Math 280: Quantum Probability
Homework 3

This problem set is due Tuesday, November 20, by 8pm in my mailbox.

3.1. Recall from class and homework 1 that the state region $M^A_2$ is a round ball. Consider a linear map
\[ E : M^\#_2 \rightarrow M^\#_2 \]
that preserves the uniform state in the center of $M^A_2 \subseteq M^\#_2$ and rescales $M^A_2$ by a factor of $t \in \mathbb{R}$. The condition that $E$ is TPP says that $E(M^A_2) \subseteq M^A_2$, which thus means that $t \in [-1, 1]$. In order for $E$ to be a quantum map, it needs to be TPCP, not just TPP. Show that $E$ is TPCP when $t \in [-\frac{1}{3}, 1]$.

To do this problem, you can use without proof a simplification of the CP condition. The full condition is that a linear map $E : (M_2^\# \otimes \mathcal{C})^\# \rightarrow (\mathcal{B} \otimes \mathcal{C})^\#$ is positive for all $\mathcal{C}$. It suffices to let $\mathcal{C} = M_2$ and check that $E(\rho_{AB}) \in (M_2 \otimes \mathcal{C})^+$, where
\[ \rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}| \quad \psi_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \]
is a Bell state.

3.2. In this question we consider a quantum map $G = D \circ U \circ E$ which is a composition as follows:
\[ (d\mathbb{C})^A \xrightarrow{E} M^A_d \xrightarrow{U} M^A_d \xrightarrow{D} (d\mathbb{C})^A. \]
Let $\{[1], [2], \ldots, [d]\}$ be the deterministic states of $(d\mathbb{C})^A$ and let $|1\rangle, |2\rangle, \ldots, |d\rangle$ be an orthonormal basis of states of the Hilbert space $\mathcal{H} = \mathbb{C}^d$ on which $M_d$ acts. We let
\[ E(|k\rangle) = |k\rangle \langle k| \quad U(\rho) = u\rho u^* \quad D(\rho) = \sum_k \langle k|\rho|k\rangle |k\rangle \langle k|, \]
where $u$ is some unitary matrix, and $D$ is the indicated measurement in the standard basis. So the interpretation of the composition $F$ is that we start with a classical $d$-state digit, encode it in a qudit, apply a unitary, and then decode back to a classical digit by measuring the qudit.

(a) Confirm that $G$ is a doubly stochastic matrix given by $G_{jk} = |u_{jk}|^2$.

(b) Show that if $d = 2$, then every doubly stochastic matrix $G$ is induced by some unitary matrix $u$.

*(c) Show that if $d = 3$, then not every doubly stochastic matrix $G$ comes from a unitary matrix $u$ in this manner.
3.3. The (simplest) no-cloning theorem: Prove that if \( d \geq 2 \), then there does not exist a quantum map

\[
E : M_d^\Delta \rightarrow (M_d \otimes M_d)^\Delta
\]

such that

\[
E(|\psi\rangle\langle\psi|) = |\psi,\psi\rangle\langle\psi,\psi| \quad |\psi,\psi\rangle = |\psi\rangle \otimes |\psi\rangle
\]

for every pure state \( |\psi\rangle \in \mathbb{C}^d \). (Hint: \( E \) can’t even be linear. Think about \( d = 2 \) first.)

**3.4.** Consider a composition of quantum maps \( F = D \circ E \) of the form

\[
(3\mathbb{C})^\Delta \xrightarrow{E} M_2^\Delta \xrightarrow{D} (3\mathbb{C})^\Delta.
\]

So \( E \) encodes a classical trit into a qubit and \( D \) decodes it back again. TPP implies TPCP for both \( D \) and \( E \), so they are simply any affine-linear maps between the triangle \((3\mathbb{C})^\Delta\) and the round ball \( M_2^\Delta \). Show that if one of the three configurations \([k]\) of \((3\mathbb{C})^\Delta\) is chosen at random, then the average probability that \( F([k]) \) is in state \([k]\) is at most \( 2/3 \). (Interpretation: You cannot encode a trit into a qubit any more reliably than you can encode a trit into a bit. The pigeonhole principle generalizes to this case.)

3.5. Let \( \mathcal{H} = \mathbb{C}^d \) be the standard \( d \)-dimensional Hilbert space with basis \( |1\rangle, |2\rangle, \ldots, |d\rangle \), and suppose that it is houses two bosons who total Hilbert space is \( S^2(\mathcal{H}) \subseteq \mathcal{H} \otimes \mathcal{H} \). To be more explicit, the Hilbert space \( S^2(\mathcal{H}) \) has a basis

\[
|k\rangle \otimes |k\rangle \quad \text{and} \quad \frac{|j\rangle \otimes |k\rangle + |k\rangle \otimes |j\rangle}{\sqrt{2}}, j < k.
\]

(a) Suppose that the two bosons are given the uniform state \( \rho_{\text{unif}} \) in \( \mathcal{L}(S^2(\mathcal{H}))^\Delta \). Suppose state we measure the first boson in the standard basis and it is found to be in state \(|1\rangle\). What is the probability that the second boson, if also measured, is in any given state \(|k\rangle\)? (Warning: The answer depends on both \( k \) and \( d \). Hint: Recall that the uniform state on \( \mathbb{C}^d \) is given by \( \rho_{\text{unif}} = \sum_{j} |j\rangle \langle j| / d \).

*(b)* Generalize part (a) to \( n \) bosons whose total Hilbert space is \( S^n(\mathbb{C}^d) \). If the \( n \) bosons are in the uniform state and if the first \( \ell \) are measured to be states \(|k_1\rangle, |k_2\rangle, \ldots, |k_\ell\rangle\), then what is the probability distribution for the measurement of the \( \ell + 1 \)st boson?