [LSS], the configuration of swell sites dominates a product measure on $\mathbb{Z}^3$ with parameter $1 - \widehat{q}'$, where $\widehat{q}' = \widehat{q}'(\widehat{q}) \to 0$ as $\widehat{q} \to 0$.

Next, we apply Proposition 5 and translation invariance to construct a swell curtain that lies close to the translated half-space $(L,L,L) + H$, rather than to H as in the proposition. To be precise, translate the configuration of swell sites by $-(L,L,L)$, construct the set D according to (2) and (3) in Section 2, but replacing open sites with swell sites in the definition of a permissible path, and then translate it back by (L,L,L) to obtain a set $\widehat{D}$ of swell sites that lies in $((L,L,L)+H)^C = \{x : x_1 + x_2 + x_3 > 3L\}$.

Let E_1 be the event that the set $\widehat{D}$ constructed in the above paragraph contains the site $\widehat{x}_0 = (1,1,0) + (L,L,L)$ (in which case Lemma 4 implies that it is a curtain). By the construction of $\widehat{D}$ in Section 2, E_1 is an increasing event with respect to the configuration of swell sites. (This follows because, in the notation of that section, the set A of sites reachable from H via permissible paths is decreasing). Therefore, by the first and second claims in Proposition 5 and the domination from [LSS] mentioned above, there exists $q_1 > 0$ such that, if $\widehat{q} < q_1$, then $\mathbb{P}(E_1) > 1 - \epsilon$.

Moreover, by the last claim of Proposition 5, together with the stochastic domination and translation invariance, for any deterministic $x = (x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = 3L$, we have

$$\mathbb{P}\Big(\widehat{D} \cap \big\{x + (t,t,0) : t > k\big\} \neq \emptyset\Big) \leq e^{-ck}, \qquad k > 0, \tag{5}$$

where $c > 0$ is an absolute constant. (Indeed, this event is decreasing in the configuration of swell sites, by the construction of $\widehat{D}$.) Now let

$$\widehat{S} := \widehat{D} \cap \widehat{B}$$

be the restriction of our set to the proto-brick. Let E_2 be the event that every $y \in \widehat{S}$ satisfies $y_1 + y_2 + y_3 < 4L$. Then (5) with $k = L/2$ and a union bound over every $x \in \widehat{B}$ with $x_1+x_2+x_3 = 3L$ imply that $\mathbb{P}(E_2) \geq 1-16L^2 \exp(-cL/2)$. Since $L \to \infty$ as $p \to 0$ (i.e. as $\widehat{p} \to 0$), for $\widehat{p}$ sufficiently small we have $\mathbb{P}(E_2) \geq 1 - \epsilon$.

We have shown that $\widehat{p}$ and $\widehat{q}$ are both sufficiently small then $\mathbb{P}(E_1 \cap E_2) \geq 1-2\epsilon$. On $E_1 \cap E_2$, the set $\widehat{S}$ satisfies properties (G1)–(G4) in the definition of a good proto-brick.

Now we address property (G5). So far we have not considered initially occupied sites (although the parameter p has appeared in the definition of the length scale L). One way to sample the auxiliary configuration is as follows. First declare each site closed independently with probability $\widehat{q}$. Then, conditional on the resulting configuration, declare each open site to be initially occupied independently with probability $\widehat{p}/(1-\widehat{q})$ $(\geq \widehat{p})$. Let E_3 be the event that $\widehat{S}$ satisfies property (G5). On $E_1 \cap E_2$, each intersection with a layer $\widehat{S} \cap \Lambda_k$ for $k \in [0, 2L-1)$ contains at least L sites (by (G3) and (G4)). Moreover, all sites in $(\widehat{S} \cap \Lambda_k) + e_3$ are open (by (G1)). Hence,

$$\mathbb{P}\big(E_3 \mid E_1 \cap E_2\big) \geq 1 - 2L(1-\widehat{p})^L \geq 1 - 2L\exp(-\widehat{p}L).$$

Since $L \sim \widehat{p}^{-2}$, this is at least $1 - \epsilon$ for $\widehat{p}$ sufficiently small.

We have shown that for $\widehat{p}$ and $\widehat{q}$ sufficiently small, with probability at least $1-3\epsilon$ the set $\widehat{S}$ satisfies (G1)–(G5) and contains $\widehat{x}_0$. Finally, recalling the definition of the auxilliary configuration, we deduce that for p and q sufficiently small, the brick B is good and has a sail containing x_0 (since it lies in $\texttt{cell}(\widehat{x}_0)$) with probability at least $1 - 3\epsilon$. □

4. Activation

Recall that a sail of a good brick is said to be activated if its head is fully occupied (at some time). To enable our renormalization argument, we now show that for appropriately placed good bricks, activation of one sail leads to activation of another.

Let B be the brick in standard position as before, and let B' be a brick with dimensions $(32L, 4L, 16L)$ such that the centroid of its tail coincides with the centroid of the tip of B. (The idea is that the tip of B cuts the tail of B' in two.) There are eight possible choices of B': two possible unoriented boxes that share a tail, each with four possible orientations – recall that a brick is an oriented box. See Figure 4 in the next section for examples.) Then we write $B \rhd B'$. Similarly for any isometry η of $\mathbb{Z}^3$ we write $\eta(B) \rhd \eta(B')$.

Proposition 7. *Let B and B' be any bricks such that $B \rhd B'$. Suppose that they are both good and let S and S' be any respective sails. In the modified bootstrap percolation model, if S is activated by some time, then S' is activated by some later time.*

We separate the proof into the following four lemmas, starting with the underlying growth mechanism.

Lemma 8. *Suppose that the proto-brick $\widehat{B}$ is good, and let $\widehat{S}$ be any set satisfying the conditions in the definition of good. Assume also that the intersection $\widehat{S} \cap \Lambda_0$ with the bottom layer is entirely occupied initially, and that $\mathbb{Z}^3 \setminus \sigma(\widehat{S})$ is entirely closed. Then $\widehat{S}$ is entirely occupied at some time.*

Proof. The argument is essentially the same as for Lemma 3, except that one must verify that the relevant sites lie in the proto-brick. We prove by induction on $k = 0, \dots, 2L-1$ that the layer $\widehat{S} \cap \Lambda_k$ is eventually occupied. For $k = 0$ this holds by assumption.

Fix $k \geq 1$ and let $Y = (\widehat{S} \cap \Lambda_{k-1}) + e_3$ be the set above the intersection with the layer below. By (G4), Y is a path with steps e_1 and $-e_2$, no three consecutive steps of the same type, and its start and end on the coordinate axes of layer Λ_k.

The set Y becomes occupied, since it is connected and open, it contains an occupied site, and it is adjacent to $\widehat{S} \cap \Lambda_{k-1}$ which becomes occupied by the inductive hypothesis.

Now consider a site $z \in \widehat{S} \cap \Lambda_k$. By (G2), either $z \in Y$ (in which case z becomes occupied as verified above) or $z = y - (1, 1, 0)$ where $y \in Y$. In the latter case, by the above properties of the path Y, there exist with $a, b \in [0, 3]$ such that $z + ae_1, z + be_2 \in Y$ — these two sites lie in $\widehat{B}$ because of the condition about the start and end of the path and the fact that $y, z \in \widehat{B}$. (See Figure 1.) Now each of the sites $z + ie_1$ and $z + je_2$ for $i \in (0, a)$ and $j \in (0, b)$ belong to $\sigma(\widehat{S})$ and hence are open. By considering these sites in decreasing order of i and j, we see that they each become occupied, hence so does z. □

The following comparison lemma states that cutting off part of a configuration only increases the eventually occupied set, provided we make the cut surface occupied. This will enable us to make use of sails that intersect each other.

Lemma 9. *Consider a set of sites A, and a subset $F \subseteq A$. Let B be a connected component of $A \setminus F$. Suppose that every site in A^C is closed but that the initial configuration is otherwise arbitrary. Now alter the initial configuration by making F initially occupied but $A \setminus (F \cup B)$ closed. The alteration (weakly) increases the set of eventually occupied sites in B.*

Proof. We proceed by induction on the time step. Suppose that at all times prior to t, the set of occupied sites of B in the altered dynamics dominates the set in the original dynamics. Assume that a site $x \in B$ becomes occupied in the original dynamics at time t. Any neighbor of x that was occupied in the original dynamics at time $t-1$ either lies in B, in which case it is also occupied in the altered dynamics by the induction hypothesis, or it lies in F, in which case it was *initially* occupied in the altered dynamics. Thus x also becomes occupied in the altered dynamics. □

Lemma 10. *For any configuration on a brick B, consider the auxiliary configuration on the proto-brick $\widehat{B}$, and perform the modified bootstrap percolation dynamics from the auxiliary configuration with all sites outside $\widehat{B}$ closed. If x becomes occupied in the auxiliary dynamics, then* `cell`(x) *becomes fully occupied in the original dynamics.*

Proof. This follows by induction on the time steps of the dynamics started from the auxiliary configuration. Suppose that a site x in the auxiliary configuration becomes occupied at step t. Then, according to the modified bootstrap rule, it was open and had occupied neighbors in at least two coordinate directions at step $t-1$. Therefore `cell`(x) is fully open, and, by the inductive hypothesis, two adjacent cells in different directions were fully occupied at some time in the original dynamics. It follows that `cell`(x) will become fully occupied in the original dynamics (at most $16+4-1$ steps later). □

Next we state a geometric fact about sails. Let $A \subseteq \mathbb{Z}^3$ and let F, B_1, B_2 be disjoint subsets of A. We say that F **separates** B_1 and B_2 in A if $A \setminus F$ contains no nearest-neighbor path from B_1 to B_2.

Lemma 11. *Suppose that the standard brick B is good. Then any sail S for B separates, in the tip of B, the two faces of the tip $\{0\} \times [0, 4L) \times [16L, 32L)$ and $\{4L-1\} \times [0, 4L) \times [16L, 32L)$.*

Proof. By property (G4) of a good proto-brick, the intersection of S with a layer Λ_k is an oriented path, thickened by conversion of sites to cells. Therefore its complement $\Lambda_k \setminus S$ clearly has two components, U_k and V_k say, which contain the intersections of the first and second faces respectively with Λ_k, by (G3).

It remains to check that no site of U_{k-1} is adjacent to a site of V_k, and likewise for V_{k-1} and U_k, for $k = 1, \ldots, 4L-1$. Since such adjacent sites would differ by e_3, this is easily verified from property (G2). (Also see Figure 1). □

Now we prove the main result of this section.

Proof of Proposition 7. Consider the dynamics started from the altered configuration in which the base of S' is initially occupied, $(B')^C$ is closed, and all other sites retain their original initial states. By Lemmas 8 and 10, S' becomes fully occupied, and in particular its head becomes fully occupied.

Now we will use Lemma 9 to compare the altered configuration with the original one. Let H be the head of S. By Lemma 11, H separates the base of B' from the head of B' in B'. Moreover, H is fully occupied at some time. Let K be

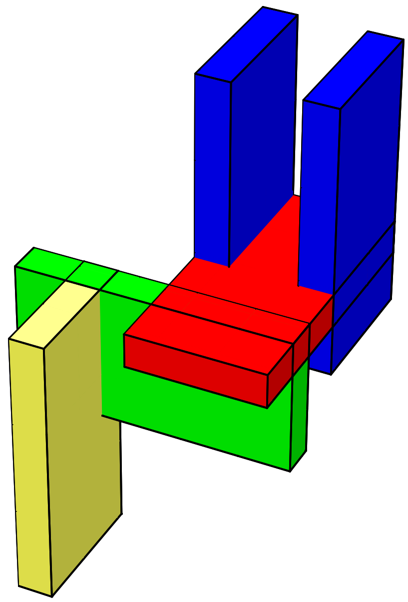

FIGURE 4. Illustration of the proof of Lemma 12. The initial brick B (at lower left) is yellow, and $B_1 = B'_1$ is green. The red box depicts both B_2 and B'_2, which are the same box but with different orientations. Finally, the two blue bricks are B_3 and B'_3. These are translations of B by $(10, 22, 22)L$ and $(22, 22, 22)L$.

the component of $B' \setminus H$ that contains the head of S'. By Lemma 9, the final set of occupied sites in K is (weakly) larger than it was starting from the altered configuration, so in particular the head of S' becomes fully occupied. □

5. Renormalization

In this section we prove the main result, Theorem 1, as well as Corollary 2. We start with a simple geometric ingredient. Recall that B is the brick in standard position.

Lemma 12. *There exist bricks B_i, B'_i, for $i = 1, 2, 3$, with*

$$B \rhd B_1 \rhd B_2 \rhd B_3,$$
$$B \rhd B'_1 \rhd B'_2 \rhd B'_3,$$

such that B, B_3, and B'_3 are distinct and have the same orientation. Furthermore, there exist vectors $u, u' \in \mathbb{Z}^3$, neither of them depending on L, such that $B_3 = B + Lu$ and $B'_3 = B + Lu'$ (so in particular Lu and Lu' are the distinguished corners of B_3 and B'_3 respectively).

Proof. See Figure 4. Recall that B has dimensions $(4L, 16L, 32L)$. We choose B_1 and B'_1 equal to each other, with dimensions $(32L, 4L, 16L)$, and satisfying

$B \triangleright B_1$. Then take B_2 and B_2' to be the same box as each other, with dimensions $(16L, 32L, 4L)$, but with different orientations and in particular different tips. Finally take B_3 and B_3' to be suitable translations of B, as determined by these tips. □

Proof of Theorem 1. As discussed in the introduction, it suffices to prove the case of the modified bootstrap model on $\mathbb{Z}^3$.

We will compare with oriented percolation in $\mathbb{Z}^2$. Let $L = \lceil p^{-128} \rceil$ and let $u, u' \in \mathbb{Z}^3$ be as in Lemma 12. Also fix $\epsilon > 0$. For $a = (a_1, a_2) \in \mathbb{Z}^2$, define the associated brick

$$B(a) := B + La_1 u + La_2 u'.$$

Call the site a **excellent** if the translations by $La_1 u + La_2 u'$ of the seven bricks B, B_i, B_i' of Lemma 12 are all good.

Suppose that a and b are excellent, so that in particular the bricks $B(a)$ and $B(b)$ are good. Suppose also that there is a path of excellent sites in $\mathbb{Z}^2$ from a to b consisting of steps e_1 and e_2. (We call a path with these steps **oriented**.) Then by Lemma 12 and Proposition 7, if some sail of $B(a)$ is activated then any sail of $B(b)$ is activated at some later time.

Let E be the event that there exists an excellent bi-infinite oriented path π in $\mathbb{Z}^2$ containing $\mathbf{0} = (0,0)$, and that moreover $B = B(\mathbf{0})$ has a good sail containing $x_0 := (L{+}1, 4L{+}4, 16L)$ in its head. The event that site a is excellent depends only on the initial states of sites within distance CL of La, for some absolute constant C. Therefore, the random configuration of excellent sites is m-dependent for some fixed m not depending on L. Therefore, by [LSS], Proposition 6, and the fact that oriented percolation on $\mathbb{Z}^2$ has a nontrivial phase transition (see e.g. [Gri]), if p and q are sufficiently small then $\mathbb{P}(E) \geq 1 - \epsilon$.

It remains to show that *some* sail on the path is activated, for which a rather crude sprinkling argument will suffice. Assuming $2p + q < 1$, we consider two coupled initial configurations. The **level-1** configuration has parameters p and q as before. Conditional on the level-1 configuration, the **level-2** configuration is obtained by adding some further occupied sites; specifically, we declare each open site that was not initially occupied at level 1 to be initially occupied at level 2 independently with probability $p/(1{-}p{-}q)$, and leave the configuration otherwise unchanged. The law of the level-2 configuration is simply a product measure with parameters $2p$ and q. Now condition on the level-1 configuration, and suppose that it is such that E occurs at level 1. Fix an excellent oriented path π as in the definition of E, and let π^+ and π^- be the forward and backward halves of π that start at 0 and end at 0 respectively. Then for each site a of π^-, *all* open sites in the brick $B(a)$ are initially occupied at level 2 with probability at least $p^{|B|}$, independently for each such a. Therefore, conditionally almost surely, some site a on π^- has this property, which implies in particular that any sail of the associated brick $B(a)$ is activated at level 2.

We conclude that if $2p$ and q are sufficiently small then with probability at least $1-\epsilon$ there exists an infinite sequence of distinct activated sails, each intersecting the next, one of which contains x_0 in its head. By translation invariance we conclude that with probability at least $1-\epsilon$, the origin lies in an infinite connected eventually occupied set, as required. $\square$

Proof of Corollary 2. It follows from Theorem 1 that $\lim_{q\to 0^+}\phi^+(q)=1$, which implies that $q_c>0$. As ϕ is a decreasing function, it is positive on $[0,q_c)$. Our remaining task is to prove the claimed upper bound on q_c, for which it suffices to consider the *standard* (as opposed to modified) bootstrap model with $r=2$ on $\mathbb{Z}^d$ for $d\geq 3$.

Call a site **3-open** if it is open and has at least 3 open sites among its $2d$ neighbors. Let p_c' be the critical probability for existence of an infinite connected set of 3-open sites in $\mathbb{Z}^d$. Then clearly $p_c'\geq p_c^{\text{site}}$. For $d=3$, the method of essential enhancements [AG, BBR] shows that $p_c'>p_c^{\text{site}}$. (The strict inequality is expected to hold for $d\geq 4$ also, but no complete proof is available – see [BBR].)

For a set $Z\subseteq\mathbb{Z}^d$, let the external boundary ∂Z be the set of sites in $\mathbb{Z}^d\setminus Z$ that have a neighbor in Z. Note that $|\partial Z|\leq 2d|Z|$. Assume that every site in Z is open, but no site in ∂Z is 3-open, and no site in $Z\cup\partial Z$ is initially occupied. Then we claim that no site in $Z\cup\partial Z$ is ever occupied. Indeed, suppose on the contrary that $x\in Z\cup\partial Z$ is a first site in the set to become occupied, say at time $t\geq 1$. Then $x\in\partial Z$, and so x has at most 2 open neighbors, of which at least one is in Z, which by assumption is unoccupied at time $t-1$. So x has at most one occupied neighbor at time $t-1$. Since $r=2$, this contradicts the assumption that x becomes occupied at time t.

Now let $q>1-p_c'$. Given the random configuration on $\mathbb{Z}^d$, create an **adjusted** configuration by converting all closed sites among the origin and its $2d$ neighbors to open (but not initially occupied) sites. Let $\mathcal{Z}$ be the maximal connected set of 3-open sites containing the origin in the adjusted configuration. Clearly, $0<|\mathcal{Z}|<\infty$ almost surely. Then we have

$$\begin{aligned}
&\mathbb{P}(0 \text{ is eventually occupied})\\
&\leq \mathbb{P}(0 \text{ is eventually occupied starting from the adjusted configuration})\\
&\leq \mathbb{P}(\mathcal{Z}\cup\partial\mathcal{Z} \text{ contains an initially occupied site})\\
&\leq \sum_{k=1}^{\infty}\mathbb{P}\big(|\mathcal{Z}|=k\big)\big(1-(1-p)^{k+2dk}\big).
\end{aligned}$$

This tends to 0 as $p\to 0^+$, by dominated convergence. $\square$

6. Open Problems

Recall that $\phi(p,q)$ is the probability that the origin is eventually occupied with densities p and q of initially occupied and closed sites respectively, and that we define $\phi^+(q)=\phi(0^+,q)$ and $q_c=\inf\{q:\phi^+(q)=0\}$.

(i) For which dimensions and thresholds $d > r \geq 3$ is it the case that $\phi(p, q) \to 1$ as $(p, q) \to (0, 0)$? As conjectured in [Mor], the answer "all" seems plausible. (The current paper proves the cases $d > r = 2$, while the conclusion fails for $d = r$.)

(ii) Where the convergence in (i) above does not hold (presumably, only for $d = r$), suppose that $p, q \to 0$ in such a way that $\log q / \log p \to \alpha$. For which α does ϕ converge to 0, or to 1? The articles [GM2] and [GHS] address $d = r = 2$ and $d = r = 3$ respectively.

(iii) Is ϕ^+ continuous at q_c?

(iv) Consider the critical value $q_c = q_c(d)$ as a function of dimension (with $r = 2$, say). Does q_c approach 1 as $d \to \infty$ and, if so, at what rate?

(v) Let T be the first time the origin is occupied. What is the asymptotic behavior of T as $p, q \to 0$? (For example, find "close" functions f and g of p and q for which $f \leq T \leq g$ with high probability).

(vi) For $r = 2$ and $d = 3$, consider

$$\gamma(q) := \limsup_{p \to 0+} p^{-1} \phi(p, q).$$

Is there a $q > q_c$ for which this is infinite? If so, this would distinguish the phase transition in the case $r = 2$ from that of the case $r = 1$ (where $q_c = 1 - p_c^{\text{site}}(\mathbb{Z}^d)$, and $\gamma(q)$ is finite for all $q > q_c$.)

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References

[AFP] H. Amini, N. Fountoulakis, and K. Panagiotou. Bootstrap percolation in inhomogeneous random graphs. arXiv:1402.2815.

[AG] M. Aizenman and G. Grimmett. Strict monotonicity for critical points in percolation and ferromagnetic models. *J. Statist. Phys.*, 63(5-6):817–835, 1991.

[AL] M. Aizenman and J. L. Lebowitz. Metastability effects in bootstrap percolation. *J. Phys. A*, 21(19):3801–3813, 1988.

[BBDM] J. Balogh, B. Bollobás, H. Duminil-Copin, and R. Morris. The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.*, 364(5):2667–2701, 2012.

[BBR] B. Bollobás, P. Balister, and O. Riordan. Essential enhancements revisited. arXiv:1402.0834.

[BDGM1] G. J. Baxter, S. N Dorogovtsev, A. V Goltsev, and J. Mendes. Heterogeneous k-core versus bootstrap percolation on complex networks. *Physical Review E*, 83(5):051134, 2011.

[BDGM2] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes. Bootstrap percolation on complex networks. *Phys. Rev. E*, 82:011103, Jul 2010.

[BDMS] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith. Universality of two-dimensional critical cellular automata. *Proceedings of London Mathematical Society.* To appear.

[BT] T. Bodineau and A. Teixeira. Interface motion in random media. *Comm. Math. Phys.*, 334(2):843–865, 2015.

[CLR] J. Chalupa, P. L. Leath, and G. R. Reich. Bootstrap percolation on a Bethe lattice. *Journal of Physics C: Solid State Physics*, 12(1):L31, 1979.

[DDG$^+$] N. Dirr, P. W. Dondl, G. R. Grimmett, A. E. Holroyd, and M. Scheutzow. Lipschitz percolation. *Electron. Commun. Probab.*, 15:14–21, 2010.

[DDS] N. Dirr, P. W. Dondl, and M. Scheutzow. Pinning of interfaces in random media. *Interfaces Free Bound.*, 13(3):411–421, 2011.

[DEK$^+$] M. Damron, S. M. Eckner, H. Kogan, C. M. Newman, and V. Sidoravicius. Coarsening dynamics on $\mathbb{Z}^d$ with frozen vertices. *J. Stat. Phys.*, 160(1):60–72, 2015.

[DvE] H. Duminil-Copin and A. C. D. van Enter. Sharp metastability threshold for an anisotropic bootstrap percolation model. *Ann. Probab.*, 41(3A):1218–1242, 2013.

[GG] J. Gravner and D. Griffeath. First passage times for threshold growth dynamics on $\mathbf{Z}^2$. *Ann. Probab.*, 24(4):1752–1778, 1996.

[GH1] G. R. Grimmett and A. E. Holroyd. Plaquettes, spheres, and entanglement. *Electron. J. Probab.*, 15:1415–1428, 2010.

[GH2] G. R. Grimmett and A. E. Holroyd. Geometry of Lipschitz percolation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(2):309–326, 2012.

[GH3] G. R. Grimmett and A. E. Holroyd. Lattice embeddings in percolation. *Ann. Probab.*, 40(1):146–161, 2012.

[GHM] J. Gravner, A. E. Holroyd, and R. Morris. A sharper threshold for bootstrap percolation in two dimensions. *Probab. Theory Related Fields*, 153(1-2):1–23, 2012.

[GHS] J. Gravner, A. E. Holroyd, and D. Sivakoff. Polluted bootstrap percolation in three dimensions. arXiv:1706.07338.

[GM1] O. Garet and R. Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster. *ESAIM Probab. Stat.*, 8:169–199, 2004.

[GM2] J. Gravner and E. McDonald. Bootstrap percolation in a polluted environment. *J. Statist. Phys.*, 87(3-4):915–927, 1997.

[Gri] G. R. Grimmett. *Percolation.* Springer-Verlag, Berlin, second edition, 1999.

[GZH] J. Gao, T. Zhou, and Y. Hu. Bootstrap percolation on spatial networks. *Scientific reports*, 5, 2015.

[HM] A. E. Holroyd and J. B. Martin. Stochastic domination and comb percolation. *Electron. J. Probab.*, 19:no. 5, 16, 2014.

[Hol] A. E. Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields*, 125(2):195–224, 2003.

[JŁTV] S. Janson, T. Łuczak, T. Turova, and T. Vallier. Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.*, 22(5):1989–2047, 2012.

[KL] C. Koch and J. Lengler. Bootstrap percolation on geometric inhomogeneous random graphs. arXiv:1603.02057.

[LSS] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1):71–95, 1997.

[Mor] R. Morris. Bootstrap percolation, and other automata. *European Journal of Combinatorics.* To appear.

[MP] B. Morris and Y. Peres. Evolving sets and mixing. In *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, pages 279–286. ACM, New York, 2003.

[Sch] R. H. Schonmann. On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.*, 20(1):174–193, 1992.

[vE] A. C. D. van Enter. Proof of Straley's argument for bootstrap percolation. *J. Statist. Phys.*, 48(3-4):943–945, 1987.

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