Weakly robust periodic solutions of one-dimensional cellular automata with random rules

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We study 2-neighbor one-dimensional cellular automata with a large number n of states and randomly selected rules. We focus on the rules with weakly robust periodic solutions (WRPS). WRPS are global configurations that exhibit spatial and temporal periodicity and advance into any environment with at least a fixed strictly positive velocity. Our main result quantifies how unlikely WRPS are: the probability of existence of a WRPS within a finite range of perods is asymptotically proportional to $1/n$, provided that a divisibility condition is satisfied. Our main tools come from random graph theory and the Chen-Stein method for Poisson approximation.

Key words: Cellular automaton, periodic solution, random rule, weak robustness.

1 INTRODUCTION

The mechanisms that cause global coordination in an evolving system of locally interacting agents have long attracted interest in diverse fields of science [14]. Cellular automata (CA) are convenient models of such phenomena, resulting in dynamics such as predator-prey oscillatory wave dynamics [12], firefly CA [9] that exhibit synchronization, and excitable media rules with periodic invariant states [2]. In the last-mentioned models, the source of periodicity are stable periodic objects that spread their influence until they entrap the states of every site into a periodic cycle. A general investigation of expanding periodic structures is therefore of interest and a natural starting point is in the simplest setting, that of one-dimensional CA. The concept of **robust periodic solutions** (RPS) was thus developed in [4], after it was briefly introduced in [3]. As these papers address behavior of finite seeds within a quiescent state, it is important that an RPS is a *finite* periodic structure, so that it can emerge within a seed.

In the present paper, we continue our study of one-dimensional cellular automata (CA) with random rules, initiated in [7]. In that paper, we focused on the probability of existence of (global) periodic solutions with given spatial and temporal periods. Here, we narrow our focus and estimate how likely it is that a random rule has a periodic solution that is able to "repair" itself after it is replaced by an arbitrary configuration on the right half-line. By preserving the periodic solution on the entire left half-line, we sidestep its emergence and are able to focus solely on its ability to expand. The desired property is thus weaker than robustness; see the formal definition below.

As in [7], we investigate CA rules with n states and one-sided neighborhood with 2 neighbors. To be precise, we consider one-dimensional CA with state space encoded by $\mathbb{Z}_n = \{0, \dots, n-1\}$, and with the dynamics determined by a 2-neighbor **rule** $f : \mathbb{Z}_n^2 \to \mathbb{Z}_n$ as follows. The (**spatial**) **configuration** of the CA at time t is a function $\xi_t : \mathbb{Z} \to \mathbb{Z}_n$, giving every $x \in \mathbb{Z}$ its state $\xi_t(x) \in \mathbb{Z}_n$ at time t. Given an initial configuration ξ_0, the evolution of the CA is generated

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iteratively by the parallel application of the local rule f:

$$\xi_{t+1}(x) = f(\xi_t(x-1), \xi_t(x)), \quad \text{for all } x \in \mathbb{Z}.$$

Thus, ξ_0 and f together determine the **trajectory** ξ_t, $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. The trajectory is commonly visually presented as a picture of its **space-time configuration**, the map $(x,t) \mapsto \xi_t(x)$ from $\mathbb{Z} \times \mathbb{Z}_+$ to $\mathbb{Z}_n$, in which sites (x,t) are squares, each painted a color according to its state $\xi_t(x)$; by convention, the temporal axis is oriented downward and the spatial axis is oriented rightward.

Assume that a CA given by the rule f starts from a periodic global configuration $\xi_0 : \mathbb{Z} \to \mathbb{Z}_n$ that satisfies $\xi_0(x) = \xi_0(x + \sigma)$, for all $x \in \mathbb{Z}$. If we also have $\xi_\tau = \xi_0$, and τ and σ are both minimal, then we have found a **periodic solution** (PS) under rule f, with **spatial period** σ and **temporal period** τ. We will not distinguish between spatial and temporal shifts of a PS. Therefore, each configuration $\xi_t \in \mathbb{Z}_n^{\mathbb{Z}}$, $t \geq 0$, characterizes the PS and is called a **PS configuration**. Within the resulting space-time configuration, any rectangle with τ rows and σ columns also characterizes the PS, and we call any such rectangle the **tile** of the PS. Thus we do not distinguish between tiles which are spatial or temporal rotations of each other.

Before we formally define the expansion property of PS under study, we first illustrate it by a few examples and provide some motivation. As in [7], we name a rule f by listing its values $f(a,b)$ for all pairs in reverse alphabetical order, from $(n-1, n-1)$ to $(0,0)$. Thus, the 3-state rule *102222210* has $f(2,2) = 1$, $f(2,1) = 0$, $f(2,0) = 2$, $\ldots$, $f(0,0) = 0$. Figure 1(a)(b)(c) exhibits three pieces of the space-time configurations under the this rule. In these examples, the PS is given by the tile

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 \end{matrix}$$

and is seen to expand after the PS configuration is replaced by three different configurations to the right of some site in $\mathbb{Z}$. In fact, given any such replacement, with an arbitrary configuration, the periodic configuration will advance to the right with at least a minimal velocity $v > 0$ as time increases, uniformly over the perturbed environment. By contrast, the rule *102122210* differs by a single assignment $f(1,2) = 1$, has the same PS, but now there is a perturbed environment, say the one given in Figure 1(d), which stops the advance of this PS.

Proceeding to the formal definition, let ξ_0 be a PS configuration under a CA rule, and η_0 be any initial configuration that agrees with ξ_0 on all $x \leq y$, for some $y \in \mathbb{Z}$; by default, $y = 0$. Adapting the definition from [4], we call such initial configurations **proper** for the PS ξ_0. Let ξ_t and η_t be the configurations obtained by running f starting with ξ_0 and η_0, respectively. Let

$$s_t = s_t(\eta_0) = \sup\{x \in \mathbb{Z} : \eta_t(z) = \xi_t(z) \text{ for all } z \leq x\}$$

be the rightmost location up to which η_t and ξ_t agree at time t. Then the **expansion velocity in the initial environment** η_0 is

$$v(\eta_0) = \liminf_{t \to \infty} \frac{s_t}{t},$$

which describes the rate at which spatial periodicity expands. The **expansion velocity**

$$v = \inf\{v(\eta_0) : \eta_0 \text{ is proper for } \xi_0\}$$

then measures uniformity over all environments. If $v > 0$, then the PS ξ_t is **weakly robust** (WRPS). With this terminology, we distinguish this property from the more restrictive robustness from [4], which, as already mentioned, requires that η_0 transitions to a background state at finite distance to the left of the origin. In Figure 1(a)(b)(c), we see the behavior of a WRPS in three different environments: in the environment (a), the expansion velocity is 1; in the environment (b), it is $4/7$; and we conjecture (but do not have a proof) that the expansion velocity in the environment (c) is again 1, almost surely. See Section 3 for a brief discussion on estimation of v for WRPS; in particular, we are

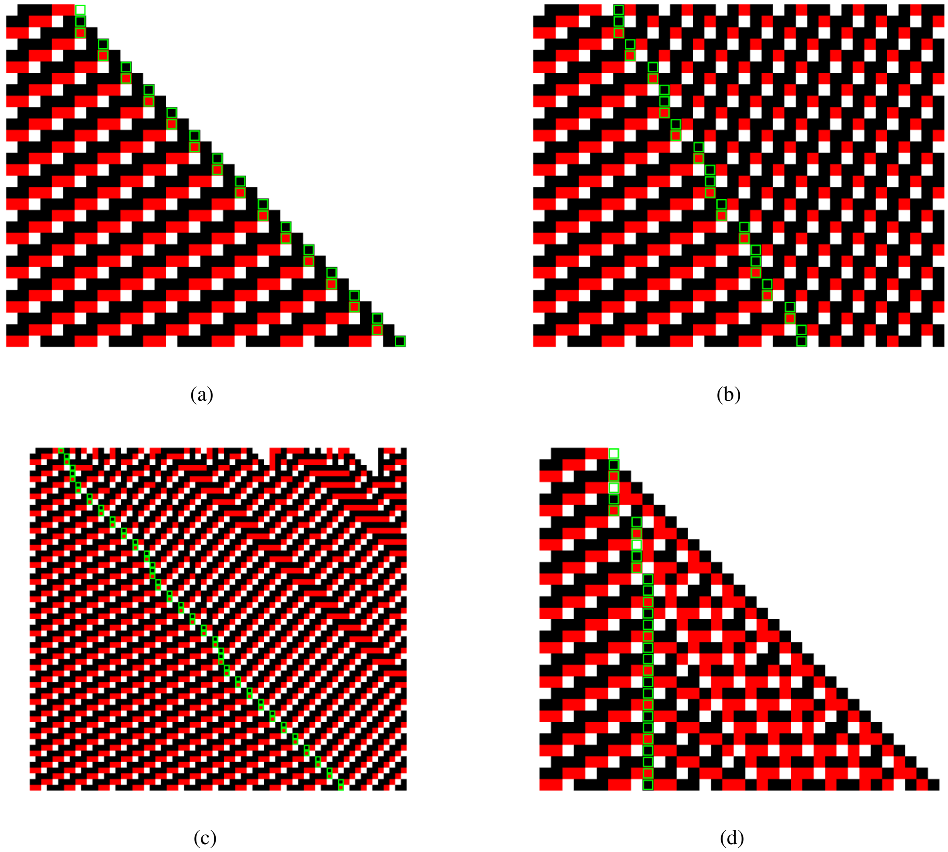

FIGURE 1: Panels (a), (b), and (c): pieces of the space-time configuration of the 3-state rule *102222210*. The starting configuration is $(022211)^\infty$ on $(-\infty, 0]$, while on $[1, \infty)$ it is (a) 0^∞, (b) $(02122)^\infty$, and (c) an independent and uniformly random selection of states. Observe that the PS expands in these three cases, as required by weak robustness. Panel (d): started from the same initial state as in (a), but now under the 3-state rule *102122210*, the expansion of the same PS eventually terminates, therefore the PS is not weakly robust. The rightmost site of agreement s_t is outlined in green. The random case (c) is run twice as long as others, as the convergence is less apparent.

able to prove that for this example $v \in [9/17, 4/7]$. By contrast, Figure 1(d) provides an example of a PS which has $v = 0$ and is therefore not weakly robust.

From a theoretical standpoint, the importance of PS with the described expansion property also stems from their relation to the stable limit cycles in continuous dynamical systems. Limit cycles, also known as isolated closed trajectories, are such that neighboring trajectories either spiral toward or away from them. In the former case, when a perturbation of a limit cycle converges back, the limit cycle is called stable [13]. Thus we consider an analogous stability property for CA: after a one-sided perturbation of a periodic configuration, the dynamics make the configuration converge back. In keeping with the terminology from [4], we refer to such stability as robustness. We remark that the minimal velocity v gives the minimal exponential rate of convergence to the PS in the standard metric, by which the distance between $\xi, \eta \in \mathbb{Z}_n^{\mathbb{Z}}$ is $\mathfrak{m}(\xi, \eta) = 2^{-n}$, where $n = \inf\{|x| : \xi(x) \neq \eta(x)\}$.

As in [7], we choose an n-state 2-neighbor rule f uniformly at random, that is, with each of n^{n^2} choices having equal probability. To formulate our main result on the probability of existence of WRPS, we introduce the following notation. For two sets $\mathcal{T}, \Sigma \subset \mathbb{N} = \{1, 2, \ldots\}$, let $\mathcal{R}_{\mathcal{T},\Sigma}$ be the (random) set of WRPS of a randomly selected n-state rule f, with temporal period τ and spatial period σ satisfying $(\tau, \sigma) \in \mathcal{T} \times \Sigma$. While our results for existence of PS [7] are valid for arbitrary finite $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$, we impose a divisibility restriction for our result on WRPS.

Theorem 1.1. *Select a two-neighbor CA rule f uniformly at random among all n^{n^2} rules. Let $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$ be fixed and finite. If there exists $(\tau, \sigma) \in \mathcal{T} \times \Sigma$ such that $\sigma \mid \tau$, then $\mathbb{P}(\mathcal{R}_{\mathcal{T},\Sigma} \neq \emptyset) = c(\mathcal{T}, \Sigma)/n + o(1/n)$, where $c(\mathcal{T}, \Sigma)$ is a constant depending only on $\mathcal{T}$ and Σ.*

We give the explicit formula for $c(\mathcal{T}, \Sigma)$ in equation (1) in Section 5.

In addition to [7], we have investigated periodic solutions for cellular automata in [5, 6] with the emphasis on maximal temporal periods; some further results and conjectures on robustness are in [8]. As already indicated, much of the motivation for the present paper came from [4], where all the 64 one-dimensional binary 3-neighbor edge CA rules and their RPS are studied. To our knowledge, robustness of PS is first addressed for the *Exactly 1* rule, i.e., the elementary CA *Rule 22*, in [3].

The rest of the paper is organized as follows. In the next section, we recall some preliminary results from [7]. While we summarize major definitions and tools, we omit the proofs and refer the reader to [7] for a more detailed discussion. In Section 3, we introduce the property of a tile that distinguishes a WRPS from a PS, i.e., the decidability of labels in a tile. We establish the probability that a label exhibits such property for a randomly selected rule in Section 4 and give the proof of Theorem 1.1 in Section 5. In the final section, we discuss the possible directions and methods to extend and generalize our results.

2 PRELIMINARIES

The main purpose of this section is to gather the relevant definitions and results from [7]. All lemmas are restatements of results in [7], where the proofs are provided.

2.1 Tiles of PS

We may express a tile with periods τ and σ as $T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$, once we fix an element in T to be placed at the position $(0, 0)$. We use the notation $\texttt{row}_i(T)$ and $\texttt{col}_j(T)$ to denote the ith row and jth column of a tile T and use $a_{i,j}$ to denote the element at the ith row and jth column of T, where we always interpret the two subscripts modulo τ and σ, respectively.

Let T_1 and T_2 be two tiles and $a_{i,j}$, $b_{k,m}$ be the corresponding elements. If $(a_{i,j}, a_{i,j+1}) \neq (b_{k,m}, b_{k,m+1})$ for $i, j, k, m \in \mathbb{Z}_+$, then T_1 and T_2 are called **orthogonal**, denoted by $T_1 \perp T_2$. In this case, we observe that two assignments $(a_{i,j}, a_{i,j+1}) \mapsto a_{i+1,j+1}$ and $(b_{k,m}, b_{k,m+1}) \mapsto b_{k+1,m+1}$ occur independently for a uniformly chosen random CA rule. We say that T_1 and T_2 are **disjoint**, and denote this property by $T_1 \cap T_2 = \emptyset$, if $a_{i,j} \neq b_{k,m}$, for $i, j, k, m \in \mathbb{Z}_+$. Clearly, every pair of disjoint tiles is orthogonal, but not vice versa.

The following quantities associated with a tile play a important role in the sequel. We define the **assignment number** of T to be $p(T) = |\{(a_{i,j}, a_{i,j+1}) : a_{i,j}, a_{i,j+1} \in T\}|$, i.e., the number of values of the rule f specified by T. Also, let $s(T) = |\{a_{i,j} : a_{i,j} \in T\}|$ be the number of different states in the tile. Clearly, $p(T) \geq s(T)$, so we define $\ell = \ell(T) = p(T) - s(T)$ to be the **lag** of T.

The following lemma from [7] lists two immediate properties of the tile of a PS.

Lemma 2.1 (Lemma 4 in [7]). *Let $T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$ be the tile of a PS with periods τ and σ. Then T satisfies the following properties:*

1. *Uniqueness of assignment: if $(a_{i,j}, a_{i,j+1}) = (a_{k,m}, a_{k,m+1})$, then $a_{i+1,j+1} = a_{k+1,m+1}$.*
2. *Aperiodicity of rows: each row of T cannot be divided into smaller identical pieces.*

We remark that for a tile of a PS that is not weakly robust, there *may* exist periodic columns; see Figure 2(d) in [7] for an example of a tile of a PS with $\tau = 4$, whose first column has period 2. However, in Section 3, we will show that, if T is a tile of a WRPS, its columns are necessarily aperiodic.

2.2 Circular Shifts

We also recall the concept of circular shifts operation on Z_n^σ (or Z_n^τ), the set of words of length σ (or τ) from the alphabet $\mathbb{Z}_n$, which will be used in Section 2.5.

Definition 2.2. *Let $\mathbb{Z}_n^\sigma$ consist of all length-σ words. A **circular shift** is a map $\pi : \mathbb{Z}_n^\sigma \to \mathbb{Z}_n^\sigma$, given by an $i \in \mathbb{Z}_+$ as follows: $\pi(a_0 a_1 \dots a_{\tau-1}) = a_i a_{i+1} \dots a_{i+\sigma-1}$, where the subscripts are modulo σ. The **order** of a circular shift π is the smallest k such that $\pi^k(A) = A$ for all $A \in \mathbb{Z}_n^\sigma$, and is denoted by $\mathrm{ord}(\pi)$. Circular shifts on $\mathbb{Z}_n^\tau$ will also appear in the sequel and are defined in the same way.*

Lemma 2.3 (Lemma 6 in [7])**.** *Let π be a circular shift on $\mathbb{Z}_n^\sigma$ and let $A \in \mathbb{Z}_n^\sigma$ be an aperiodic length-σ word from alphabet $\mathbb{Z}_n$. Then: (1) $\mathrm{ord}(\pi) \mid \sigma$; and (2) for any $d \mid \sigma$,*

$$|\{B \in \mathbb{Z}_n^\sigma : A = \pi(B) \textit{ for some } \pi \textit{ with } \mathrm{ord}(\pi) = d\}| = \varphi(d).$$

Two words A and B of length σ are **equal up to a circular shift** if $B = \pi(A)$ for some circular shift π.

2.3 Directed Graph on Labels

In our study of PS in [7], we extended the notion of label trees from [4] to define the **label digraph**. As this object is also of relevance to WRPS, we recall its definition in this subsection.

Definition 2.4. *Let $A = a_0 \dots a_{\tau-1}$ and $B = b_0 \dots b_{\tau-1}$ be two words from alphabet $\mathbb{Z}_n$, which we call **labels** of length τ. (While it is best to view them as vertical columns, we write them horizontally for reasons of space, as in [4].) We say that A **right-extends to** B if $f(a_i, b_i) = b_{i+1}$, for all $i \in \mathbb{Z}_+$, where (as usual) the indices are modulo τ, and we write $A \to B$. We form the **label digraph** associated with a given τ by forming an arc from a label A to a label B if A right-extends to B.*

To give an example, in the PS presented in Figure 1, the label 021 right-extends to the 221 under both rules. We also remark that only right-extension is considered because the rules being investigated are one-sided, i.e., only left neighbors are taken into account in the evolution. The right extension relation is the basis for the Algorithm 2.5 below for finding all the PS with temporal period τ.

Algorithm 2.5.

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input : *Label digraph $D_{\tau,f}$ of f with temporal period τ*
output: *The set of all PS tiles*

Find all the directed cycles in $D_{\tau,f}$
for *each cycle $A_0 \to A_1 \to \cdots \to A_{\sigma-1} \to A_0$* **do**
  form the tile T by placing labels $A_0, A_1, \dots, A_{\sigma-1}$ on successive columns.
  if *both spatial and temporal periods of T are minimal* **then**
    print *T as a PS*
  end
end

Proposition 2.6 (Proposition 12 in [7])**.** *All PS of temporal period τ of f can be obtained by the Algorithm 2.5.*

2.4 Chen-Stein Method for Poisson Approximation

The most useful tool in proving Poisson convergence is the Chen-Stein method [1]. The local version stated below (Theorem 4.7 from [10]) was instrumental in [7] and continues to play a similar role in the present paper.

Let Poisson(λ) be a Poisson random variable with expectation λ, and let d_{TV} be the total variation distance between measures on $\mathbb{Z}_+$. Assume that I_i, $i \in \Gamma$, are indicators of a finite family of events, $p_i = \mathbb{E}(I_i)$, $W = \sum_{i\in\Gamma} I_i$, $\lambda = \sum_{i\in\Gamma} p_i = \mathbb{E}W$, and $\Gamma_i = \{j \in \Gamma : j \neq i,\ I_i \text{ and } I_j \text{ are not independent}\}$.

Lemma 2.7. *We have*

$$d_{TV}\left(W, \text{Poisson}\left(\lambda\right)\right) \leq \min\left(1, \lambda^{-1}\right)\left[\sum_{i\in\Gamma} p_i^2 + \sum_{i\in\Gamma, j\in\Gamma_i}\left(p_i p_j + \mathbb{E}\left(I_i I_j\right)\right)\right].$$

2.5 Simple Tiles

If a tile T has zero lag, we call T **simple**. In [7], we show that while a tile for PS may not be simple, the leading term in the asymptotics of probability of existence of PS is given by simple tiles. In Section 5, we show that simple tiles also dominate the probability of existence of WRPS.

Lemma 2.8 (Lemma 15 in [7]). *Suppose that we have a CA and T is a tile of its PS. Assume that*

$$T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$$

is a simple tile. Then

1. *the states on each row of T are distinct;*
2. *if two rows of T share a state, then they are circular shifts of each other;*
3. *the states on each column of T are distinct; and*
4. *if two columns of T share a state, then they are circular shifts of each other.*

Let $T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$ be a simple tile. Let

$$i(T) = \min\{k = 1, 2, \ldots, \tau - 1 : \mathtt{row}_k(T) = \pi(\mathtt{row}_0(T)),\ \text{for some circular shift } \pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma\}$$

be the smallest i such that $\mathtt{row}_i(T)$ is a circular shift of $\mathtt{row}_0(T)$, and let $i = 0$ if and only if T does not have circular shifts of $\mathtt{row}_0$ other than this row itself. Then this circular shift satisfies $\mathtt{row}_{(j+i) \mod \tau}(T) = \pi(\mathtt{row}_j(T))$, for all $j = 0, \ldots, \tau - 1$ we denote this circular shift by π_T^r. We denote by π_T^c the analogous circular shift for columns.

Lemma 2.9 (Lemma 16 in [7]). *Let T be a simple tile of a PS, and let $d_1 = ord\ (\pi_T^r)$ and $d_2 = ord\ (\pi_T^c)$. Then d_1 and d_2 are equal and divide $\gcd(\tau, \sigma)$.*

Lemma 2.10 (Lemma 17 in [7]). *An integer $s \leq n$ is the number of states in a simple tile T of PS if and only if there exists $d \mid \gcd(\tau, \sigma)$, such that $s = \tau\sigma/d$.*

The above lemma gives the possible values of $s(T)$ for a simple tile T and the next one enumerates the number of simple tiles of PS containing s different states.

Lemma 2.11 (Lemma 18 in [7]). *The number of simple tiles of PS with temporal periods τ and spatial period σ containing s states is*

$$\varphi(d)\binom{n}{s}(s-1)!\,,$$

where $d = \tau\sigma/s$.

Consider two different simple tiles T_1 and T_2 under the rule. The following lemma provides a lower bound on the combined number of values of the rule f assigned by T_1 *and* T_2, in terms of the number of states.

Lemma 2.12 (Lemma 19 in [7])**.** *Let T_1 and T_2 be two different simple tiles for the same rule. If T_1 and T_2 have at least one state in common, then there exist $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$ such that $a_{i,j} = b_{k,m}$ and $a_{i,j+1} \neq b_{k,m+1}$.*

As a result, if $s(T_1) = s_1$, then $p(T_1) \geq s_1$, i.e., there are at least s_1 values assigned by T_1. If there are s_2' states in T_2 that are not in T_1, then there are at least s_2' additional values to assign. With the above lemma, a lower bound of the number of values to be assigned in T_1 and T_2 is $s_1 + s_2' + 1$.

3 DECIDABILITY AND WRPS

In order for a PS to be weakly robust, we need one more condition on the directed cycle in the label digraph, which requires that each label *decides* its unique child. To be more accurate, let A and B be two labels. Assume that at a site $k \in \mathbb{Z}$ the temporal evolution of the states, arranged vertically, is the repeated label A: $a_0 \ldots a_{\tau-1} a_0 \ldots a_{\tau-1} \ldots$. Suppose that the states at site $k+1$ eventually converge to repetition of B: $b_0 \ldots b_{\tau-1} b_0 \ldots b_{\tau-1} \ldots$, regardless of the initial state at site $k+1$. In this case, we say that A decides B, and then it is clear that A does not decide C for any other length-τ label C that is not equal to B up to a circular shift. We now provide a more formal definition.

Definition 3.1. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two length-τ labels. We call that label A **decides** B, denoted as $A \Rightarrow B$, if the following two conditions are satisfied:*

1. *label A right-extends to B, i.e., $A \rightarrow B$;*
2. *for an arbitrary $c_0 \in \mathbb{Z}_n$, recursively define $c_{j+1} = f(a_{j \bmod \tau}, c_j)$; then there exists a $j \geq 0$ such that $c_{j \bmod \tau} = b_{j \bmod \tau}$.*

The following proposition, analogous to Proposition 2.2 in [4], provides an algorithm to verify whether a PS is weakly robust.

Proposition 3.2. *A tile is a WRPS if and only if each column decides the column to its right.*

Proof. Assume that a tile $T = (a_{i,j})$ is a WRPS with columns A_j, $j = 0, \ldots, \sigma - 1$. Let η be the initial configuration formed by doubly infinite repetition of $a_{0,0} \ldots a_{0,\sigma-1}$. If $A_j = a_{0,j} \ldots a_{\tau-1,j}$ does not decide $A_{j+1} = a_{0,j+1} \ldots a_{\tau-1,j+1}$, for some $j = 0, \ldots, \tau - 1$, then there exists a $c_0 \in \mathbb{Z}_n$ such that in the position to the right of A_j, the states do not converge to a repetition of A_{j+1}. Now, construct an initial configuration η' by replacing one $a_{0,j+1}$ by c_0 in η. Then η' is proper for η, but the advance of the spatial period is stopped, thus $v(\eta') = 0$ and T cannot be weakly robust.

Conversely, note that if label A_j decides A_{j+1}, then for any $c_0 \in \mathbb{Z}_n$ to the right of $a_{0,j}$, the label converges to A_{j+1} within $n\tau$ iterations. Thus the expansion velocity must be at least $1/(\tau n)$. □

Recall that by Lemma 2.1, a tile of a PS does not have periodic rows. The following lemma concludes that a periodic label cannot be a part of WRPS tile, since otherwise the temporal period of the WRPS is reduced.

Lemma 3.3. *If T is a tile of WRPS of period τ, then every column has minimal period τ.*

Proof. Assume that A is a label of length τ that is formed by concatenating shorter label A' that has length τ'. It is clear that if $A \Rightarrow B = b_0 \ldots b_{\tau-1}$, A also decides the circular shift $b_{\tau'} b_{\tau'+1} \ldots b_{\tau} b_0 \ldots b_{\tau'-1}$. This implies that $b_0 = b_{\tau'}$, $b_1 = b_{\tau'+1}$, etc. That is, B is also periodic with period τ'. By induction, every label in T is periodic with period τ', thus T is temporally reducible. □

In a label digraph $D_{\tau,f}$, we call an arc $A \rightarrow B$ **deciding arc** if $A \Rightarrow B$ and a directed cycle **deciding cycle** if all the arcs contained in this cycle are deciding arcs. Algorithm 3.4 finds all WRPS of temporal period τ for rule f.

Algorithm 3.4.

input : *Label digraph $D_{\tau,f}$ of f with temporal period τ*
output: *The set of all WRPS tiles*

Find all deciding cycles in $D_{\tau,f}$
for *each deciding cycle $A_0 \Rightarrow A_1 \Rightarrow \cdots \Rightarrow A_{\sigma-1} \Rightarrow A_0$* **do**
  form the tile T by placing labels $A_0, A_1, \ldots, A_{\sigma-1}$ on successive columns.
  if *both spatial and temporal periods of T are minimal* **then**
    print *T as a WRPS*
  end
end

Decidability of a label can be interpreted as a property of a certain τ-partite graph. Specifically, we construct **label assignment digraph** (LAD) $G_{\tau,n}(f, A)$ of a label A under a rule f in the following manner.

An LAD $G_{\tau,n}(f, A)$ is a τ-partite digraphs with the ith part denoted by $(i, *) = \{(i, j) : j = 0, \ldots, n-1\}$, $i = 0, \ldots, \tau - 1$, so that its vertex set is $\cup_{i=0}^{\tau-1}(i, *)$. The arcs of the digraph $G_{\tau,n}(f, A)$ are determined as follows: for all $i = 0, \ldots, \tau - 1$ and $j = 0, \ldots, n-1$, there is an arc $(i, j) \to (i+1, j')$ if $f(a_i, j) = j'$. In particular, every node in a LAD has out-degree 1 since it represents a deterministic (local) CA dynamics. As usual, we identify $i = \tau$ with $i = 0$, $i = \tau + 1$ with $i = 1$, etc. We next state the conditions for $G_{\tau,n}(f, A)$ that characterize when $A \to B$ and when $A \Rightarrow B$.

Definition 3.5. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two labels. Consider the following conditions on a τ-partite graph G:*

1. *G contains the cycle $(0, b_0) \to (1, b_1) \to \cdots \to (\tau - 1, b_{\tau-1}) \to (0, b_0)$;*

2. *there is a directed path in G from (i, j) to $(0, b_0)$ for all $i = 0, \ldots, \tau - 1$ and $j = 0, \ldots, n - 1$.*

Let $\mathcal{E}(A, B)$ be the set of all τ-partite digraphs G, which satisfy condition (1) and let $\mathcal{D}(A, B)$ be the set of all such digraphs G that satisfy both conditions (1) and (2).

Lemma 3.6. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be any two labels. Then $A \to B$ if and only if $G_{\tau,n}(f, A) \in \mathcal{E}(A, B)$ and $A \Rightarrow B$ if and only if $G_{\tau,n}(f, A) \in \mathcal{D}(A, B)$.*

We skip the proof as it follows immediately from the definitions, and instead give two examples for different rules by Figure 2. For the reader's convenience, we denote a node $(i[a_i], j)$ instead of (i, j) as in the definition. The two labels are $A = 12$ and $B = 00$ in both cases, where $n = 3$ states are assumed and the states are 0, 1 and 2. Under the rule that generates the left LAD, $A \to B$, but $A \not\Rightarrow B$, i.e., $G_{\tau,n}(f, A) \in \mathcal{E}(A, B) \setminus \mathcal{D}(A, B)$; under the rule that generates the right LAD, $A \Rightarrow B$, i.e., $G_{\tau,n}(f, A) \in \mathcal{D}(A, B)$.

A WRPS tile gives σ successive pairs of columns and thus σ LADs. In each of these LADs, count the number of vertices on the longest path from a vertex to the unique oriented cycle, which are not on the cycle. (In the example on the right of Figure 2, this number is 4.) Then let ℓ be the largest such number among all LADs. Then

$$v \geq 1/\ell \geq 1/((n-1)\tau).$$

This straightforward lower bound on v is typically rather poor and computing v precisely appears to be a highly nontrivial task. As a result, we do not attempt to study the distribution of expansion velocities in random rules. However, with the aid of the computer, we can often get quite good bounds on v for a particular WRPS. For illustration, we estimate the WRPS in Figure 1(a)(b)(c), by a variation of the method introduced in [4]. Fix a WRPS, and choose

FIGURE 2: Two LADs of label $A = 12$ under two different rules. We use $(i[a_i], j)$ to represent a node for the reader's convenience. In the left one, $A \to 00$ but $A \not\Rightarrow 00$; in the right one, $A \Rightarrow 00$.

an integer $a \geq 1$. Consider the set $\mathcal{I}$ of all proper initial configurations η_0 for this PS, in which we replace the PS by an arbitrary configuration on $[1, \infty)$. We let T_a to be the smallest time t such that $s_t(\eta_0) \geq a$ for all $\eta_0 \in \mathcal{I}$. For any $\eta_0 \in \mathcal{I}$, iteration gives $s_{\ell T_a}(\eta_0) \geq \ell a$ for any integer $\ell \geq 0$, and then

$$\frac{s_t(\eta_0)}{t} \geq \frac{s_{\lfloor t/T_a \rfloor T_a}(\eta_0)}{t} \geq \frac{\lfloor t/T_a \rfloor a}{t} \to \frac{a}{T_a}.$$

We conclude that

$$v \geq a/T_a.$$

Computation of T_a is finite, albeit costly: we need only compute the states in the interval $[1, a]$, therefore the number of initial configurations that need to be checked is $\sigma\tau \cdot n^a$. (We need to choose the tile rotation with the upper right corner at the origin, and the states in $[1, a]$.) In our example, a computer verification gives $T_9 = 17$ and therefore $v \geq 9/17 \approx 0.5294$. To obtain nontrivial upper bounds, we consider proper configurations η_0 which are periodic on $[1, \infty)$, and in addition are such that the dynamics keeps the interface between the two spatially periodic configurations uniformly bounded in time (and thus periodic), so that $v(\eta_0)$ is computable. An upper bound is simply the smallest of the resulting $v(\eta_0)$ over all such η_0 we manage to find. For our example, the best η_0, obtained by a computer search, is the one in Figure 1(b), which gives $v \leq 4/7 \approx 0.5714$.

4 DECIDABILITY PROBABILITY

We call a label $A = a_0 \ldots a_{\tau-1}$ **simple** if $a_i \neq a_j$ for $i \neq j$. We next prove the main result regarding the probability of the decidability of simple labels.

Theorem 4.1. *Fix a number of states n and a $\tau \leq n$. Let $A = a_0 \ldots a_{\tau-1}$ be a simple label with length τ and $B = b_0 \ldots b_{\tau-1}$ be any other label (not necessarily simple) of length τ. Then*

$$\mathbb{P}(A \Rightarrow B) = \frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau}.$$

The theorem is proved in three lemmas below. At the heart of the argument is a calculation of the probability that a random τ-partite graph LAD, introduced in the previous section, is a directed **pseudo-tree**, i.e., a weakly connected directed graph that has at most one directed cycle.

Fix a label $A = a_0 \ldots a_{\tau-1}$. The LAD $G_{\tau,n}(f, A)$ becomes a random graph if the rule f is selected randomly and we are interested in $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{E}(A, B))$ and $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B))$. The case that A is simple is easier as we can take advantage of independence of assignments of f. To be precise, let A be a simple label with length τ and B be an arbitrary label with the same length. We clearly have that $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{E}(A, B)) = 1/n^\tau$, as the assignments on (a_j, b_j)'s are independent.

Next, we find $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B))$ for simple label A thus complete the proof of Theorem 4.1. We start by the following observation, which follows from symmetry.

Lemma 4.2. *If A and A' are simple labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A' \Rightarrow B)$ for any label B; if B and B' are labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A \Rightarrow B')$ for any simple label A.*

To find $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{D}(A, B)\right)$, we adapt the counting techniques in [11] to enumerate $\mathcal{D}(A, B)$. We start by proving the following combinatorial result.

Lemma 4.3. *Let $A_{k,\ell} = \binom{n-1}{k}(\ell+1)^k(n-1-\ell)^{n-1-k}$, and assume that k_{m+1} is a non-negative integer. Then*

$$\begin{aligned} S_m &:= \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}} \ldots \left[\sum_{k_2=0}^{n-1} A_{k_2,k_3}\left[\sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2}\right]\right] \\ &= n^{(m+1)(n-2)}\left[P_{m+1} + k_{m+1}(n-1)^m\right], \end{aligned}$$

where $P_m = n^m - (n-1)^m$.

Proof. We use induction on m. Assume $m = 1$. Observe that

$$A_{k,\ell} = n^{n-1}\mathbb{P}\left(\text{Binomial}\left(n-1, \frac{\ell+1}{n}\right) = k\right).$$

Therefore,

$$\begin{aligned} &\sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2} \\ &= n^{n-2} \cdot n^{n-1} \cdot \left[1 + (n-1)\frac{k_2+1}{n}\right] \\ &= n^{2(n-2)}\left[P_2 + k_2(n-1)\right]. \end{aligned}$$

Now, by the induction hypothesis

$$\begin{aligned} S_m &= \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}} S_{m-1} \\ &= n^{m(n-2)} \sum_{k_m=0}^{n-1} \binom{n-1}{k_m}(k_{m+1}+1)^{k_m}(n-1-k_{m+1})^{n-1-k_m}\left[P_m + k_m(n-1)^{m-1}\right] \\ &= n^{m(n-2)}\left[n^{n-1}P_m + (n-1)^m(k_{m+1}+1)n^{n-2}\right] \\ &= n^{(m+1)(n-2)}\left[nP_m + k_{m+1}(n-1)^m + (n-1)^m\right] \\ &= n^{(m+1)(n-2)}\left[P_{m+1} + k_{m+1}(n-1)^m\right], \end{aligned}$$

which is the desired result. □

Now, we are ready to prove the key combinatorial result.

Lemma 4.4. *Let A and B be labels with length τ and let A be simple. Then $|\mathcal{D}(A, B)| = n^{\tau(n-2)}(n^\tau - (n-1)^\tau)$.*

Proof. The argument we give partly follows the proof of Theorem 1 in [11]. Applying Lemma 4.2, we may assume that $B = 0\ldots0$, without loss of generality. Observe that the LAD then has a cycle $(0,0) \to (1,0) \to \cdots \to (\tau-1,0) \to (0,0)$. We need to count the number of LADs such that every other vertex has a direct path to this cycle.

In the first half of the proof, we assign arcs from $(\tau-1,*)$ to $(0,*)$, $(\tau-2,*)$ to $(\tau-1,*)$, ..., $(1,*)$ to $(2,*)$. In the second half, we assign arcs from $(0,*)$ to $(1,*)$. As any cycle must go though all the parts, it is the second half that guarantees the uniqueness of the cycle.

First, choose a $k_{\tau-1} \in \{0, \ldots, n-1\}$, pick $k_{\tau-1}$ nodes in $(\tau-1, *) \setminus \{(\tau-1, 0)\}$, and form $k_{\tau-1}$ arcs from those nodes to the node $(0, 0)$. There are $\binom{n-1}{k_{\tau-1}}$ choices for a fixed $k_{\tau-1}$. Denote this subset of $(\tau-1, *)$ together with $(\tau-1, 0)$ as $(\tau-1, *)'$; thus, $(\tau-1, *)' \subset (\tau-1, *)$ are the nodes in $(\tau-1, *)$ that are mapped to $(0, 0)$. Assign the images of the nodes in $(\tau-1, *) \setminus (\tau-1, *)'$ to $(0, *) \setminus \{(0, 0)\}$, for which there are $(n-1)^{n-1-k_{\tau-1}}$ choices. So, for a fixed $k_{\tau-1}$ to assign the image of nodes in $(\tau-1, *)$, there are

$$\binom{n-1}{k_{\tau-1}}(n-1)^{n-1-k_{\tau-1}}$$

choices.

Second, we need to assign the image of the nodes in $(\tau-2, *)$ to $(\tau-1, *)$. Choose a $k_{\tau-2} \in \{0, \ldots, n-1\}$, pick $k_{\tau-2}$ nodes in $(\tau-2, *) \setminus (\tau-2, 0)$, and form $k_{\tau-2}$ arcs from those nodes to the nodes in $(\tau-1, 0)'$. There are $\binom{n-1}{k_{\tau-2}}$ choices to choose those nodes for a fixed $k_{\tau-2}$ and $(k_{\tau-1}+1)^{k_{\tau-2}}$ choices to assign the images. Denote this subset of $(\tau-2, *)$ together with $(\tau-2, 0)$ as $(\tau-2, *)'$. Now, the images of the nodes in $(\tau-2, *) \setminus (\tau-2, *)'$ should be in $(\tau-1, *) \setminus (\tau-1, *)'$, for which there are $(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$ choices. Hence, for fixed $k_{\tau-1}$ and $k_{\tau-2}$, to assign the image of the nodes in $(\tau-2, *)$ to $(\tau-1, *)$, there are

$$\binom{n-1}{k_{\tau-2}}(k_{\tau-1}+1)^{k_{\tau-2}}(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$$

choices.

Repeat the above steps for $(\tau-3, *), \ldots, (1, *)$. To complete the construction, we assign the images of the nodes in $(0, *) \setminus \{(0, 0)\}$. We choose a $t \in \{0, \ldots, n-2\}$, and add t arcs from $(0, *) \setminus \{(0, 0)\}$ to $(1, *) \setminus (1, *)'$ consecutively as specified below, making sure to avoid creating a cycle that does not include $(0, 0)$.

In the evolving digraph, a **component** is a weakly connected component, obtained by ignoring the orientation of edges. First note that there are n components in the current digraph; more precisely, each node of $(0, *)$ belongs to a different component (possibly consisting of a single node).

To select the first arc, pick a $b \in (1, *) \setminus (1, *)'$ ($n-1-k_1$ choices). There is one component that contains $(0, 0)$ and one other component containing b. As a result, there are $n-2$ other components and among each of them, there is a node in $(0, *) \setminus \{(0, 0)\}$ with zero out-degree. Among these $n-2$ nodes, we select one and connect it to b. Therefore, there are $(n-2)(n-1-k_1)$ choices for the first arc. The addition of this arc decreases the number of components by one.

To assign the second arc, again pick a $b \in (1, *) \setminus (1, *)'$ (again $n-1-k_1$ choices). Now there are exactly $n-3$ components, among which there is a node in $(0, *) \setminus \{(0, 0)\}$ with zero out-degree. We again select one and connect it with this b, leading to $(n-3)(n-1-k_1)$ choices.

In subsequent steps, we add an arc from a to b, where $b \in (1, *) \setminus (1, *)'$ is arbitrary, while $a \in (0, *) \setminus \{(0, 0)\}$ is a unique node with zero out-degree in any component not containing b in the graph already constructed. The algorithm guarantees that the number of components decreases by one after each arc is added, i.e., that a cycle not including $(0, 0)$ is never created.

In the above steps we add t arcs, with the number of choices, in order: $(n-2)(n-1-k_1), (n-3)(n-1-k_1) \ldots, (n-t-1)(n-1-k_1)$. As any order in which they are assigned produces the same digraph, there are

$$\begin{aligned}
&\frac{(n-2)(n-1-k_1)(n-3)(n-1-k_1)\cdots(n-t-1)(n-1-k_1)}{t!} \\
&= \binom{n-2}{t}(n-1-k_1)^t
\end{aligned}$$

choices. Finally, we assign the remaining $n-1-t$ arcs to $(1, *)'$, for which we have $(k_1+1)^{n-1-t}$ choices. Hence, for a fixed k_1, to assign the arcs originating from $(0,*) \setminus \{(0,0)\}$, there are

$$\sum_{t=0}^{n-2} \binom{n-2}{t} (n-1-k_1)^t (k_1+1)^{n-1-t} = (k_1+1)n^{n-2}$$

choices, in total. Lastly, we use Lemma 4.3 to get

$$\begin{aligned}
|\mathcal{D}(A,B)| &= \sum_{k_{\tau-1}=0}^{n-1} \binom{n-1}{k_{\tau-1}} (n-1)^{n-1-k_{\tau-1}} \\
&\quad \cdot \left[\sum_{k_{\tau-2}=0}^{n-1} A_{k_{\tau-2},k_{\tau-1}} \cdots \left[\sum_{k_2=0}^{n-1} A_{k_2,k_3} \left[\sum_{k_1=0}^{n-1} A_{k_1,k_2} (k_1+1) n^{n-2} \right] \right] \cdots \right] \\
&= n^{(\tau-1)(n-2)} \sum_{k_{\tau-1}=0}^{n-1} \binom{n-1}{k_{\tau-1}} (n-1)^{n-1-k_{\tau-1}} [P_{\tau-1} + k_{\tau-1}(n-1)^{\tau-2}] \\
&= n^{(\tau-1)(n-2)} \left[n^{n-1} P_{\tau-1} + (n-1)^{\tau-1} n^{n-2} \right] \\
&= n^{\tau(n-2)} P_\tau,
\end{aligned}$$

as claimed. □

Now, proof of Theorem 4.1 is straightforward.

Proof of Theorem 4.1. It is clear that the number of LADs $G_{\tau,n}(f,A)$ is $n^{\tau n}$. Then, by Lemma 4.4,

$$\mathbb{P}(A \Rightarrow B) = \mathbb{P}(G_{\tau,n}(f,A) \in \mathcal{D}(A,B)) = \frac{n^{\tau(n-2)}[n^\tau - (n-1)^\tau]}{n^{\tau n}} = \frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau},$$

as claimed. □

By Theorem 4.1, assuming that A is simple and B is any label of the same length τ, we have

$$\mathbb{P}(A \Rightarrow B \mid A \to B) = \frac{n^\tau - (n-1)^\tau}{n^\tau} = \frac{\tau}{n} + o\left(\frac{1}{n}\right).$$

The case when A is not simple is much harder, since the parts of $G_{\tau,n}(f,A)$ are no longer independent from each other for a random rule f. While it is possible to obtain the deciding probability for a specific label using a similar method as in Theorem 4.1, it is hard to find a general formula or even to prove this probability is always $\mathcal{O}(1/n)$. We are, however, able to obtain the following weaker result that the probability goes to 0.

Theorem 4.5. *Let $A = a_0 \dots a_{\tau-1}$ and $B = b_0 \dots b_{\tau-1}$ be two fixed labels (not necessarily simple) with length τ. Then*

$$\mathbb{P}\left(G_{\tau,n}(f,A) \in \mathcal{D}(A,B) \,\middle|\, G_{\tau,n}(f,A) \in \mathcal{E}(A,B)\right) = o(1).$$

Equivalently, we have

$$\mathbb{P}\left(A \Rightarrow B \,\middle|\, A \to B\right) = o(1).$$

Proof. Again, we assume that $B = 0 \dots 0$. We remark that, unlike Theorem 4.1, label B here does affect the deciding probability. However, the case of general B does not significantly alter the proof but it makes it transparent, so we choose this B for readability.

Let $a'_0, \ldots, a'_{\ell-1}$ be the different states in A and m_i be the repetition numbers of a_i's, for $i = 0, \ldots, \ell - 1$. Clearly, $\sum_{i=0}^{\ell-1} m_i = \tau$. Let ζ be the cycle $(0,0) \to (1,0) \to \cdots \to (\tau-1,0) \to (0,0)$. It suffices to show that

$$\mathbb{P}(\text{there are no other cycles in } G_{\tau,n}(f,A) \mid \zeta \in G_{\tau,n}(f,A)) = o(1).$$

To accommodate the conditional probability, our probability space will be a uniform choice of a digraph from $\mathcal{E}(A,B)$ for the remainder of the proof.

Fix an integer $K \geq 1$. Call a cycle $\zeta' = (0, j_0) \to (1, j_1) \to \cdots \to (0, j_0)$ **simple with respect to** ζ if:

1. ζ' contains no parallel arcs, i.e., if (i,j) and (i',j) are nodes in ζ', then $a_i \neq a_{i'}$; and
2. if (i,j) is on ζ and (i',j') on ζ', then $(a_{i'}, b_{j'}) \neq (a_i, b_j)$.

Let Y_k be the random number of simple cycles with respect to ζ with length exactly τk and $Z_K = \sum_{k=1}^{K} Y_k$ be the random variable that counts the number of such cycles with length less than or equal to τK. We will show that, for any K, $\lim_{n\to\infty} \mathbb{P}(Z_K \geq 1) = 1 - \exp\left(-\sum_{k=1}^{K} 1/k\right)$, converging to 1 as $K \to \infty$. As a consequence, the LAD has another simple cycle asymptotically almost surely (in n), and this will conclude the proof.

We first compute the expectation of Y_k:

$$\mathbb{E}Y_k = \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k}}{k} \cdot \frac{1}{n^{\tau k}} \to \frac{1}{k}, \quad \text{as } n \to \infty.$$

Here and in the sequel, we use the falling factorial notation $(x)_n = x(x-1)\cdots(x-n+1)$. The first factor counts the number of simple cycles with respect to ζ and the second factor is the probability that a fixed simple cycle with length τk is formed.

Now, let $\lambda_K = \mathbb{E}Z_K = \sum_{k=1}^{K} \mathbb{E}Y_k$. We use the notation Γ^k to denote the set of all possible simple cycles with length τk and define $\Gamma = \bigcup_{1\leq k\leq K} \Gamma^k$ as set of such cycles with length less than or equal to τK. The set Γ_i consists of cycles in Γ that has at least one node in common with the cycle i. The random variable I_i is the indicator that the cycle $i \in \Gamma$ is formed and $p_i = \mathbb{E}I_i$.

We use Lemma 2.7 to find an upper bound for $d_{\text{TV}}(Z_K, \text{Poisson}(\lambda_K))$. For the first term $\sum_{i\in\Gamma} p_i^2$, we have

$$\sum_{i\in\Gamma} p_i^2 = \sum_{k=1}^{K} \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k}}{k} \frac{1}{n^{2\tau k}} = \mathcal{O}\left(\frac{1}{n^\tau}\right).$$

To obtain an upper bound for $\sum_{i\in\Gamma} \sum_{j\in\Gamma_i} p_i p_j$, we note that if i is the index of a simple cycle of length τr, then we may count the number of length-τk simple cycles that have no common vertex with the cycle i, that is

$$|\Gamma^k \setminus \Gamma_i| = \frac{(n-1-r)_{m_1 k} \cdots (n-1-r)_{m_\ell k}}{k}.$$

It immediately follows that,

$$\begin{aligned}
&|\Gamma^k \cap \Gamma_i| \\
&= \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k} - (n-1-r)_{m_1 k} \cdots (n-1-r)_{m_\ell k}}{k} \\
&= \mathcal{O}\left(n^{\tau k - 1}\right),
\end{aligned}$$

as the highest powers of n in the numerator cancel. Hence, for a fixed r and k, we have

$$\begin{aligned}
&\sum_{i\in\Gamma^r}\sum_{k\in\Gamma_i\cap\Gamma^k} p_i p_j \\
&= \frac{(n-1)_{m_1 r}\cdots(n-1)_{m_\ell r}}{r}\cdot|\Gamma^k\cap\Gamma_i|\cdot\frac{1}{n^{\tau r}}\cdot\frac{1}{n^{\tau k}} \\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore, the total sum

$$\sum_{i\in\Gamma}\sum_{j\in\Gamma_i} p_i p_j = \mathcal{O}\left(\frac{K^2}{n}\right).$$

For the last term in the upper bound in Lemma 2.7 , we observe that $\mathbb{E}I_iI_j = 0$ if two cycles have shared vertices.

Now, by Lemma 2.7,

$$\mathbb{P}\left(Z_K = 0\right) \le e^{-\lambda_K} + \mathcal{O}\left(\frac{K^2}{n}\right) \le \frac{1}{K+1} + \mathcal{O}\left(\frac{K^2}{n}\right).$$

Sending $n \to \infty$ and noting that K is arbitrary conclude the proof. □

5 PROOF OF THEOREM 1.1

Let T be a tile with τ rows and σ columns. Define the **rank** of T to be the largest x such that there exist x columns of T with distinct $x\tau$ states. We denote the rank of a tile as rank(T). For example, the tiles

$$T_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{matrix}, \qquad T_2 = \begin{matrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{matrix}.$$

have rank$(T_1) = 2$ and rank$(T_2) = 1$.

Analogously to the notation in [7], we denote by $\mathcal{R}^{(\ell)}_{\tau,\sigma,n}$ as the set of tile of WRPS that has lag ℓ. Thus the set of simple WRPS is $\mathcal{R}^{(0)}_{\tau,\sigma,n}$. We also use the notation $\mathcal{R}^{(0,y)}_{\tau,\sigma,n} \subset \mathcal{R}^{(0)}_{\tau,\sigma,n}$ to denote the set of WRPS whose tile is simple and has rank y. We use $\mathcal{T}_{\tau,\sigma,n}$ to denote the set of all PS tiles; to be more precise, this is the set of all $\tau\times\sigma$ arrays T with state space $\mathbb{Z}_n$ that satisfy properties 1 and 2 in Lemma 2.1, so that there exists a CA rule with a PS given by T. We also use $\mathcal{T}^{(0)}_{\tau,\sigma,n}$ and $\mathcal{T}^{(0,y)}_{\tau,\sigma,n}$ to denote the tiles in $\mathcal{T}_{\tau,\sigma,n}$ that are simple, and that are simple with rank y, respectively.

Our first step is to study the probability that $\mathcal{R}^{(0,x)}_{\tau,\sigma,n}$ is not empty, where $x = \sigma/\gcd(\tau,\sigma)$. Before we advance, we state two lemmas on simple tiles.

Lemma 5.1. *Let T be a simple tile. Then*

1. rank$(T) \ge \sigma/\gcd(\sigma,\tau)$*;*
2. rank$(T) = y$ *if and only if* $s(T) = \tau y$*. In particular,* rank$(T) = \sigma/\gcd(\sigma,\tau)$ *if and only if* $s(T) = \tau\sigma/\gcd(\sigma,\tau) = $ lcm(σ,τ).

Proof. By Lemma 2.8, the states on each column of T are distinct and two columns either share no common states or are circular shifts of each other. As a result, rank$(T) \ge s(T)/\tau$. Together with Lemma 2.10, this proves (1) and implication ($\Longrightarrow$) of (2). The reverse implication in (2) follows from $s(T) \ge \tau\cdot$ rank(T). □

In the sequel, we write

$$d = \gcd(\tau,\sigma), k = \text{lcm}(\sigma,\tau), x = \sigma/d.$$

By Lemma 5.1, k is the number of distinct states in a simple tile with rank x. As before, φ is the Euler totient function. We index the tiles in $\mathcal{T}_{\tau,\sigma,n}^{(0,x)}$ in an arbitrary way. Let

$$\mathfrak{T}_m = \left\{ (T_i, T_j) \subset \mathcal{T}_{\tau,\sigma,n}^{(0,x)} \times \mathcal{T}_{\tau,\sigma,n}^{(0,x)} : i < j \text{ and } T_i, T_j \text{ have } m \text{ states in common} \right\}.$$

The following lemma gives the cardinality of these sets.

Lemma 5.2. *The following enumeration results hold:*

1. *the set $\mathcal{T}_{\tau,\sigma,n}^{(0,x)}$ has cardinality $\varphi(d)\binom{n}{k}(k-1)!$;*
2. *if $m < k$, the set $\mathfrak{T}_m$ has cardinality*
$$\frac{1}{2}\varphi(d)\binom{n}{k}(k-1)!\varphi(d)\binom{k}{m}\binom{n-k}{k-m}(k-1)! = \mathcal{O}\left(n^{2k-m}\right);$$
3. *if $m = k$, the set $\mathfrak{T}_m$ has cardinality*
$$\frac{1}{2}\varphi(d)\binom{n}{k}(k-1)!\left(\varphi(d)(k-1)! - 1\right) = \mathcal{O}\left(n^{k}\right).$$

Proof. Part (1) follows directly from Lemma 2.11. Then, part (2) follows from (1). Part (3) also follows from (1), after we note that once we select T_i, we have all k colors fixed and we are not allowed to select T_j equal to T_i. $\square$

We will also need the following consequence of Theorem 4.1.

Lemma 5.3. *Let T be a simple tile and* $\text{rank}(T) = y$. *Let $A_0, \ldots, A_{\sigma-1}$ be the labels in T. Then we have*

$$\mathbb{P}\left(A_i \Rightarrow A_{i+1}, \textit{ for } i = 0, \ldots, \sigma-1 \;\middle|\; A_i \to A_{i+1}, \textit{ for } i = 0, \ldots, \sigma-1\right) = \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^y.$$

Proof. Assume that the y columns with $y\tau$ states have indices in $I \subset \{0, \ldots, \sigma-1\}$ and let those columns have labels A_i, $i \in I$. As A_i's do not share any states, the events $\{A_i \to A_{i+1}\}$, $i \in I$ are independent, and so are $\{A_i \Rightarrow A_{i+1}\}$, $i \in I$. We use Lemma 2.8 and Theorem 4.1 to get

$$\begin{aligned}
&\mathbb{P}\left(A_i \Rightarrow A_{i+1}, \text{ for } i = 0, \ldots, \sigma-1 \;\middle|\; A_i \to A_{i+1}, \text{ for } i = 0, \ldots, \sigma-1\right) \\
&= \frac{\mathbb{P}\left(A_i \Rightarrow A_{i+1}, \text{ for } i \in I\right)}{\mathbb{P}\left(A_i \to A_{i+1}, \text{ for } i \in I\right)} \\
&= \frac{\prod_{i \in I} \mathbb{P}\left(A_i \Rightarrow A_{i+1}\right)}{\prod_{i \in I} \mathbb{P}\left(A_i \to A_{i+1}\right)} \\
&= \left(\frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau}\right)^y \Big/ \left(\frac{1}{n^\tau}\right)^y \\
&= \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^y,
\end{aligned}$$

as desired. $\square$

Theorem 1.1 will now be established through next three propositions, the first one of which deals with existence of WRPS with zero lag and minimal rank $x = \sigma/d$.

Proposition 5.4. *Recall that $x = \sigma / \gcd(\tau, \sigma)$. We have*

$$\mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

where

$$c(\tau,\sigma) = \frac{\varphi(\gcd(\tau,\sigma))\tau^{\sigma/\gcd(\tau,\sigma)}}{\operatorname{lcm}(\tau,\sigma)}.$$

In particular, when τ and σ are coprime, $c(\tau,\sigma) = \tau^{\sigma-1}/\sigma$.

Proof. We first find an upper bound by Markov inequality.

By Lemma 5.2, we have that $|\mathcal{T}^{(0,x)}_{\tau,\sigma,n}| = \varphi(d)\binom{n}{k}(k-1)!$. The probability that a tile in $\mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ forms a PS is $1/n^k$ and the probability that the desired decidability, thus weak robustness, holds is $(\tau/n + o(1/n))^x$ by Lemma 5.3. As a result, we have

$$\mathbb{E}\left(|\mathcal{R}^{(0,x)}_{\tau,\sigma,n}|\right) = \varphi(d)\binom{n}{k}(k-1)!\frac{1}{n^k}\left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^x = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

as an upper bound.

To find an asymptotically matching lower bound, we use the Bonferroni's inequality

$$\mathbb{P}\left(\bigcup_i A_i\right) \geq \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right).$$

Here, A_i is the event that $T_i \in \mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ is formed as a simple WRPS, for $i = 1, \ldots, \varphi(d)\binom{n}{k}(k-1)!$. Clearly, $\sum_i \mathbb{P}(A_i) = \mathbb{E}\left(|\mathcal{R}^{(0,x)}_{\tau,\sigma,n}|\right)$. Then it suffices to show that $\sum_{i<j} \mathbb{P}(A_i \cap A_j) = o\left(1/n^x\right)$.

Recall the definition of $\mathfrak{T}_m$ given before Lemma 5.2. For a pair of tiles $(T_i, T_j) \in \mathfrak{T}_m$, there are $2k - m$ different colors in $T_i \cup T_j$. By Lemma 2.12, there is at least one additional restriction on the number of maps. Using this lemma, the enumeration result Lemma 5.2, and Lemma 5.3, we have

$$\begin{aligned}
\sum_{i<j} \mathbb{P}(A_i \cap A_j) &= \sum_{m=0}^{k} \sum_{i<j} \mathbb{P}\left(A_i \cap A_j \cap \{(T_i, T_j) \in \mathfrak{T}_m\}\right) \\
&= \sum_{m=0}^{k} \mathcal{O}\left(n^{2k-m}\right) \frac{1}{n^{2k-m+1}} \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^x \\
&= \mathcal{O}\left(\frac{1}{n^{x+1}}\right).
\end{aligned}$$

□

Next, we consider *all* simple tiles and show that among simple tiles, the WRPS with rank x provide the leading order in the probability of existence of WRPS.

Proposition 5.5. *We have*

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

where $c(\tau,\sigma)$ is defined in Proposition 5.4.

Proof. First, we note the following bounds for $\mathbb{P}(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset)$,

$$\mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{y} \mathbb{P}\left(\mathcal{R}^{(0,y)}_{\tau,\sigma,n} \neq \emptyset\right),$$

where the last sum is over $y = \sigma/d'$ for $d' \mid \gcd(\tau,\sigma)$ and $d < \gcd(\tau,\sigma)$. As $x < y$, we have from Lemmas 5.1–5.3,

$$\begin{aligned}\mathbb{P}\left(\mathcal{R}^{(0,y)}_{\tau,\sigma,n} \neq \emptyset\right) &\leq \mathbb{E}\left(|\mathcal{R}^{(0,y)}_{\tau,\sigma,n}|\right) \\ &= \varphi(d_y)\binom{n}{k_y}(k_y-1)!\frac{1}{n^{k_y}}\left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^y \\ &= o\left(\frac{1}{n^x}\right),\end{aligned}$$

where, $k_y = \tau y$ is the number of states in a tile in $\mathcal{R}^{(0,y)}_{\tau,\sigma,n}$ and $d_y = \sigma/y$. The conclusion now follows from Proposition 5.4. ☐

Lemma 5.6. *If $\ell > 0$, then*

$$\mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) = o\left(\frac{1}{n}\right).$$

Proof. For a fixed ℓ, let $g_{\tau,\sigma,\ell}(s)$ count the number of tiles of lag ℓ with periods τ and σ, and s different *fixed* states. For a fixed such tile, $1/n^{s+\ell}$ is the probability that it is a tile of a PS, as there are $s+\ell$ assignments to make by a random map, and each assignment occurs independently with probability $1/n$. After we know it is a tile of a PS, Theorem 4.5 implies that it is a tile of a WRPS with probability $o(1)$. Thus,

$$\begin{aligned}\mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{E}\left(|\mathcal{R}^{(\ell)}_{\tau,\sigma,n}|\right) &= \sum_{s=1}^{\tau\sigma}\binom{n}{s} g_{\tau,\sigma,\ell}(s)\frac{1}{n^{s+\ell}} \cdot o(1) \\ &= o\left(\frac{1}{n^\ell}\right) = o\left(\frac{1}{n}\right).\end{aligned}$$

☐

Next, we extend Proposition 5.5 to cover non-simple tiles. It is here that we impose the condition that $\sigma \mid \tau$.

Proposition 5.7. *If $\sigma \mid \tau$, then*

$$\mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n} + o\left(\frac{1}{n}\right).$$

Proof. First, note that $\sigma \mid \tau$ implies that $x = \sigma/\gcd(\tau,\sigma) = 1$ and as a result of Proposition 5.5, we have

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n} + o\left(\frac{1}{n}\right).$$

The desired result now follows from the bounds

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \sum_{\ell=0}^{\tau\sigma} \mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right)$$

and Lemma 5.6. ☐

Let

$$c(\mathcal{T},\Sigma) = \sum_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} c(\tau,\sigma), \tag{1}$$

where $c(\tau,\sigma)$ is defined in Proposition 5.4.

Proof of Theorem 1.1. If $\sigma \nmid \tau$, then $x = \sigma/\gcd(\tau,\sigma) > 1$, and by Proposition 5.5 and Lemma 5.6,

$$\mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{\ell=1}^{\tau\sigma} \mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right).$$

These bounds, together with Proposition 5.7, now give the desired result:

$$\begin{aligned}\frac{c(\mathcal{T},\Sigma)}{n} + o\left(\frac{1}{n}\right) &= \sum_{\sigma|\tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset)\\ &\leq \mathbb{P}(\mathcal{R}_{\mathcal{T},\Sigma,n} \neq \emptyset)\\ &\leq \sum_{\sigma|\tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) + \sum_{\sigma\nmid\tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) \leq \frac{c(\mathcal{T},\Sigma)}{n} + o\left(\frac{1}{n}\right).\end{aligned}$$

□

6 DISCUSSION

Inspired by [4], we prove that the probability that a randomly chosen CA has a weakly robust periodic solution with periods in the finite set $\mathcal{T} \times \Sigma$ is asymptotically $c(\mathcal{T},\Sigma)/n$, provided that $\mathcal{T} \times \Sigma$ contains a pair (τ,σ) with $\sigma \mid \tau$. A natural first question is whether the divisibility condition may be removed.

Question 6.1. *Let $\mathcal{R}_{\tau,\sigma,n}$ be the set of WRPS with periods τ and σ from a random rule f. Do we have*

$$\mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

where $x = \sigma/\gcd(\tau,\sigma)$?

A possible strategy to answer Question 6.1 affirmatively is through proving the following two conjectures, the first of which provides a lower bound of the rank of a tile. Recall that $x = \sigma/\gcd(\tau,\sigma)$.

Conjecture 6.2. *Let T be a tile of a WRPS of period τ and σ and $\ell = p(T) - s(T)$. Then* $\mathrm{rank}(T) \geq x - \ell$.

We recall that a tile of a WRPS satsifies the properties stated in Lemmas 2.1 and 3.3. The next conjecture presents an asymptotic property similar to the one in Theorem 4.5. In its formulation, we assume validity of Conjecture 6.2: for a tile T of a WRPS, we let $I = I(T) \subset \{0,\ldots,\sigma-1\}$ be the index set with $|I| = x - \ell$, such that the labels indexed by I are the leftmost $x - \ell$ labels without a repeated state.

Conjecture 6.3. *Assume that T is a tile of a WRPS. Then there exists a label A_j with index $j \notin I$ so that*

$$\mathbb{P}\left(A_j \Rightarrow A_{j+1} \;\middle|\; \{A_i \Rightarrow A_{i+1} \text{ for all } i \in I\}\right) = o(1).$$

If there exists a label j that does not share any state with A_i, for any $i \in I$, the conjecture can be proved in the same way as Theorem 4.5. To see how Question 6.1 is settled in the case that both of the conjectures are satisfied, use again the bounds

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{\ell} \mathbb{E}\left(|\mathcal{R}^{(\ell)}_{\tau,\sigma,n}|\right),$$

and then, with $g_{\tau,\sigma}(s)$ as in the proof of Lemma 5.6, and using Lemma 5.3,

$$\mathbb{E}\left(|\mathcal{R}^{(\ell)}_{\tau,\sigma,n}|\right) = \sum_{s=1}^{\tau\sigma} \binom{n}{s} g_{\tau,\sigma}(s) \frac{1}{n^m} \cdot \mathcal{O}\left(\frac{1}{n^{x-\ell}}\right) \cdot o(1) = o\left(\frac{1}{n^x}\right).$$

To provide some modest evidence for the validity of Conjecture 6.2, we prove that it holds when $\sigma = 2$ or $\tau = 2$. Conjecture 6.3 remains open even in these cases. We begin by the following lemma.

Lemma 6.4. *Let T be a tile of a WRPS with $\sigma = 2$ and odd τ. Fix an arbitrary row as the 0th row. Let $\mathcal{M}_t = \{$maps up to t th row$\}$, $\mathcal{S}_t = \{$states up to t th row$\}$ and $\ell_t = |\mathcal{M}_t| - |\mathcal{S}_t|$, for $t = 0, 1, \ldots, \tau - 1$. Assume the $(t+1)$th row of the tile is ab. Then:*

1. *if $a \in \mathcal{S}_t$ and $b \in \mathcal{S}_t$, $\ell_{t+1} - \ell_t = 2$;*
2. *if exactly one of a and b is in $\mathcal{S}_t$, then $\ell_{t+1} - \ell_t = 1$; and*
3. *if $a \notin \mathcal{S}_t$ and $b \notin \mathcal{S}_t$, $\ell_{t+1} - \ell_t = 0$.*

Proof. Write $\ell_{t+1} - \ell_t = (|\mathcal{M}_{t+1}| - |\mathcal{M}_t|) - (|\mathcal{S}_{t+1}| - |\mathcal{S}_t|)$. Observe that $a \neq b$, as otherwise the spatial period of the tile is reducible. In addition, $(a, b) \notin \mathcal{M}_t$, as otherwise T is temporally reducible, and $(b, a) \notin \mathcal{M}_t$, as otherwise τ is even. Hence, $|\mathcal{M}_{t+1}| - |\mathcal{M}_t| = 2$, which implies the claim. □

Proof of Conjecture 6.2 when $\sigma = 2$. If τ is even, we need to show that $\text{rank}(T) \geq 1 - \ell$. This is trivial if $\ell \geq 1$, and follows from Lemma 2.8 when $\ell = 0$.

If τ is odd, we must show that $\text{rank}(T) \geq 2 - \ell$. We may assume $\ell = 1$ as otherwise this is immediate (as above). Then there exists exactly one $t \in \{0, \ldots, \tau - 1\}$ at which Case 2 of Lemma 6.4 happens, and otherwise Case 3 happens. If $a \in \mathcal{S}_t$, then column with b has no repeated state, and vice versa. □

Proof of Conjecture 6.2 when $\tau = 2$. We will prove this for any tile that satisfies the properties stated in Lemmas 2.1 and 3.3. We assume that no two different labels of T are rotations of each other; otherwise the argument is similar.

We use induction on the lag. If $\ell(T) = 0$, T is simple and Lemma 2.8 applies. Suppose now the statement is true for any tile T with $\ell(T) = \ell \geq 0$. Now, consider a tile T with $\ell(T) = \ell + 1$. As $\ell(T) \geq 1$, there is at least one repeated state, say a. Consider two appearance of a and its neighbors:

$$bac \quad \text{and} \quad b'ac'.$$

As $\tau = 2$ and T has no rotated columns, $b \neq b'$ and $c \neq c'$. Now replace the a in bac by an arbitrary state not represented in T, say z, and denote the new tile by T'. Note that T' also satisfies the properties in Lemmas 2.1 and 2.8. Moreover, $p(T') = p(T)$ and $s(T') = s(T) + 1$ imply that $\ell(T') = \ell$. By inductive hypothesis, $\text{rank}(T') \geq \sigma / \gcd(\sigma, \tau) - \ell$. Among $\text{rank}(T')$ labels of T' without a repeated state, at most one has the state z. Excluding this label, if necessary, we conclude that $\text{rank}(T) \geq \sigma / \gcd(\sigma, \tau) - (\ell + 1)$. □

Besides the above two special cases, we are also able to prove Conjecture 6.2 for a special class of tiles, which may give a hint about the general case. Within T, fix an arbitrary row as the 0th row and find the smallest $\tilde{\tau}$ such that $\texttt{row}_{\tilde{\tau}}$ is a cyclic permutation of $\texttt{row}_0$. It is likely that such $\tilde{\tau}$ does not exist, in which case define $\tilde{\tau} = \tau$. We call T **semi-simple** if $p(T) = \tilde{\tau}\sigma$; i.e., within the first $\tilde{\tau}$ rows in T, there are no repeated states. We omit the proof of our last lemma, as it is very similar to the argument above.

Lemma 6.5. *A semi-simple tile T has rank at least $\sigma / \gcd(\tau, \sigma) - \ell$.*

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