Reverse Shapes in First-Passage Percolation and Related Growth Models

JANKO GRAVNER
Mathematics Department
University of California
Davis CA 95616
gravner@feller.ucdavis.edu

DAVID GRIFFEATH
Mathematics Department
University of Wisconsin
Madison WI 53706
griffeat@math.wisc.edu

Final version, October 1998

Abstract. Over the past 40 years, it has been observed that many of the simplest random and deterministic local growth dynamics expand at a linear rate in each radial direction, and attain an asymptotic geometry. *Shape theorems* to this effect have been proved in several instances. In a similar manner, initially very large holes within supercritical local dynamics may be expected to attain a characteristic shape as they shrink, a while before disappearing. We describe a general theory of *reverse shapes* which formalizes this phenomenology, and then apply it to first-passage percolation and related deterministic and stochastic growth models. As an application, we analyze the last holes of such models started from sparse product measures.

1991 *Mathematics Subject Classification*. Primary 60K35, 52A10.
Key words and phrases. First-passage percolation, growth model, cellular automaton, shape theory.
Running head : REVERSE SHAPES IN GROWTH MODELS

Reverse Shapes in First-passage Percolation and Related Growth Models

JANKO GRAVNER AND DAVID GRIFFEATH

1. Introduction

Since the pioneering work of Broadbent and Hammersley **[BH]** and Eden **[Ede]** in the late 1950s, there has been a steady stream of research on *shape theory*, the principles whereby "crystals" formed from many of the simplest spatial growth algorithms attain an asymptotic geometry as they spread. Starting from a bounded set A of occupied sites, if the dynamics are local, homogeneous and supercritical, then one typically finds that the occupied crystal A_t at time t satisfies

$$t^{-1} A_t \stackrel{H}{\to} \mathcal{L}\,, \tag{1.1}$$

($\stackrel{H}{\to}$ denotes convergence in the Hausdorff metric), for some *asymptotic shape* $\mathcal{L}$ which is independent of the initial seed A. The first rigorous result of type (1.1) was obtained by Richardson **[Ric]**, using subadditivity arguments, for various random growth models on the d-dimensional integers $\mathbb{Z}^d$. The subadditivity approach to shape theory was subsequently extended to a variety of stochastic interacting particle systems (IPS) in **[BrG1-2]**, **[DL],** and **[DG]**. See **[Lig]** and **[Dur]** for authoritative accounts of subadditivity and its application to additive growth models, and **[BoG]** for an extension to some random systems which are not additive. In the context of first-passage percolation **[HW], [CD]**, if $\tau(x)$ denotes the passage time from the origin to site x, then under quite general conditions $A_t = \{x : \tau(x) \le t\}$ obeys the limit (1.1). Indeed, some of Richardson's processes may be viewed as special cases of first-passage growth, having passage times which are geometric with parameter p, while Eden's original crystal model can be interpreted as the asynchronous limit of such systems as $p \to 0$. Recent results for first-passage growth appear in **[NP]**.

Shape theory for cellular automata (CA) and related deterministic dynamics was initiated by Willson **[Wil]**, and more recently has been studied in considerable detail by the authors **[GG1-4]**. Some particularly simple growth rules evolve recursively, and subadditivity applies as in the random case, but the most illuminating route to (1.1) for cellular automata is based on a polar transform representation of

$\mathcal{L}$ in terms of half-space velocities, the so-called "Wulff construction" (cf. **[TCH]**). In the course of our earlier work on Excitable CA systems **[FGG1-2],** and in the study of various first-passage and dynamic tiling problems **[GG2-3]**, it became evident that rare nucleating crystals pass through three stages as they fill space: initially individual droplets spread with characteristic shape $\mathcal{L}$; later on large droplets collide to create wedge-shaped interstices often governed by new forces; and then eventually the last isolated empty pockets are filled. An analysis of convex corners and their complements, as in **[GG2]**, is crucial to understanding the intermediate time regime. One goal of this paper is to formulate a general *reverse shapes* approach to the final process of filling in large but bounded holes.

Although the Wulff recipe for asymptotic shape also seems quite broadly applicable to IPS models, only in a few cases (cf. **[Sep]**) are there techniques currently available to handle the difficult issue of stochastic interface fluctuations, and thereby establish the polar representation of $\mathcal{L}$ rigorously. For Richardson models which admit a first-passage representation, as for a large class of well-behaved first-passage percolation models, Kesten's groundbreaking work **[Kes2]** supplies the needed large deviation estimates. The polar formula is well-suited for Monte Carlo simulation (cf. **[GG4]**, Section 7), but otherwise offers little advantage over subadditivity unless half-space velocities can be computed effectively, which is virtually never the case with random dynamics. However, subadditivity does not seem directly applicable to the second and third stages of crystallization described above, in which case half-space analysis can provide genuinely new insights. As motivating examples, we consider two problems – reverse shapes and last holes – for the following one parameter family of systems.

Synchronously, at each discrete update, if x is any empty site which has at least one of eight nearest neighbors ($y : \|y - x\|_\infty = 1$) occupied, then x becomes occupied with probability p, independently of all other sites. This is *Richardson's model*, as studied in **[Ric]**, **[DL]**, **[Dur]**, except that we choose the box instead of diamond ($\|\cdot\|_1$) neighborhood, since we will also briefly discuss *Threshold Growth* rules in which "one of eight" is replaced by "two of eight," "three of eight," or "four of eight."

The problem of reverse shapes starts such dynamics from a large hole of prescribed shape, i.e., $A_0 = (mA)^c$ for some fixed A and large m. We will establish a reverse limit shape $\mathcal{R}$ for the shrinking hole, ϵm time units before it vanishes, as $m \to \infty$ and then $\epsilon \to 0$. An important feature distinguishes reverse limits from the "forward" result (1.1): since the initial hole mA becomes arbitrarily large, $\mathcal{R}$ typically depends on the geometry of A.

For the last holes problem, begin by populating $\mathbb{Z}^d$ according to Bernoulli product measure ν_δ, with density δ of occupied sites. For small δ, the vast majority of initially occupied sites are isolated singletons which spawn autonomous spreading crystals that approximate the shape $\mathcal{L}$ of (1.1) as they grow. Later, on a time scale of δ^{-1}, these droplets interact and combine, leaving isolated pockets of unoccupied sites which eventually shrink and disappear. What, then, is the *shape* of the last remaining holes a while before they vanish? To be precise, fix attention on the origin, say, denote the occupation time of $\mathbf{0}$ by T, and let $\mathcal{C}_n$ be the connected cluster, in the usual site percolation sense, of unoccupied sites containing $\mathbf{0}$. Thus $\{\mathcal{C}_n = \emptyset\} = \{T \le n\}$. Denote the law of $\frac{1}{\epsilon n}\mathcal{C}_{T-\epsilon n}$ under the conditional measure $P(\,\cdot\,|\,T > n)$ as $\mu_{\delta,\epsilon,n}$. Then a random set $\mathcal{H}_\delta$ with associated distribution $\mu_{\mathcal{H}_\delta}$ is called the *last hole* if

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} d(\mu_{\delta,\epsilon,n}, \mu_{\mathcal{H}_\delta}) = 0. \tag{1.2}$$

(Here d is the Prohorov distance between random compact sets corresponding to the Hausdorff metric d_H on compact subsets of $\mathbb{R}^2$.)

Using the reverse shapes formalism, we will see that for symmetric additive deterministic dynamics, and for any $\delta \in (0,1)$, $\mathcal{H}_\delta$ ***equals the forward shape*** $\mathcal{L}$ of (1.1) with probability one. For example, this result applies to the deterministic Richardson model above with $p = 1$, in which case the forward shape and last holes are squares . Our analysis will also show that for certain quasi-additive systems (to be defined later), $\mathcal{H}_\delta$ depends on δ but converges to $\mathcal{L}$ as $\delta \to 0$. For instance, the "two of eight" solidification CA has last holes which are approximately diamond shaped for small δ. More generally, e.g., for CA rules such as Bootstrap Percolation **[AL]**, or "three of eight" solidification, $\mathcal{H}_\delta$ most likely consists of a random assortment of holes, possibly nonconvex or with volume 0. In additive random models, the last hole as defined in (1.2) turns out to be $\{\mathbf{0}\}$, but for first passage times n with $\delta^{-\frac{1}{d}} \ll n \ll \delta^{-\frac{1}{d-1}}$, holes of shape $\mathcal{L}$ are observed. The nature of $\mathcal{H}_\delta$ for general random dynamics remains an enigma, as the event $\{T > n\}$ seems very difficult to understand in detail.

The rest of the paper is organized as follows. Section 2 begins with an abstract framework in $\mathbb{R}^d$, convenient for development of the basic theory of deterministic growth. Then we formulate precisely the general problem of reverse shapes, state and prove two reverse limit theorems, give corresponding formulas for the limit, and derive various consequences of those formulas. In Section 3 we offer six examples of Threshold Growth automata, on box neighborhoods of various ranges ρ, and with various thresholds θ, to illustrate some of the subtleties of reverse shape theory. These CA rules are chosen for their relative ease of computation, but the reader should understand that many of the same phenomena can be expected to arise in stochastic growth models. Section 4 applies the results of Section 2 to the problem of last holes for deterministic growth started from random seedings ν_δ. Finally, Section 5 details the modifications to our arguments which are needed to prove reverse shape and last hole results for additive stochastic solidification dynamics such as Richardson's model and first-passage percolation.

2. Reverse Shape Theory for Deterministic Growth

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^d$. A map $\mathcal{T}\colon \mathcal{B} \to \mathcal{B}$ is a *crystal growth transformation* (CGT) if it satisfies the following five properties:

(*a*) *absorption*: $\mathcal{T}\emptyset = \emptyset$;

(*b*) *translation invariance*: for all $A \in \mathcal{B}$ and $x \in \mathbb{R}^d$, $\mathcal{T}(x + A) = x + \mathcal{T}A$;

(*c*) *locality*: there exists a *neighborhood radius* $R > 0$ so that for every $A \in \mathcal{B}$,

$$\mathcal{T}A = \{x \in \mathbb{R}^d : x \in \mathcal{T}(A \cap B_2(x, R))\} ;$$

(We use the standard notation for Euclidean balls: $B_p(x, r) = \{y \in \mathbb{R}^d : \|y - x\|_p \leq r\}$. When $d = 2$, we will refer to the shapes $B_\infty(0, 1), B_2(0, 1)$, and $B_1(0, 1)$ as *box*, *circle*, and *diamond*, respectively.)

(*d*) *monotonicity*: for every $A, B \in \mathcal{B}$, $A \subset B \Rightarrow \mathcal{T}A \subset \mathcal{T}B$;

(*e*) *solidification*: for every $A \in \mathcal{B}$, $A \subset \mathcal{T}A$.

The iterates of a crystal growth transformation, given by $\mathcal{T}^n A_0$ for a fixed initial set A_0 are called *crystal growth dynamics*. Write $\mathcal{T}^\infty A_0 = \bigcup_{n \geq 0} \mathcal{T}^n A_0$. Such dynamics are *supercritical* if $\mathcal{T}^\infty B_2(0, R) = \mathbb{R}^d$ for a sufficiently large R, *subcritical* if $\mathcal{T}^\infty B_2(0, R)^c \neq \mathbb{R}^d$ for a sufficiently large R, and *critical* otherwise. *Completely symmetric* sets are invariant with respect to permutation of coordinates and switching signs of coordinates. Also, $\mathcal{T}$ is *completely symmetric* if it commutes with all such operations on coordinates.

Threshold growth dynamics, with parameters μ, θ, where μ is a measure on $\mathbb{R}^d$ with compact support, and $\theta \geq 0$ is the threshold, are given by $\mathcal{T}A = \{x \in \mathbb{R}^d : \mu(A - x) \geq \theta\}$. *Additive dynamics* with parameter a bounded set $\mathcal{N}$, are defined simply by $\mathcal{T}A = A + \mathcal{N}$. These are the two basic classes of crystal growth transformations we have in mind, but many other examples may be obtained by taking unions and intersections. Indeed, if $\mathcal{T}_1, \mathcal{T}_2, \ldots$ have the same neighborhood radius, then one can define new CGTs $\bigcup_n \mathcal{T}_n$ and $\bigcap_n \mathcal{T}_n$ by

$$(\bigcup_n \mathcal{T}_n)A = \bigcup_n (\mathcal{T}_n A), \quad (\bigcap_n \mathcal{T}_n)A = \bigcap_n (\mathcal{T}_n A).$$

Note that any crystal growth transformation $\mathcal{T}$ must translate half-spaces: if $u \in S^{d-1}$ is a unit vector and $H_u^- = \{x \in \mathbb{R}^d : \langle x, u \rangle \leq 0\}$, then there exists a *speed* $w(u) \geq 0$ such that either $\mathcal{T}H_u = H_u^- + [0, w(u))u$, or $\mathcal{T}H_u = H_u^- + [0, w(u)]u$. Now, introduce the (possibly unbounded) set $K_{1/w} = \bigcup_u [0, w(u)]u \subset \mathbb{R}^d$. Locality of $\mathcal{T}$ implies that $w : S^{d-1} \to \mathbb{R}$ is Lipshitz continuous.

Conversely, it is not hard to show that any such Lipshitz w is the speed function of a CGT, and that if $K_{1/w}$ is completely symmetric, then $\mathcal{T}$ can be chosen completely symmetric as well.

Continuity of w immediately implies that supercriticality is equivalent to $w > 0$ (**[GG1]**). On the other hand, subcriticality is not so easy to characterize. To see this, note that threshold growth with μ uniform on *box* and $\theta = 2$ is critical, while it has $w \equiv 0$. On the other hand, we have proved in **[GG2]** that the two-dimensional *discrete* threshold model, in which μ is a counting measure on a finite subset of $\mathbb{Z}^2$ is subcritical if and only if $w \equiv 0$.

In the supercritical case, assuming that the iterates of a bounded initial A_0 eventually cover any compact set, the *(Forward) Shape Theorem* **[GG1]** states that

$$\frac{1}{n}\mathcal{T}^n A \to \mathcal{L} = K^*_{1/w} \quad \text{as } n \to \infty, \tag{2.1}$$

where $K^* = \{y : \langle x, y\rangle \le 1 \text{ for every } x \in K\}$ is the polar transform of the set K.

We now proceed to the formulation and proof of two *Reverse Shape Theorems*, beginning with the simpler case. Since we will be concerned, in part, with infinite limit sets, let us specify precisely what is meant by convergence. In the present context, it is most convenient to say that $\Lambda_n \subset \mathbb{R}^d$ converges to a closed set $\Lambda \subset \mathbb{R}^d$ if $\Lambda_n \cap B_2(0, R)$ converges in the Hausdorff metric d_H to $\Lambda \cap B_2(0, R)$ for all sufficiently large $R > 0$. For notational convenience, we write $\lim_{\epsilon \to 0} \lim_{m\to\infty} \Lambda_{\epsilon,m} = \Lambda$ if for every R,

$$\lim_{\epsilon \to 0} \limsup_{m\to\infty} d_H(\Lambda_{\epsilon,m} \cap B_2(0, R), \Lambda \cap B_2(0, R)) = 0.$$

Throughout the paper, A will be denote a prescribed compact set which contains a neighborhood of $\mathbf{0}$. As in Section 1, set $A_0 = (mA)^c$ for some large m. Let $T = T(m)$ be the first time the origin is occupied by the dynamics:

$$T = \inf\{\, n : \mathbf{0} \in \mathcal{T}^n((mA)^c)\}.$$

Our first result establishes a reverse limit $\mathcal{R}(A)$ in the case of *convex* A.

Theorem 2.2. Assume that $w \ge 0$ is not identically 0 on S^{d-1}, and that A is convex. Then there is a unique nonempty convex proper subset $\mathcal{R}(A)$ of $\mathbb{R}^d$ such that

$$\lim_{\epsilon \to 0} \lim_{m\to\infty} \frac{1}{\epsilon m}\mathcal{T}^{T-\epsilon m}(mA^c) = \mathcal{R}(A)^c. \tag{2.3}$$

If $\mathcal{T}$ and A_0 are completely symmetric, then $\mathcal{R}(A)$ is bounded.

Remark. The weak double limit formulation in (2.3) for the reverse shape property holds quite generally for CGTs, and also for stochastic growth models, as we shall see in Section 5. In deterministic lattice dynamics, the rapid convergence methods developed by **[Wil]** imply that ϵm can be replaced by a large M and the double limit by $\lim_{M\to\infty} \lim_{m\to\infty}$.

Proof. If the boundary of mA has small curvature, then $\mathcal{T}^n((mA)^c)$ is well-approximated by the union of iterates over all exterior half spaces **[GG1-2]**. Hence, for m and n large,

$$\text{(2.4)} \qquad \mathcal{T}^n((mA)^c) \approx \cup \Big\{ \Big((m\alpha_A(u) - n\frac{w(v)}{\langle u,v\rangle})u + H^-_{-v} \Big) : u, v \in S^{d-1},\, v \in \nu_A(u) \Big\}.$$

Here $A = \{\lambda u : u \in S^{d-1},\, 0 \le \lambda \le \alpha_A(u)\}$, and $\nu_A(u)$ is the set of exterior normals to A at $\alpha_A(u)\,u$. The right side of (2.4) defines a transformation $\overline{\mathcal{T}}_{n,m}(A)$. For $u \in S^{d-1}$, $v \in \nu_A(u)$, define

$$\phi(u,v) = \frac{w(v)}{\alpha_A(u)\langle u,v\rangle} = w(v)\alpha_{A^*}(v),$$

$$M = \max_{u,\,v} \phi(u,v).$$

Then, if one chooses $n = (M^{-1} - \delta)m$ in (2.2), one gets

$$\lim_{m\to\infty} \frac{T(m)}{m} = \frac{1}{M}, \quad \text{and}$$

$$\text{(2.5)} \qquad \mathcal{R}(A) = \cap\{ M\alpha_A(u)u + H^-_v : v \in \nu_A(u), \phi(u,v) = M\}.$$

The proof of (3.2) in **[GG2]** shows that $\mathcal{R}(A)$ is invariant for $\overline{\mathcal{T}}_{n,m}$, i.e.,

$$\overline{\mathcal{T}}_{n,m}(\mathcal{R}(A)) = (m - Mn)\,(\mathcal{R}(A)^c),$$

and so a good candidate for the reverse shape.

Rigorous justification of (2.5) hinges on three observations. First, the $\supset$ inclusion in (2.4) follows from monotonicty. Secondly, one can approximate *all* the of iterates $\overline{\mathcal{T}}_{n,m}$ by sets of uniformly small curvature uniformly within ϵn. As in Section 4 of **[GG1],** this implies that $\mathcal{T}$ is well-approximated by $\overline{\mathcal{T}}$. Thirdly,

$$\lim_{\epsilon\to 0} \lim_{m\to\infty} \frac{\overline{\mathcal{T}}_{\epsilon m,m}(A)}{\epsilon m} = \mathcal{R}(A)^c,$$

by direct computation, since only u, v which maximize ϕ determine either side. □

Another characterization of $\mathcal{R}(A)$ follows from (2.5).

Corollary 2.6. Let

$$t_0 = \max\{t : tA^* \subset K_{1/w}\}.$$

Then

$$\mathcal{R}(A) = \big(t_0 A^* \cap \partial K_{1/w}\big)^*. \tag{2.7}$$

In particular, starting from *circle*, with $\overline{w} = \max\{ w(u) : u \in S^{d-1}\}$,

$$\mathcal{R}(\textit{circle}) = \cap \{\overline{w}\, u + H_u^- : w(u) = \overline{w}\, \}. \tag{2.8}$$

Proof. Since $\alpha_{A^*}(v) = (\alpha_A(u)\langle u, v\rangle)^{-1}$, (2.5) can be rewritten as

$$\mathcal{R}(A) = \cap \{ \frac{1}{\alpha_{A^*}(v)}\, v + H_v^- \,:\, v \in \nu_A(u), \alpha_{A^*}(v) = \frac{M}{w(v)}\, \}.$$

The condition $v \in \nu_A(u)$ is superfluous, and (2.7) follows. In the case $A = \textit{circle}$, note that $M = \overline{w}$. □

One unsatisfactory feature of Theorem 1 is the rather severe restriction to convex initial A. Unfortunately, nonconvex holes typically arise in disordered systems, and the general mechanism by which they shrink is quite complicated. As an illustration of the added complexity with nonconvex initial sets, consider addititive dynamics with $\mathcal{N} = \textit{box}$, and the completely symmetric stars A_x with boundary consisting of 8 line segments, one of which connects $(1,1)$ with any point $(x, 0)$, where $1 \le x \le \infty$. All of these sets are invariant, and all except the square are nonconvex. We call the dynamics $\mathcal{T}$ ***quasi-additive*** if $K_{1/w}$ is convex, a property enjoyed by additive $\mathcal{T}$, since $K_{1/w} = \mathcal{N}^*$ in that case, but otherwise not often satisfied (cf. **[GG1]**, **[Gra1]**). Only the two-dimensional quasi-additive case is simple enough to permit a thorough understanding of reverse shapes starting from fairly general sets. The construction leading to the following result is rather involved, so we will only provide a sketch in the additive case with convex $\mathcal{N}$.

Theorem 2.9. Assume $d = 2$, that $\mathcal{T}$ is quasi-additive, that $w > 0$ on S^1, and that the boundary of A consists of a finite number of piecewise differentiable curves. Then the double limit in (2.3) exists, and $\mathcal{R}(A)$ is nonempty, although it need not be convex.

Sketch of the proof. We begin with a recipe for $\mathcal{R}(A)$. For additive dynamics, the first-passage time of the origin can be represented as

$$T = \sup\{t : t\mathcal{N} \cap \partial A = \emptyset\}.$$

For each $x \in T\mathcal{N} \cap \partial A$, we define a wedge or half-space W_x as follows. By convention, none of the wedges or half spaces in the following construction include **0**. If ∂A is differentiable at x, then W_x is the tangent half plane to ∂A at x. If ∂A is nondifferentiable at x, then there are two rays tangent to ∂A at x. Denote the wedge determined by those rays as $W_x^{'}$. The ray from x through **0** intersects $W_x^{'} + \mathcal{N}$ at some point y. W_x is the half plane or wedge at x which is parallel to the half plane or wedge tangent to $\boldsymbol{\partial}\,(W_x^{'} + \mathcal{N})$ at y. Finally, we claim that

$$\mathcal{R}(A) = \Big(\bigcup_{x \,\in\, T\mathcal{N} \,\cap\, \partial A} W_x\Big)^c. \tag{2.10}$$

To justify this formula, note first that only an infinitesimal neighborhood of $T\mathcal{N} \cap \partial A$ plays a role in the reverse shape since other points of A lag linearly behind. Also, the prescription is correct if $T\mathcal{N} \cap \partial A$ consists of a single point. Finally, the general case follows by additivity.

Formula (2.10) also holds for any $\mathcal{T}$ and initial A satisfying the hypotheses of the theorem provided $\mathcal{N}$ is replaced by $\mathcal{L}$ in the construction, but the proof is somewhat more involved so we omit it. The formula shows that $\mathcal{R}(A)$ is convex for every A if and only $\mathcal{L}$ is C^1. □

Key to our analysis of last holes for symmetric, additive and quasi-additive CA crystals in Section 4 is the following special case of Corollary 2.6, which asserts that $\mathcal{L}$ is *invariant*, by which we mean that the forward asymptotic shape is its own reverse limit. In other words, the hole obtained by growing a large crystal A_t, and then reversing the occupied and empty sites of $\mathbb{R}^d$, retains its shape as it shrinks.

Corollary 2.11 If $\mathcal{T}$ is quasi-additive, then $\mathcal{R}(\mathcal{L}) = \mathcal{L}$.

Proof. Since $K_{1/w}$ is convex, $A^* = \mathcal{L}^* = K_{1/w}$, so that in (2.7) evidently $t_0 = 1$ and $t_0 A^* \cap \partial K_{1/w} = \partial K_{1/w}$. Hence $\mathcal{R}(\mathcal{L}) = (\partial K_{1/w})^* = \mathcal{L}$, as claimed. □

For the remainder of this section we assume that $d = 2$. Let us first try to answer the following question: is there a way to center convex initial A so that $\mathcal{R}(A)$ is bounded? More precisely, call $x_0 \in \mathbb{R}^2$ a $\mathcal{T}$*-center* for A if $\mathcal{R}(x_0 + A)$ is bounded. If A is not lattice symmetric, then the center need not exist. To see this, write $e_1 = (1,0)$, and consider the case with $K_{1/w} = B_\infty(e_1, 1) \cup B_\infty(-e_1, 1)$, corresponding to additive dynamics with $\mathcal{N} = \{(\pm 1, 0), (0, \pm \frac{1}{2})\}$. Choose $A =$ *box*, in which case different speeds in the horizontal and vertical directions produce increasingly oblong shrinking holes. However such anomalies cannot occur in completely symmetric cases.

Proposition 2.12. Under the assumptions of Theorem 1, and complete symmetry of $\mathcal{T}$ and A, the origin is the unique $\mathcal{T}$-center for A.

Proof. Since w is finite, and positive in some direction u, t_0 in Corollary 1 must be positive and finite. By symmetry, $t_0A^* \cap \partial K_{1/w}$ must contain at least 4 symmetrically situated points. Hence the dual of this set, $\mathcal{R}(A)$, is bounded with nonempty interior containing the origin. Thus **0** is a $\mathcal{T}$-center. For $x_0 \neq$ **0,** the dynamics from $x_0 + A$ are the translate by x_0 of the dynamics from A. So when $\mathcal{T}^n((m(x_0 + A))^c)$ hits the origin, its diameter is bounded below by a constant multiple of m. □

For simplicity, then, we will assume complete symmetry of $\mathcal{T}$, and complete symmetry and convexity of all sets A, in the following analysis of shape dependence on initial seeds. For a typical $K_{1/w}$, different A yield many different reverse shapes. As we will see, most of them are unstable, since small perturbations produce large differences in $\mathcal{R}(A)$. It is important to note that the following notion of stability is formulated ***only*** within the restricted class of convex and completely symmetric sets A.

Definitions. A has a ***weakly stable*** reverse shape $\mathcal{R}(A)$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_H(A, A') < \delta$ implies $d_H(\mathcal{R}(A), \mathcal{R}(A')) < \epsilon$. $\mathcal{R}(A)$ is ***stable*** if there exists a $\delta > 0$ such that $d_H(A, A') < \delta$ implies $d_H(\mathcal{R}(A), \mathcal{R}(A')) = 0$.

In the stability criterion which follows,

$$\Omega = \{(x_1, x_2) : 0 \leq x_2 \leq x_1\}, \quad R = \{(x_1, x_2) : x_1 > 0,\, x_1 + x_2 > 0\} \cup \{\mathbf{0}\}.$$

Proposition 2.13. The reverse shape $\mathcal{R}(A)$ is weakly stable if and only if $|t_0A^* \cap \partial K_{1/w}|$ is either 4 or 8 (in which case $\mathcal{R}(A)$ has 4 or 8 sides, respectively). $\mathcal{R}(A)$ is stable if and only if it is weakly stable and, for any $x_0 \in t_0A^* \cap \partial K_{1/w}$ which lies in the closed first octant, there is a $\delta > 0$ such that $B_2(x_0, \delta) \cap \partial K_{1/w} \cap \Omega \subset x_0 + R$.

In other words, stability means that in the first octant, t_0A^* only meets parts of the boundary of $K_{1/w}$.which lie inside the translate of cone R. Note that in quasi-additive cases the forward shape $\mathcal{L}$, which is its own reverse shape by Corollary 2.11, can ***never*** be even weakly stable.

Proof. We apply Corollary 2.6, using the fact that small perturbations of A produce small changes in t_0. Assume first that t_0A^* intersects $K_{1/w}$ in 8 points, one in the interior of each octant by symmetry. Fix an $\epsilon > 0$, and let B_ϵ denote the ϵ-neighborhood of these 8 points. There exists an $\eta > 0$ such that $d_H(B_\epsilon^c \cap K_{1/w}, t_0A^*) > \eta$. Therefore, for A' sufficiently close to A, the corresponding t_0' is close to t_0, and $d_H(B_\epsilon^c \cap K_{1/w}, t_0'(A')^*) > \eta/2$. By symmetry, $t_0'(A')^* \cap \partial K_{1/w}$ contains at least one point in each octant. It follows that the d_H-distance between $t_0'(A')^* \cap \partial K_{1/w}$ and $t_0A^* \cap \partial K_{1/w}$ is at most ϵ. Continuity of the polar transform establishes weak stability. The remaining cases of 4 or 8 point intersections are similar. Conversely, if $\Omega \setminus \{x_1 = x_2\}$ contains two points of intersection, then the point in the interior of Ω can be removed by an arbitrarily small perturbation.

As before, we consider only the case of one intersection point x interior to Ω. If $t_0 A^*$ only meets $\partial K_{1/w}$ in $x + R$, then instability of $\mathcal{R}(A)$ would imply the existence of 8 symmetrically located points in $t_0 A^*$, having x interior to their convex hull. This contradicts the definition of t_0. Conversely, if the stability condition is violated, then the octagon with one vertex at x is not stable. ☐

A particularly strong form of stability is ***uniqueness***. The rare cases in which it holds are as follows.

Proposition 2.14. All reverse shapes $\mathcal{R}(A)$ agree if $\mathcal{R}($*diamond*$) = \mathcal{R}($*box*$)$. Moreover, if the reverse shape is unique, then it must be a diamond , a box, or an octagon.

Proof. By Corollary 2.6, if the reverse shapes for diamond and box agree, then there exist t_0 and t_0' such that $\partial K_{1/w}$ meets t_0 ***diamond*** $\cup\, t_0'$ ***box*** only in $S = t_0\, \partial$***diamond*** $\cap\, t_0'\, \partial$***box***. Any completely symmetric convex set A such that $S \subset \partial A$ is included in t_0 ***diamond*** $\cup\, t_0'$ ***box***. This proves uniqueness. The reverse shape is then determined by S, which consists of 4 points in a box, 4 points in a diamond, or 8 points in an octagon. ☐

Necessary and sufficient conditions for this unique $\mathcal{R}$ to be a diamond or box are $\mathcal{R}($***box***$) =$ ***diamond***, and $\mathcal{R}($***diamond***$) =$ ***box***, respectively. For example, let $\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2,$ where $\mathcal{T}_1, \mathcal{T}_2$ correspond to additive dynamics with $\mathcal{N}_1 = \{(\pm 1, 0), (0, \pm \frac{1}{3})\}$ and $\mathcal{N}_2 = \{(0, \pm 1), (\pm \frac{1}{3}, 0)\},$ respectively. Then $K_{1/w} = \{0, (\pm 2, 0), (0, \pm 2)\} +$ ***box***, and ***diamond*** is the unique shape. Rotate by $45°$ to get a (rescaled) box as the unique shape. The first example of the next section illustrates octagonal uniqueness.

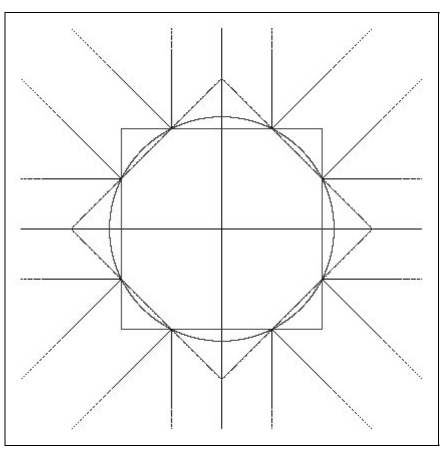
3. Examples

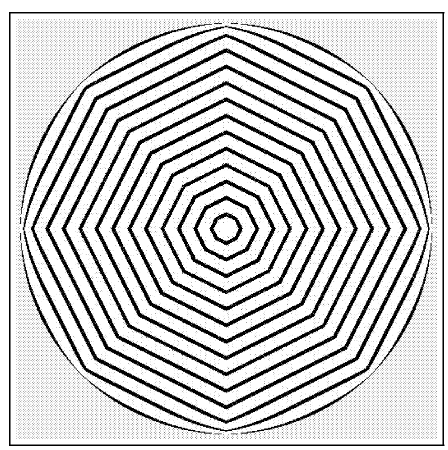
Let us now present five illustrative examples of Theorem 2 for deterministic Threshold Growth on the two dimensional integers. Such dynamics have already been studied elsewhere in considerable detail (see **[Boh]**, **[GG1-5]**). Within our general framework, these cases have μ a counting measure on the range ρ lattice box $\mathcal{N} = B_\infty(0, \rho) \cap \mathbb{Z}^2$, and θ an integer-valued threshold. In words, an empty site x joins the occupied set if it sees at least θ occupied cells within its range ρ box neighborhood. Such dynamics are supercritical for $\theta \le \rho(2\rho + 1)$, and critical for $\rho(2\rho + 1) < \theta \le 2\rho(\rho + 1)$. The discrete deterministic setting makes computation of the half-space velocities $w(u)$ straightforward, and hence, starting from completely symmetric convex initial data A, a complete analysis of the possible reverse shapes $\mathcal{R}(A)$ follows from Corollary 2.6. The solidification process called *Hickerson's Diamoeba* **[GG4]** provides a beautiful example of distinct subsequential reverse shapes when CGT axiom (***d***) is lacking.

For each of the following examples we specify the range ρ, threshold θ, vertices of $K_{1/w}$ in the first octant $\{0 \leq y \leq x\}$ (the others are determined by symmetry), and the maximal number of sides in any (polygonal) reverse shape $\mathcal{R}$. In each case we show that graph of $K_{1/w}$, including the largest inscribed diamond, circle, and square. Their points of contact with $K_{1/w}$ in (2.7) characterize the dual of the reverse shape for a square, diamond and circle, respectively (recall that the diamond and square are dual and the circle self-dual). A second graphic for all but one of the examples shows the evolution of a shrinking hole, in an alternating black and white palette, for a representative initial A. Finally, a brief description summarizes the distinctive features of each case. Octagons play a central role in the stability analysis of our examples, so we denote by O_m the completely symmetric shape with sides of slope $\pm m$, $\pm m^{-1}$. For simplicity we identify all dilations of $\mathcal{R}$ in the discussion, ignoring the *size* (as opposed to *shape*) of the limit. Thus the notation $\mathcal{R} \propto O_m$ means that $\mathcal{R} = c\, O_m$ for a suitable constant $c > 0$ which we choose not to compute. However c is easily determined from $K_{1/w}$.

Some "diophantine" aspects of these deterministic lattice examples presumably do not arise in more general growth models, especially those with random dynamics. But other features should be shared by many nonlinear systems, deterministic or stochastic, with space and time either discrete or continuous, since it would seem common for anisotropic neighborhoods to generate nonconvex $K_{1/w}$. We shall return to this point in Sections 4 and 5.

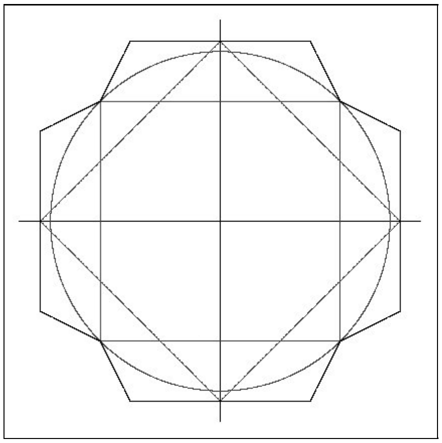
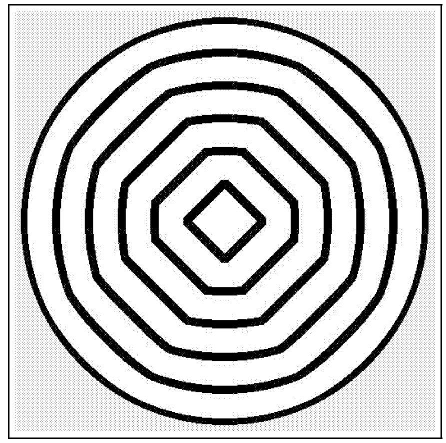
Example 3.1. $\rho = 1, \theta = 4$.
$K_{1/w}$ has vertices $(\frac{1}{0}, 0), (2, 1), (\frac{1}{0}, \frac{1}{0})$. $\mathcal{R}$ is always a rescaling of O_2.





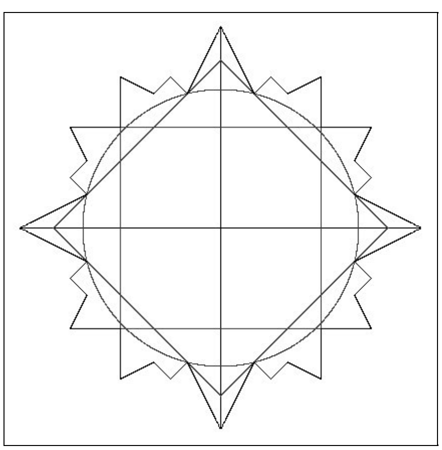
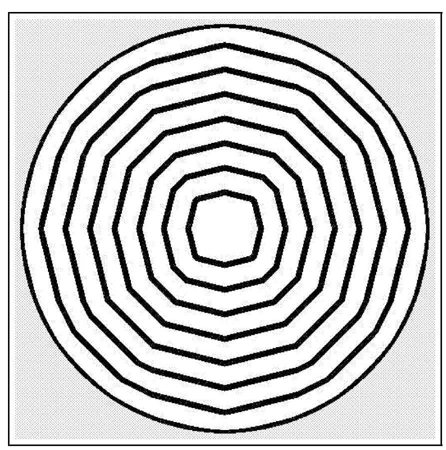
This critical rule seems to be the only lattice example of reverse shape uniqueness with box neighborhood, for any range ρ. Note that $K_{1/w}$ is infinite in the horizontal, vertical and diagonal directions, a manifestation of criticality. The figure on the right shows convergence to $\propto O_2$ from a lattice circle, after 530 steps from 'radius' 250).

Example 3.2. $\rho = 2, \theta = 3$.
$K_{1/w}$ has vertices $(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{3})$. $\mathcal{R}$ has at most 12 sides.

O_m octagons are invariant (up to rescaling) and weakly stable for $m \geq 3$, but converge to a multiple of *diamond* for $1 \leq m < 3$. The right figure shows $\mathcal{R}(\textit{circle}) \propto \textit{diamond}$. (A hole of 'radius' 250 fills at time 118.)

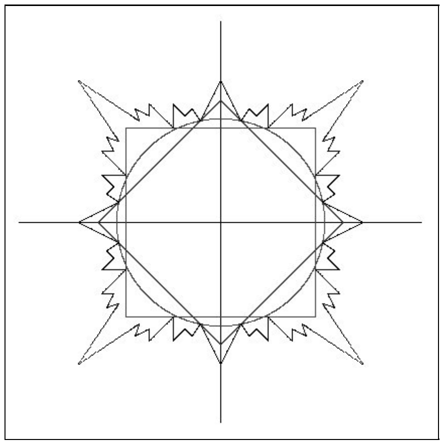
Example 3.3. $\rho = 2, \theta = 6$.
$K_{1/w}$ has vertices $(1, 0), (\frac{2}{3}, \frac{1}{6}), (\frac{3}{4}, \frac{1}{4}), (\frac{2}{3}, \frac{1}{3}), (\frac{3}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$. $\mathcal{R}$ has at most 12 sides.

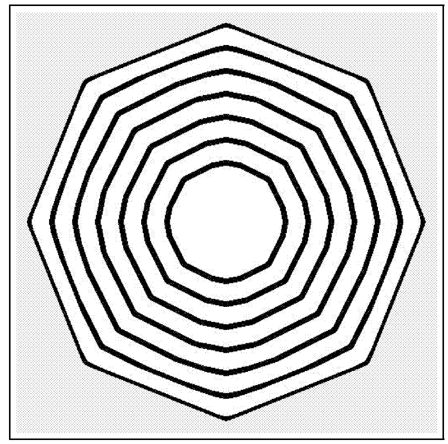



O_2 is unstable with three distinct limit shapes in its neighborhood: $\propto$ *diamond* (stable), $\propto O_4$ (stable), and a rescaling of the unstable 12-gon with vertices $(51, 0)$, $(44, 28)$. The right figure illustrates convergence to $\propto O_4$ from a lattice circle (at time 150 from 'radius' 250).

Example 3.4. $\rho = 3, \theta = 16$.

$K_{1/w}$ has vertices $(\frac{5}{7}, \frac{1}{7}), (\frac{2}{3}, \frac{1}{3})$, and many others irrelevant to reverse shapes. $\mathcal{R}$ has at most 16 sides.



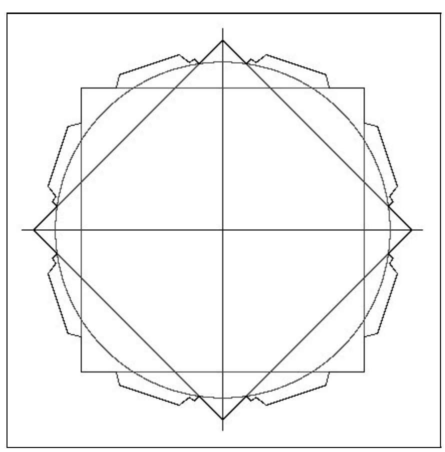


O_2 and O_5 are invariant up to rescaling, and stable. O_m tends to $\propto O_2$ for $1 \leq m \leq \frac{5}{2}$, to $\propto O_5$ for $m > \frac{5}{2}$. $O_{\frac{5}{2}}$ tends to a rescaling of the unstable 16-gon with vertices (234,0), (215,95), (175,175). The left figure shows convergence of $O_{\frac{5}{2}}$ to the unstable 16-gon (at time 125 from 'principal radius' 245).

Example 3.5. $\rho = 4, \theta = 11$.

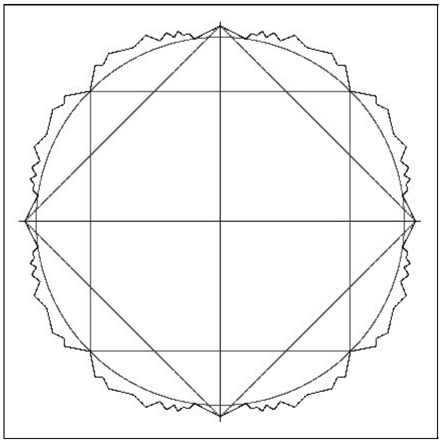
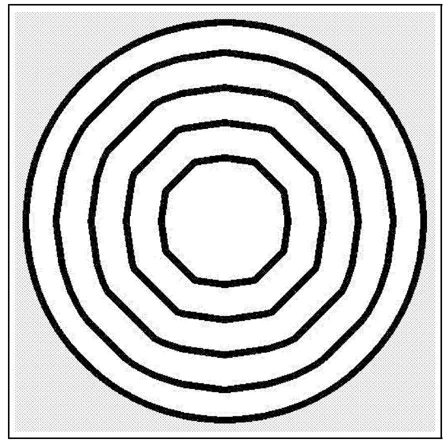
$K_{1/w}$ has vertices $(\frac{1}{3}, 0), (\frac{7}{24}, \frac{1}{24}), (\frac{3}{10}, \frac{1}{20}), (\frac{5}{17}, \frac{1}{17}), (\frac{4}{13}, \frac{1}{13}), (\frac{3}{11}, \frac{2}{11}), (\frac{3}{16}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4})$.

$\mathcal{R}$ has at most 32 sides.



Flat edges of $K_{1/w}$ give rise to a continuum of unstable reverse shapes; here an example of an unstable 32-gon, invariant up to rescaling, has polar determined by (7/24,1/24), (9/31,4/31), (2/7,1/7), (3/16,1/4).

Example 3.6. $\rho = 5, \theta = 15$.
$K_{1/w}$ has vertices $(\frac{7}{30}, \frac{1}{30}), (\frac{2}{9}, \frac{1}{9}), (\frac{3}{13}, \frac{1}{13}), (\frac{1}{6}, \frac{1}{6}),$ and *many* more irrelevant to reverse shapes. $\mathcal{R}$ has at most 24 sides.

From a circle, the reverse limit shape $\mathcal{R}(\textit{circle})$ is an unstable rescaling of the 12-gon with vertices (150,0), (140,70). To obtain a reverse shape with more than 8 sides starting from a circle, by (2.8) there must be two points x_1 and x_2 in the first octant of the boundary of $K_{1/w}$ such that $\|x_1\|_2 = \|x_2\|_2$, as in the present case where $\|(\frac{7}{30}, \frac{1}{30})\|_2 = \|(\frac{1}{6}, \frac{1}{6})\|_2$. Since edge speeds are a nonlinear function of system parameters, such examples are *very* rare; this is the only case with small range, and perhaps the only one for any choice of ρ and θ. The figure on the right shows convergence to the unstable 12-gon from a lattice circle (at time 42 from 'radius' 250).

4. Last Holes for Deterministic Growth on $\mathbb{Z}^d$

We now turn to the study of last holes (1.2), as described in the Introduction. Let $\Pi = \Pi(\delta)$ be a random configuration of occupied sites in $\mathbb{Z}^d$ distributed according to the Bernoulli product measure ν_δ with density δ. Recall that T is the occupation time of $\mathbf{0}$ starting from Π. Throughout this section, let $\mathcal{T}$ be a CGT satisfying (*a*) - (*e*) of Section 2, and also

(*f*) *symmetry*: for every A, $\mathcal{T}(-A) = -\mathcal{T}A$. (In the additive case, $\mathcal{N} = -\mathcal{N}$).

For additive supercritical transformations, we are able to identify the shape of last holes starting from any nontrivial $\Pi(\delta)$. Roughly, if the origin is not occupied at time n, then the set $n\mathcal{L}$ initially has no seeds, but if $n^d \gg \delta^{-1}$, then there are lots of seeds everywhere around the boundary. Since $\mathcal{L}$ is an invariant reverse shape by Corollary 2.11, the hole around $\mathbf{0}$ at time $T - \epsilon n$ is approximately $\epsilon n\mathcal{L}$.

Theorem 4.1. For any $\delta \in (0,1)$, additive supercritical growth satisfies (1.2), with the last hole $\mathcal{H}_\delta$ equal to the (forward) asymptotic shape $\mathcal{L}$ of (1.1). That is, for every $0 < \delta < 1,$ and every $\eta > 0$,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} P_\delta\big(d_H(\frac{\mathcal{C}_{T-\epsilon n}}{\epsilon n}, \mathcal{L}\big) > \eta \mid T > n) = 0. \tag{4.2}$$

Proof. Writing $A_n = \mathcal{T}^n\{\mathbf{0}\},$ symmetry implies that

$$\{T > n\} = \{A_n \cap \Pi = \emptyset\}. \tag{4.3}$$

Conditioning on $\{T > n\}$ therefore has no effect on sites in A_n^c. By (2.1), for large n and any $\eta_1 > 0$,

$$(1-\eta_1)\, n\mathcal{L} \subset A_n \subset (1+\eta_1)\, n\mathcal{L}\,.$$

In particular, the event $\{T > n\}$ and events involving sites outside $(1+\eta_1)\, n\mathcal{L}$ are independent. With overwhelming probability, for every x belonging to $(1+2\eta_1)\, n\mathcal{L} \setminus (1+\eta_1)\, n\mathcal{L}$,

$$\Pi \cap B_\infty(x, \eta_1 n) \neq \emptyset\,.$$

Moreover, on $\{T > n\}$, $\Pi \cap (1-\eta_1)\, n\mathcal{L} = \emptyset$. The last two conditions, together with Corollary 2.11, imply that, on $\{T > n\}$, $\mathcal{C}_{T-\epsilon n}$ differs from $\epsilon n\mathcal{L}$ by $(C\eta_1 + o(\epsilon))\, n$ for some constant $C = C(\mathcal{N})$. Choosing $\eta_1 = o(\epsilon)$, (4.2) follows. ☐

An elaboration of the last argument shows that the last holes of *quasi-additive* dynamics from sparse product measures have shape approximately $\mathcal{L}$. For simplicity we assume $d = 2$. For instance, our next result applies to “two of eight” solidification, or to Threshold Growth on $\mathbb{Z}^2$ with $\theta = 2$ and any range ρ box neighborhood **[GG2]**. New complications arise due to the presence of failed nucleation centers around the origin (e.g., isolated singletons in our $\theta = 2$ examples) which slightly alter the process whereby holes are filled in. For this reason, $\mathcal{L}$ in only achieved in the limit as $\delta \to 0$. To ensure that failed centers are well-behaved, in addition to (*a*) - (*f*) we assume

(*g*) *strong omnivoracity* : Let R be such that $\mathcal{T}^\infty B_\infty(0,R) = \mathbb{R}^d$. There exists an N so that for every A_0 such that $\mathcal{T}^{N+1}A_0 \neq \mathcal{T}^N A_0,$ $B_\infty(x,R) \subset \mathcal{T}^N A_0$ for some x.

This uniform version of the *omnivorous* assumption in **[GG2]** holds for all the threshold 2 models mentioned above, and can be verified by computer for other simple crystals.

Theorem 4.4. If $\mathcal{T}$ is quasi-additive, supercritical, and strongly omnivorous, then the last hole converges to $\mathcal{L}$ as $\delta \to 0$, in the sense that, for every $\eta > 0$,

$$\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \limsup_{n \to \infty} P_\delta\big(d_H(\frac{\mathcal{C}_{T-\epsilon n}}{\epsilon n}, \mathcal{L}\big) > \eta \mid T > n) = 0.$$

Proof. Call site x is a ***nucleus*** if at the time N of (*g*), $B_2(x,R) \subset \mathcal{T}^N\Pi$. Let $r > 0,$ and introduce comparison dynamics $\mathcal{T}_\delta,$ as on p.1771 of **[GG2]** . This process supposes a nucleus at a single site, but nowhere else, and models growth in a sparse random environment with effects of range at most R. Lemma 6.4 of **[GG2]** shows that for every $\eta_1 > 0,$ there is a δ sufficiently small that

$$P\left(\mathcal{T}_\delta^n\{x\} \not\subset (1+\eta_1)\,n\mathcal{L}\right) \le e^{-n}.$$

Let G be the following event, which depends on $\Pi|_{(1+2\eta_1)\,n\mathcal{L}}$. On $G,$ impose a boundary condition of all 1's outside $(1+2\eta_1)\,n\mathcal{L},$ and require that every nucleus x is such that $\mathcal{T}_\delta^k\{x\} \cap (1+2\eta_1)n\mathcal{L} \subset x + (1+\eta_1)\,k\mathcal{L}$ for every $k > \eta_1 n$. By the FKG inequality,

$$P\left(G^c \,|\, T > n\right) \le P(G^c) \le 8\,C^2 n^3 e^{-\eta_1 n}.$$

Since the dynamics are deterministic, on $\{T > n\}$ there is no nucleus within $(1-\eta_1)\,n\mathcal{L}$. Moreover, on $G,$ no points outside $(1+2\eta_1)\,n\mathcal{L}$ can effect $\{T > n\}$.

From this point on, the proof is a minor modification of the additive argument. Write

$$G_1 = \{\forall x \in (1+3\eta_1)\,n\mathcal{L} \setminus (1+3\eta_1)\,n\mathcal{L}), B_\infty(x,\eta_1 n) \text{ contains a nucleus}\},$$
$$G_2 = \{\text{there is no nucleus in } (1-\eta_1)\,n\mathcal{L}\}.$$

If n is very large compared to δ^{-1}(say, of order δ^{-8R^2}), then $P\left(G_1^c\right)$ is very small. Hence, since G_1^c and $\{T > n\}$ are conditionally independent given $G,$ $P\left(G_1^c \,|\, T > n\right)$ is also very small for large n. Moreover, G_2 occurs deterministically on $\{T > n\}$. On $G \cap G_1 \cap G_2,$ as in the additive case, $d_H(\mathcal{C}_{T-\epsilon n}, \epsilon n\mathcal{L}) \le (C\eta_1 + o(\epsilon))\,n.$ □

Without quasi-additivity a host of new issues arise, so we conclude this section with speculation about two of the simplest cases.

Example 4.5. For “three of eight” solidification, forward shape $\mathcal{L}$ is the completely symmetric octagonal region with vertices $(\frac{1}{2},0), (\frac{1}{3},\frac{1}{3})$, and $\mathcal{R}(\mathcal{L})$ is the square $B_\infty(0,\frac{1}{2})$. But there are five holes A smaller than that square, hence more prevalent in ν_δ, which are invariant and such that mA reaches the origin at the same time T. These candidates for last holes are the nonconvex completely symmetric star with vertices $(\frac{1}{2},0), (\frac{1}{6},\frac{1}{6})$, and the asymmetric nonconvex quadrilateral with vertices (couterclockwise) $(\frac{1}{2},0), (-\frac{1}{2},-\frac{1}{2}), (0,-\frac{1}{2}), (\frac{1}{4},\frac{1}{4})$ and its rotations by 90°, 180° and 270°, all with area $\frac{1}{3}$. We suspect that no smaller invariant hole has the same $T,$ and hence conjecture that

$$\lim_{\delta\to 0}\ \limsup_{\epsilon\to 0}\ \limsup_{n\to\infty}\ d(\mu_{\delta,\epsilon,n}, \mu_{\mathcal{H}_\delta}) = 0$$

(in the manner of (1.2)), where $\mu_{\mathcal{H}_\delta}$ is some measure concentrated on the above five shapes. Several hurdles would need to be overcome in order to establish this result, and even so, the relative weight of last holes assigned to stars vs. quadrilaterals would pose an extremely difficult problem

Example 4.6. "Four of eight" solidification is a *critical* CGT, convex-confined like its *bootstrap* **[AL]** relatives on the range 1 diamond neighbor set. The last hole is most likely, for every δ, a random set of dimension one. This is probably quite difficult to prove, but **[AMS]** provides good evidence: for one variant of bootstrap percolation, in which a site becomes occupied if it has one occupied nearest neighbor in the horizontal direction and one in the vertical direction, that paper shows that as $n \to \infty$,

$$\frac{\log P(T > n)}{2n \log (1 - \delta)} \to 1. \tag{4.7}$$

The denominator is the logarithm of the probability that an interval of $2n$ sites is empty. This should mean that conditioning on $\{T > n\}$ is equivalent to conditioning on such a vacant interval around **0**. For every large n (much larger than $e^{c/\delta}$), the rest of space is filled way before the interval shortens significantly from its ends. The last stage then consists of filling the interval, and the shape is obvious, but lattice symmetry dictates a random distribution of orientations. Of course, whether or not such heuristics can be substantiated, the last hole of the **[AMS]** model cannot possibly be a set with positive area, since that would violate (4.7).

5. Reverse Shapes for Additive Random Crystals

Our setting for stochastic growth on $\mathbb{Z}^d$ begins with a finite set $\mathcal{N} \subset \mathbb{Z}^d$ (the *neighborhood* of the origin) with $\mathbf{0} \in \mathcal{N}$, and a table of local transition probabilities $\pi : 2^{\mathcal{N}} \to [0,1]$. We assume that π is a *monotone solidification* map, i.e., $A \subset B \subset \mathcal{N}$ implies $\pi(A) \le \pi(B), \pi(\emptyset) = 0,$ and $\pi(\{\mathbf{0}\})$=1. Then a random crystal growth model $\xi_n \in \{0,1\}^{\mathbb{Z}^d}$ is constructed as follows. Identifying ξ_n with $\{\xi_n = 1\}$, at every update, each site x looks at the configuration of its neighborhood $x + \mathcal{N}$, and decides independently, with probability $\pi((\xi_n - x) \cap \mathcal{N})$, whether to becomes occupied. Given any $\Lambda \subset \mathbb{R}^d$, let ξ_n^Λ denote the dynamics started from $\Lambda \cap \mathbb{Z}^d$.

Although propagation of stochastic interfaces is a notoriously difficult subject, at least the formulation of half space speeds is straightforward. For $u \in S^{d-1}$, let $\xi_0 = H_u^- \cap \mathbb{Z}^d$, and introduce

$$V_n(u) = \{\lambda \in \mathbb{R} : \lambda u \in \xi_n + B_\infty(\mathbf{0}, \tfrac{1}{2})\}.$$

Call $w(u)$ the *half space velocity in direction* u if

$$w(u) = \lim_{n \to \infty} n^{-1} \max V_n(u) = \lim_{n \to \infty} n^{-1} \min (\mathbb{R} \setminus V_n(u))$$

exists with probability one. In order to apply the deterministic methods of Section 2 to random crystals, one needs a large deviations inequality of the type established in **[Kes2]** for some first-passage percolation models to be mentioned below. We formalize the required estimate as follows.

Definition. Say that ξ_n is *Kesten* if w exists and there is a strictly positive function $\Gamma : S^{d-1} \times (0,\infty) \to (0,\infty)$ such that for every $u \in S^{d-1}$ and every $\epsilon > 0$,

$$P(|\max V_n(u) - n\,w(u)| + |\min(\mathbb{R} \setminus V_n(u)) - n\,w(u)| > \epsilon n) \le e^{-\Gamma(u,\epsilon)\,n}. \tag{5.1}$$

In **[BGG]** we show that Kesten random crystals grow with an asymptotic shape $\mathcal{L}$ given by the polar formula in (2.1). The main result of this section establishes corresponding reverse shapes given by (2.4). In the statement of the next theorem, a.s. convergence refers to the basic coupling which constructs processes $\xi_n^{mA^c}$ for every m on a common probability space. Note that coupling and monotonicity imply continuity of $w : S^{d-1} \to [0,\infty)$ whenever it exists.

Theorem 5.2. Assume that ξ_n is Kesten, and that $w > 0$. Given convex $A \subset \mathbb{R}^d$, there is a unique nonempty convex proper subset $\mathcal{R}(A)$ of $\mathbb{R}^d$ such that almost surely,

$$\lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{\epsilon m} \xi_{T-\epsilon m}^{mA^c} = \mathcal{R}(A)^c. \tag{5.3}$$

Proof. Introduce

$$V_n(u,x) = \{\lambda \in \mathbb{R} : \lambda u + x \in \xi_n + B_\infty(\mathbf{0}, \tfrac{1}{2})\}.$$

Then by (5.1) ,

$$P(|\max V_n(u,x) - n\,w(u)| + |\min(\mathbb{R} \setminus V_n(u,x)) - n\,w(u)| \le \epsilon n$$
$$\text{for every } x \text{ such that } \langle x, u\rangle = 0 \text{ and } \|x\|_2 \le n^2)$$
$$\le C\,n^{2(d-1)}\,e^{-\Gamma(u,\epsilon)\,n}.$$

Assume that A is a convex set with C^2 boundary having curvature bounded above and below. In other words, there is a positive constant C_1 such that

$$C_1^{-1}\|u_2 - u_1\| \le \|\nu_A(u_2) - \nu_A(u_1)\| \le C_1\|u_2 - u_1\|. \tag{5.4}$$

For a fixed $\eta > 0$, we will select $v_1, \ldots, v_k \in S^{d-1}$ depending on η and C_1, and set

$$\gamma = \frac{1}{2} \min_i \Gamma(v_i, \eta).$$

We claim that the v_i can be chosen so that, for large $m,\ \xi_{\eta m}^{mA^c}$ differs from

$$\cup \left\{ \left((m\alpha_A(u) - \eta\, m\, w(\nu_A(u))\, \nu_A(u) + H_{\nu_A(u)} \right) : u \in S^{d-1} \right\}$$

(the right side of (2.4) with $n = \eta\, m$) by at most $C_2\, \eta^2 m,$ with probability at least $1 - e^{-\gamma\, \eta m}$. Here C_2 depends only on C_1 of (5.4).

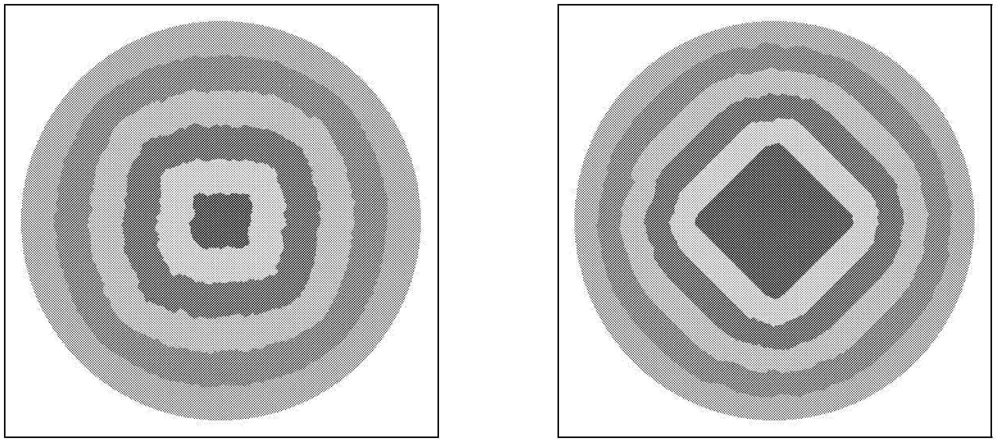
In order to verify the claim, first choose the (unique) direction u_i such that $\nu_A(u_i) = v_i$. Then there is a constant C_3 (depending only on C_1) such that within distance $\eta\, m$ of $m\, \alpha_A(u_i) u_i$ the boundary of mA lies within a cylinder of height $C_3\, \eta^2 m$. Moreover, there is a C_4 (depending only on $\mathcal{N}$) such that only sites within distance $\eta\, m$ influence how far sites within distance $C_4^{-1} \eta\, m$ advance. Hence we simply choose the v_i so that $\alpha_A(u_{i+1}) u_{i+1}$ is within the cylinder with height $C_3\, \eta^2$ and base consisting of points on the boundary of $\alpha_A(u_i) u_i + H_{v_i}^-$ within distance $C_4^{-1} \eta$.

To finish the proof, we now apply the claim repeatedly, $\frac{M-\epsilon}{\eta}$ times in all, using smooth approximations as in the proof of Theorem 2.2. □

Example 5.5. One class of Kesten random dynamics to which Theorem 5.2 applies comprises any additive growth with neighborhood $\mathcal{N}$ in which $\pi(A) = 0$ if $A = \emptyset,\ = 1$ if $0 \in A,$ and $= p$ otherwise. Such systems permit a first-passage interpretation, and then the martingale difference method of **[Kes2]** yields (5.1). More generally, Kesten's analysis and the techniques of the present paper apply to the growth of any first-passage percolation model in which the distribution function F of the time it takes to cross a site (bond) satisfies

$F(0) < p_c,$ the critical value for the site (bond) percolation, and
$\int e^{h\,x} dF(x) < \infty$ for some $h > 0$.

Example 5.6. Another instance of Theorem 5.2 is "barely supercritical random threshold growth" (**[Gra2]**, **[KS]**). In this case, $\mathcal{N}$ consists of **0** and its four closest neighbors, $\pi(A) = 1$ if $0 \in A$ or $|A| \geq 2,\ = p$ if A is a singleton other than **0**, $= 0$ otherwise. Estimate (5.1) is obtained from a last passage representation (cf. **[GK], [Gra2]**). As p decreases from 1 to 0, $K_{1/w}$ changes from $B_\infty(\mathbf{0}, 1)$ to the *cross* $\{\|x_1\| \leq 1$ *or* $\|x_2\| \leq 1\}$. Hence the reverse shape of any convex, completely symmetric A other than $B_1(\mathbf{0}, 1)$ (which is always invariant) makes a transition from a square or octagon at $p = 1$ to a diamond at $p = 0$. In particular, the reverse shape of a circle changes from square to diamond, perhaps through some intermediate regime of octagons. Level sets of one simulated sample path of the reverse dynamics, starting from a lattice circle of "radius" 250, at times 0, 50, 100, $\ldots$, 250 are shown below for parameter values $p = .7$ (left) and .3 (right).



Remark. For p close to 1 or 0, this last example illustrates weak stability of CA crystals with small random "error." Results about the forward shape $\mathcal{L}$ under perturbations appear in **[DL]** (the additive case) and **[BGG]** (nonconvex $K_{1/w}$). A representative general result for reverse shapes is the following. Given a CGT $\mathcal{T}$, let $\mathcal{T}_p$ denote the random dynamics which adjoin sites with probability p whenever they are added deterministically under $\mathcal{T}$, and let $\mathcal{R}_p(\,\cdot\,)$ be the reverse shapes for $\mathcal{T}_p$. If A is any completely symmetric, convex set which is weakly stable for $\mathcal{T}$, then as p tends to 1, $\mathcal{R}_p(A)$ can be made arbitrarily close to $\mathcal{R}(A)$.

We conclude by analyzing the last holes problem for additive random crystals with symmetric neighborhood $\mathcal{N}$ and update probability p, started from ν_δ. (Essentially the same analysis applies to the first-passage models of Example 5.5.) For large n, the most likely scenario on $\{T > n\}$ is that $\mathbf{0}$ refuses to be affected by its surroundings, leading to a trivial last hole.

Theorem 5.7. For additive, symmetric random growth, $\mathcal{H}_\delta = \{\mathbf{0}\}$.

Proof. Let A_n be the random crystal started from $\{\mathbf{0}\}$, in the standard additive graphical representation, so that (4.3) holds. By the method used to prove Theorem 4.1, the last hole must be trivial if the conditional distribution of A_n concentrates on balls of radius $o(n)$ around $\mathbf{0}$, i.e.,

(5.8) $$P_\delta\left(\text{diam}\,(A_n) \geq \eta\, n \;\middle|\; T > n\right) \to 0 \quad \text{as } n \to \infty$$

for any $\eta > 0$. Indeed, this implies that T is less than $(1+\eta)n$ with high probability, in which case $\mathcal{C}_{T-\epsilon n}$ has diameter at most $\eta\, n$.

The first step in checking (5.8) is to show, for a suitable $M < \infty$,

(5.9) $$P_\delta\left(|A_n| \geq Mn \;\middle|\; T > n\right) \to 0 \quad \text{as } n \to \infty$$

By definition, this last probability equals

$$\frac{E_\delta\left[(1-\delta)^{|A_n|}1_{\{|A_n|\geq Mn\}}\right]}{P_\delta(T>n)}.$$

The denominator here is at least $(1-\delta)(1-p)^{|\mathcal{N}\setminus\{\mathbf{0}\}|\,n}$, corresponding to the event that the origin is initially empty and $A_n = \{\mathbf{0}\}$. Of course the numerator is at most $(1-\delta)^{Mn}$, so (5.9) follows by choosing M large enough that $(1-\delta)^M \leq (1-p)^{|\mathcal{N}\setminus\{\mathbf{0}\}|}$.

The second step asserts that given any $\eta > 0$, there is a $c > 0$ such that

$$P_\delta\left(\text{diam}\,(A_n) \geq \eta\, n, |A_n| \leq Mn\right) \leq e^{-c\,n^2} \tag{5.10}$$

for n large. As noted in the previous paragraph, $\{T > n\}$ has at least exponentially small probability in n, so conditioning on that event does not appreciably alter (5.10). In combination with (5.9), this yields (5.8). Finally, to check (5.10), note that the total number of deterministic sequences $A_0 = \{\mathbf{0}\}$, $A_1, \ldots, A_n$ which lead to a possible configuration of size $\leq Mn$ is at most $(M'n)^n$. (A_n is $\mathcal{N}$-connected, and every site in A_n has at most n choices of when to first become occupied.) If diam $(A_n) \geq \eta\, n,$ then linearly many (in n) of $A_1, \ldots, A_n$ must have linear diameter, and hence linear boundary. Thus the probability of a prescribed trajectory entering into (5.10) is at most $(\max\{p, 1-p\})^{c'\,n^2}$. Summing over trajectories finishes the proof. We note that the same estimate precludes diam $(A_n) \geq n^{\frac{1}{2}+\eta}$ on $\{T > n\}$. □

In spite of the previous theorem, a "last nontrivial hole" with shape $\mathcal{L}$ arises by restricting the window of observation times n to a shorter horizon as the seeding density δ decreases..

Theorem 5.11. Under the same hypotheses, if $n = n_\delta \in [\,\delta^{-\frac{1}{d}-\alpha}, \delta^{-\frac{1}{d-1}+\alpha}]$ for some $\alpha > 0,$ then

$$\lim_{\epsilon\to 0}\limsup_{\delta\to 0} P_\delta\left(d_H\left(\frac{\mathcal{C}_{T-\epsilon n}}{\epsilon n}, \mathcal{L}\right) > \eta \mid T > n\right) = 0. \tag{5.12}$$

Proof. For a suitable constant $C = C_p$, with δ sufficiently small and n in the prescribed range,

$$P_\delta\big(T > n\big) \geq \exp\{-Cn^d\delta\} \gg e^{-C'n}$$

(any C'). Hence, the large deviation bounds established in **[Kes2]** for the forward limit shape $\mathcal{L}$ give

$$P_\delta\left((1-\eta)n\mathcal{L} \subset A_n \subset (1+\eta)n\mathcal{L} \mid T > n\right) \to 0 \quad \text{ as } \delta \to 0.$$

Also for n in the prescribed range, with overwhelming probability there are initially lots of occupied sites everywhere around the boundary of $n\mathcal{L}$, as in the argument for Theorem 4.1, and the rest of the proof is essentially the same as for that case of additive deterministic dynamics. We remark that (5.12) also holds with the conditioning event changed to $\{T = n\}$, perhaps a more natural formulation. □

Acknowledgment. We extend our profound gratitude to Harry Kesten for all he has taught us about percolation, random growth, and stochastic processes over the years, and for providing us with a preeminent role model as master of mathematical probability.

REFERENCES

[AL] M. Aizenman, J. Lebowitz, *Metastability effects in bootstrap percolation*, J. Phys. A: Math. Gen. **21** (1988), 3801-3813.

[AMS] E. Andjel, T. Mountford, R. Schonmann, *Equivalence of exponential decay rates for bootstrap percolation like cellular automata*. Ann. Inst. H. Poincaré **31** (1995), 13-25.

[Big] J. Biggins, *The asymptotic shape of the branching random walk,* Adv. Appl. Prob. **10** (1978), no. 1, 62-84.

[Boh] T. Bohman, *Discrete Threshold Growth dynamics are omnivorous for box neighborhoods*, Trans. Amer. Math. Soc. (1997), to appear.

[BoG] T. Bohman, J. Gravner, *Random threshold growth models*, (1997), submitted.

[BGG] T. Bohman, J. Gravner, D. Griffeath, *Asymptotic shapes for random threshold growth models*, (1998), in preparation.

[BrG1] M. Bramson, D. Griffeath, *On the Williams-Bjerknes Tumour Growth Model I* , Ann. Probability **9** (1981), 173-185.

[BrG2] M. Bramson, D. Griffeath, *On the Wiliams-Bjerknes Tumour Growth Model II* , Math. Proc. Cambridge Phi. Soc. **88** (1980), 339-357.

[BH] S. Broadbent, J. Hammersley, *Percolation processes. I. Crystals and mazes,* Proc. Cambridge Philos. Soc. **53** (1957), 629-641.

[CD] J. T. Cox, R. Durrett, *Some limit theorems for percolation processes with necessary and sufficient conditions,* Ann. Probability **9** (1981), 583--603.

[Dur] R. Durrett, *Lecture Notes on Particle Systems and Percolation*, Wadsworth & Brooks/Cole, 1988.

[DG] R. Durrett, D. Griffeath, *Contact processes in several dimensions*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 59 (1982), 535-552.

[DKS] R. Dobrushin, R. Kotecky, S. Shlosman, *Wulff Construction, A Global Shape from Local Interaction*, American Mathematical Society, 1992.

[DL] R. Durrett, T. Liggett, *The shape of the limit set in Richardson's growth model*, Ann. Probability **9** (1981), 186-193.

[Ede] M. Eden, p.359 in Symp. on Information Theory in Biology, ed. H. Yockey, Pergamon Press, New York, 1958.

[FGG1] R. Fisch, J. Gravner, D. Griffeath, *Threshold-range scaling of excitable cellular automata*, Statistics and Computing **1** (1991), 23-39.

[FGG2] R. Fisch, J. Gravner, D. Griffeath, *Metastability in the Greenberg-Hastings model*, Ann. Appl. Prob. **3** (1993), 935-967.

[GK] A. Gandolfi, H. Kesten, *Greedy lattice animals. II. Linear growth*, Ann. Appl. Prob. **4** (1994), 76-107.

[Gra1] J. Gravner, *The boundary of iterates in Euclidean growth models*, T.A.M.S. **348** (1996), 4549-4559.

[Gra2] J. Gravner, *Recurrent ring dynamics in two-dimensional cellular automata*, (1998), to appear.

[GG1] J. Gravner, D. Griffeath, *Threshold Growth dynamics*, T.A.M.S. **340** (1993), 837-870.

[GG2] J. Gravner, D. Griffeath, *First-passage times for discrete Threshold Growth dynamics*, Ann. Probability **24** (1996), 1752-1778.

[GG3] J. Gravner, D. Griffeath, *Multitype Threshold Growth: convergence to Poisson-Voronoi tessellations*, Ann. Appl. Prob. **7** (1997), 615-647.

[GG4] J. Gravner, D. Griffeath, *Cellular Automaton Grown on* $\mathbb{Z}^2$ *: theorems, examples and problems,* Adv. Appl. Math. **21** (1998), 241-304.

[Gri] D. Griffeath, *Primordial Soup Kitchen*, http://psoup.math.wisc.edu/kitchen.html, World Wide Web.

[HW] J. Hammersley, D. Welsh, *First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory,* 1965 Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif., 61-110. Springer-Verlag, New York.

[Kes1] H. Kesten, *First-passage percolation and a higher-dimensional generalization*, in *Particle Systems, Random Media and Large Deviations,* ed. R.Durrett, American Mathematical Society, 1984.

[Kes2] H. Kesten, *On the speed of convergence in first-passage percolation*, Ann. Appl. Prob. **4** (1994), 76-107.

[KeSc] H. Kesten, R. Schonmann, *On some growth models with a small parameter*, Probab. Th. Rel. Fields **101** (1995), 435-468.

[KrSp] J. Krug, H. Spohn, *Kinetic roughening of growing surfaces*, in Solids far from Equilibrium, ed. C. Godreche, Cambridge University Press, 1882, 479-582.

[Lig] T. Liggett, *Interacting Particle Systems*, Springer-Verlag, 1985.

[NP] C. Newman, M. Piza, *Divergence of shape fluctuations in two dimensions*, Ann Probability **23** (1995), 977-1005.

[Ric] D. Richardson, *Random growth in a tesselation*, Proc. Camb. Phil. Soc. **74** (1973), 515-528.

[Sep] T. Seppäläinen, *Hydrodynamic scaling, convex duality and asymptotic shapes of growth models,* Markov Process. Rel. Fields **4** (1998), 1-26.

[TCH] J. Taylor, J. Cahn, C. Handwerker, *Geometric models of crystal growth* (Overview no. 98-1), Acta Met. **40** (1992), 1443-1474.

[Wil] S. Willson, *On convergence of configurations*, Discrete Math. **23** (1978), no. 3, 279-300.