3.1. Recall from class and homework 1 that the state region $M^A_2$ is a round ball. Consider a linear map $E : M^A_2 \to M^A_2$ that preserves the uniform state in the center of $M^A_2 \subseteq M^A_2$ and rescales $M^A_2$ by a factor of $t \in \mathbb{R}$. The condition that $E$ is TPP says that $E(M^A_2) \subseteq M^A_2$, which thus means that $t \in [-1, 1]$. In order for $E$ to be a quantum map, it needs to be TPCP, not just TPP. Show that $E$ is TPCP when $t \in [-\frac{1}{2}, 1]$. To do this problem, you can use without proof a simplification of the CP condition. The full condition is that a linear map $E : M^A_2 \to \mathcal{B}$ is completely positive when $E \otimes I : (M^A_2 \otimes \mathcal{C}) \to (\mathcal{B} \otimes \mathcal{C})$ is positive for all $\mathcal{C}$. It suffices to let $\mathcal{C} = M_2$ and check that $E(\rho_{AB}) \in (M_2 \otimes \mathcal{C})^+$, where $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$ $\psi_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is a Bell state.

3.2. In this question we consider a quantum map $F = D \circ U \circ E$ which is a composition as follows:

$$(d\mathbb{C})^A \xrightarrow{E} M^A_d \xrightarrow{U} M^A_d \xrightarrow{D} (d\mathbb{C})^A.$$ $$E([k]) = |k\rangle \langle k| \quad U(\rho) = u \rho u^* \quad D(\rho) = \sum_k \langle k|\rho|k\rangle [k],$$

where $u$ is some unitary matrix, and $D$ is the indicated measurement in the standard basis. So the interpretation of the composition $F$ is that we start with a classical $d$-state digit, encode it in a qudit, apply a unitary, and then decode back to a classical digit by measuring the qudit.

(a) Confirm that $F$ is a doubly stochastic matrix given by $F_{jk} = |u_{jk}|^2$.

(b) Show that if $d = 2$, then every doubly stochastic matrix $F$ is induced by some unitary matrix $u$.

*(c) Show that if $d = 3$, then not every doubly stochastic matrix $F$ comes from a unitary matrix $u$ in this manner.
3.3. The (simplest) no-cloning theorem: Prove that if $d \geq 2$, then there does not exist a quantum map

$$E : \mathcal{M}_d^\Delta \to (\mathcal{M}_d \otimes \mathcal{M}_d)^\Delta$$

such that

$$E(\langle \psi \vert \psi \rangle) = \vert \psi, \psi \rangle \langle \psi, \psi \vert$$

for every pure state $\vert \psi \rangle \in \mathbb{C}^d$. (Hint: $E$ can’t even be linear. Think about $d = 2$ first.)

*3.4. Consider a composition of quantum maps $F = D \circ E$ of the form

$$\begin{align*}
(3\mathbb{C})^\Delta & \xrightarrow{E} \mathcal{M}_2^\Delta \\
& \xrightarrow{D} (3\mathbb{C})^\Delta.
\end{align*}$$

So $E$ encodes a classical trit into a qubit and $D$ decodes it back again. TPP implies TPCP for both $D$ and $E$, so they are simply any affine-linear maps between the triangle $(3\mathbb{C})^\Delta$ and the round ball $\mathcal{M}_2^\Delta$. Show that if one of the three configurations $[k]$ of $(3\mathbb{C})^\Delta$ is chosen at random, then the average probability that $F([k])$ is in state $[k]$ is at most $2/3$. (Interpretation: You cannot encode a trit into a qubit any more reliably than you can encode a trit into a bit. The pigeonhole principle generalizes to this case.)

3.5. Let $\mathcal{H} = \mathbb{C}^d$ be the standard $d$-dimensional Hilbert space with basis $\vert 1 \rangle, \vert 2 \rangle, \ldots, \vert d \rangle$, and suppose that it is houses two bosons whose total Hilbert space is $S^2(\mathcal{H}) \subseteq \mathcal{H} \otimes \mathcal{H}$. To be more explicit, the Hilbert space $S^2(\mathcal{H})$ has a basis

$$\begin{align*}
\vert k \rangle \otimes \vert k \rangle & \quad \text{and} \quad \frac{\vert j \rangle \otimes \vert k \rangle + \vert k \rangle \otimes \vert j \rangle}{\sqrt{2}}, \ j < k.
\end{align*}$$

(a) Suppose that the two bosons are given the uniform state $\rho_{\text{unif}}$ in $\mathcal{L}(S^2(\mathcal{H}))^\Delta$. Suppose state we measure the first boson in the standard basis and it is found to be in state $\vert 1 \rangle$. What is the probability that the second boson, if also measured, is in any given state $\vert k \rangle$? (Warning: The answer depends on both $k$ and $d$. Hint: Recall that the uniform state on $\mathbb{C}^d$ is given by $\rho_{\text{unif}} = \sum_j |k\rangle \langle k| / d$.)

*(b) Generalize part (a) to $n$ bosons whose total Hilbert space is $S^n(\mathbb{C}^d)$. If the $n$ bosons are in the uniform state and if the first $\ell$ are measured to be states $\vert k_1 \rangle, \vert k_2 \rangle, \ldots, \vert k_\ell \rangle$, then what is the probability distribution for the measurement of the $\ell + 1$st boson?