Infinite-dimensional Flag Varieties and other Applications of Well-ordered Filtrations

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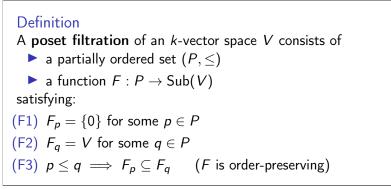
Conventions

- All rings are commutative and unital.
- If M is an R-module, we define Sub(M) to be the set of submodules of M.

• $(\operatorname{Sub}(M), \subseteq)$ is a poset.

All categories are locally small.

Filtrations



- (F, P) is **linear** if (P, \leq) is totally ordered.
- (F, P) is embedded if p ≤ q ⇔ F_p ⊆ F_q (F is an order embedding). [⇒ F is injective.]
- ▶ Linear (*F*, *I*) is maximal if *F*(*I*) is a maximal chain of Sub(*V*).

Examples of Filtrations

 $k^{\oplus\mathbb{N}} := \bigoplus_{n \in \mathbb{N}} k$ ("finitely supported sequence space")

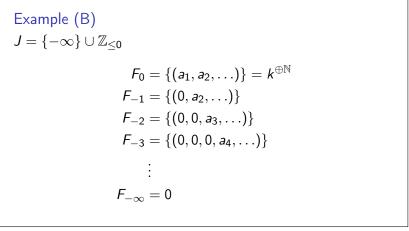
Example (A)

$$I = \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

 $E_{\infty} = \{(a_1, a_2, \ldots)\} = k^{\oplus \mathbb{N}}$
 \vdots
 $E_3 = \{(a_1, a_2, a_3, 0, \ldots)\}$
 $E_2 = \{(a_1, a_2, 0, \ldots)\}$
 $E_1 = \{(a_1, 0, \ldots)\}$
 $E_0 = 0$

• (E, I) is a maximal embedded linear filtration of $k^{\oplus \mathbb{N}}$.

Examples of Filtrations



• (F, J) is a maximal embedded linear filtration of $k^{\oplus \mathbb{N}}$.

Examples of Filtrations

$$k^{\mathbb{N}} := \prod_{n \in \mathbb{N}} k$$
 ("sequence space")

Example (C)

$$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$$

$$G_0 = \{(a_1, a_2, \ldots)\} = k^{\mathbb{N}}$$

$$G_{-1} = \{(0, a_2, \ldots)\}$$

$$G'_{-2} = \{(0, 0, a_3, \ldots)\}$$

$$G'_{-3} = \{(0, 0, 0, a_4, \ldots)\}$$

$$\vdots$$

$$G_{-\infty} = 0$$

• (G, J) is a maximal embedded linear filtration of $k^{\mathbb{N}}$.

Gradations

Definition A gradation of (F, P) is a function $S : P \to Sub(V)$ satisfying: $\blacktriangleright F_p = \bigoplus_{q \le p} S_q$ for all $p \in P$

Definition

An almost gradation of (F, P) is a function $S : P \to Sub(V)$ satisfying:

$$F_p = F_{< p} \oplus S_p \text{ for all } p \in P$$
(where $F_{< p} := \sum_{q < p} F_q$)

• Gradation \implies almost gradation.

Example (A)

$$I = \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

$$E_{\infty} = \{(r_1, r_2, \ldots)\} = k^{\oplus \mathbb{N}}$$

$$S_{\infty} = 0$$

$$\vdots$$

$$E_3 = \{(a_1, a_2, a_3, 0, \ldots)\}$$

$$S_3 = \text{span}\{e_3\}$$

$$E_2 = \{(a_1, a_2, 0, \ldots)\}$$

$$S_2 = \text{span}\{e_2\}$$

$$E_1 = \{(a_1, 0, \ldots)\}$$

$$S_1 = \text{span}\{e_1\}$$

$$E_0 = 0$$

$$S_0 = 0$$

 \blacktriangleright S is a gradation of E.

• We'll see that *every* almost gradation of E is a gradation.

Example (B)

$$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$$

$$F_{0} = \{(a_{1}, a_{2}, \ldots)\} = k^{\oplus \mathbb{N}}$$

$$F_{-1} = \{(0, a_{2}, a_{3}, \ldots)\}$$

$$F_{-2} = \{(0, 0, a_{3}, \ldots)\}$$

$$T_{-1} = \text{span}\{e_{1}\}$$

$$T_{-1} = \text{span}\{e_{2}\}$$

$$T_{-2} = \text{span}\{e_{3}\}$$

$$F_{-3} = \{(0, 0, 0, a_{4}, \ldots)\}$$

$$\vdots$$

$$F_{-\infty} = 0$$

$$T_{-\infty} = 0$$

► *T* is a gradation of *F*.

▶ Unlike before, not every almost gradation of *F* is a gradation.

Example (B') $J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$	
$F_{0} = \{(a_{1}, a_{2}, \ldots)\} = k^{\oplus \mathbb{N}}$ $F_{-1} = \{(0, a_{2}, a_{3}, \ldots)\}$ $F_{-2} = \{(0, 0, a_{3}, \ldots)\}$ $F_{-3} = \{(0, 0, 0, a_{4}, \ldots)\}$ \vdots $F_{-\infty} = 0$	$\begin{array}{rcl} T_0' &=& {\rm span}\{e_1-e_2\} \\ T_{-1}' &=& {\rm span}\{e_2-e_3\} \\ T_{-2}' &=& {\rm span}\{e_3-e_4\} \\ T_{-3}' &=& {\rm span}\{e_4-e_5\} \\ &\vdots \\ T_{-\infty}' &=& 0 \end{array}$

T' is an almost gradation of F.
T' is not a gradation of F because T'_i ⊆ ker(1, 1, ...) for all i ∈ I', so

$$\sum_{i\leq 0} T'_i \subseteq \ker(1,1,\ldots) \subsetneq k^{\oplus\mathbb{N}} = F_0.$$

Example (C)

$$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$$

$$G_0 = \{(a_1, a_2, \ldots)\} = k^{\mathbb{N}}$$

$$G_{-1} = \{(0, a_2, \ldots)\}$$

$$G'_{-2} = \{(0, 0, a_3, \ldots)\}$$

$$G'_{-3} = \{(0, 0, 0, a_4, \ldots)\}$$

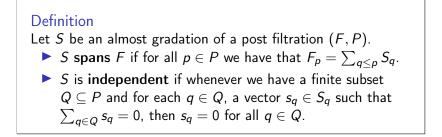
$$\vdots$$

$$G_{-\infty} = 0$$

S any almost gradation, S_i ≅ G_i/G_{i-1} ∀i ∈ Z_{≤0} which has dimension 1 ⇒ dim(∑_{i≤0} S_i) is countable, but dim(k^N) is uncountable.

$$\blacktriangleright \implies \text{There are } NO \text{ gradations of } (G, J)!$$

Spanning and Independent Almost Gradations



Guaranteeing Independence

Theorem

Every almost gradation of a linear filtration is independent.

Definition

The intersection of two poset filtrations (E, P) and (F, Q) of M is the poset filtration consisting of

- ▶ the poset $(P \times Q, \leq_{\times})$ (the product poset)
- ▶ the function $E \cap F : P \times Q \rightarrow Sub(M)$ defined by $[E \cap F]_{(p,q)} = E_p \cap F_q$.

Theorem

Every almost gradation of the *intersection of two linear filtrations* is independent.

It is NOT true that every almost gradation of the intersection of <u>three</u> linear filtrations is independent!

Guaranteeing Spanning

Definition A poset is **well-ordered** if every nonempty subset has a minimal element.

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Theorem
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Every almost gradation S of a well-ordered poset filtration (F, P) spans F.
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Proof.

Transfinite induction.

Guaranteeing Independence and Spanning

Corollary

- Every almost gradation of a linear well-ordered filtration (F, I) is a gradation.
- ► Every almost gradation of the intersection (E ∩ F, I × J) of two linear well-ordered filtrations (E, I) and (F, J) is a gradation.

Partial converse:

Theorem

Let (E, I) be a maximal embedded linear filtration of V. If every almost gradation of (E, I) is a gradation, then (I, \leq) is well-ordered. Theorem Let Q be an $A_{\infty,\infty}$ quiver. Every (not necessarily finite-dimensional) representation of Q is a direct sum of connected thin (locally 0 or 1 dimensional) sub-representations $\iff Q$ is eventually outward.

$$k \qquad \qquad k \xrightarrow{1} k$$

$$k \xrightarrow{1} k \xleftarrow{1} k \qquad \qquad k \xleftarrow{1} \dots \xrightarrow{1} k$$

Figure 2: Thin Representations

Applications: Double Complexes

Theorem

Every (not necessarily finite-dimensional or bounded) double complex supported in the first and third quadrants is a direct sum of "zig-zags" and "squares".

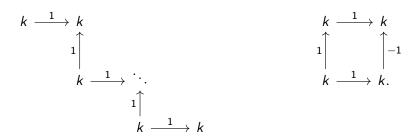


Figure 3: Zigzags and Squares

Applications: Infinite-dimensional Flag Spaces

Definition

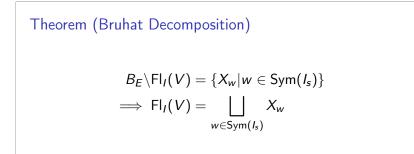
Let *I* be a well-ordered set. An *I*-flag in *k*-vector space *V* is a maximal embedded linear filtration (F, I) of *V*. Denote the set of *I*-flags of *V* by $Fl_I(V)$.

- This will be our analog of the full flag variety.
- GL(V) acts transitively on $Fl_I(V)$.
- Fix a reference flag E ∈ Fl₁(V), and denote the stabilizer of E by B_E ⊆ GL(V).
- ▶ B_E acts on Fl_I(V).
- The Bruhat Decomposition gives a nice description of the set of orbits B_E\Fl_I(V).

The Bruhat Decomposition

- ► I_s := {successor elements of I}
- Sym (I_s) := group of permutations of I_s (the Weyl group).
- For each w ∈ Sym(I_s), let X_w ⊆ FI_I(V) (the Schubert cell) denote the set of *I*-flags F whose intersection with E satisfies:

$$\dim(F \cap E)_{(i,j)}/(F \cap E)_{<(i,j)} = \begin{cases} 1 & \text{if } i = w(j) \\ 0 & \text{otherwise} \end{cases}$$



The Geometry of Finite-dimensional Flag Varieties

• Let V be finite-dimensional, $k = \overline{k}$.

To simplify our initial discussion, restrict to the case of Grassmannians: Gr_r(V) := {W ⊆ V|dim W = r}.

- $Gr_r(V)$ can be made into a variety in several different ways:
 - 1. Explicitly define charts.
 - 2. Use the Plücker embedding

$$\mathsf{Gr}_r(V) \hookrightarrow \mathbb{P}(\Lambda^r V)$$

 $\mathsf{span}(v_1, \ldots, v_r) \mapsto \mathsf{span}(v_1 \land \ldots \land v_r)$

and note that the image is Zariski-closed.

► $\overline{X_w} = \bigsqcup_{u \le b^w} X_u$, where $\le b$ denotes the **Bruhat order** on Sym $(I_s) = S_n$ where $n = \dim(V)$.

The Geometry of the Infinite-dimensional Flag Spaces

- Now let V be infinite-dimensional.
- Above approaches to defining a topology on Fl₁(V) no longer work.
 - 1. The domains of the charts are no longer affine (they are products of ind-affine ind-schemes).
 - 2. The Plücker embedding is no longer well-defined.
- ▶ Goal: Find a "natural" topology on Fl_I(V) and a "Bruhat order" on Sym(I_s) such that the closure theorem still holds.
- Understanding the geometry of the infinite flag space is a moduli problem - use Grothendieck's "functor of points" approach.

The Functor of Points: An Approach to Moduli Problems

- Moduli space the points are known, the geometric structure is not.
- Grothendieck [1]: to study a moduli space we study maps into it.
- Work in the category of schemes.
- A morphism of schemes f : X → Gr_r(kⁿ) can be interpreted as algebraically assigning to every point in X, an r-dimensional subspace of kⁿ.
- In other words if Gr_r(kⁿ) were a scheme, Mor_{Sch}(X, Gr_r(kⁿ)) would be in bijection with rank-r subbundles of the rank-n trivial bundle on X.

Representable Functors



Define a functor $\mathcal{G}r_{r,n}: \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ by sending

▶ a scheme X to the set of rank-r subbundles of the rank-n trivial bundle $\mathscr{O}_X^{\oplus n}$.

▶ a scheme morphism φ to its pullback φ^* .

Definition

For any scheme Y, we define a functor $h_Y : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ by sending

• a scheme X to the set $Mor_{Sch}(X, Y)$

• a morphism φ to pre-composition $-\circ \varphi$.

*h*_− : Sch → [Sch^{op}, Set] is a fully faithful (Yoneda Lemma)
 ⇒ if *Gr*_{r,n} ≅ *h*_Y then *Y* is unique.

The Grassmannian in Finite Dimensions

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Definition
A functor \mathcal{X} : \mathbf{Sch}^{\mathsf{op}} \to \mathbf{Set} is representable if there exists a scheme Y such that \mathcal{X} \cong h_Y.
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Theorem The functor \mathcal{G}r_{r,n} is representable.
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- Explicitly, there is a scheme Y such that the set Mor_{Sch}(X, Y) is naturally isomorphic to the set of rank r-subbundles of 𝒫^{⊕n}.
- In principle, this is what the Grassmannian should be.

The Grassmannian: A Purely Algebraic Approach

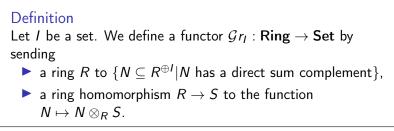
This is supposed to be an *algebra* talk

- Re-frame algebraic geometry \longrightarrow algebra!
- Schemes are locally affine.
- ▶ Spec : $Ring \rightarrow AffSch^{op}$ is an equivalence of categories.
- Serre [2]: A finite-rank vector bundle on an affine scheme Spec(R) is the same as a finitely generated projective R-module.
 - A subbundle corresponds to a module direct summand.

Note Re-define $\mathcal{G}r_{r,n}$: **Ring** \rightarrow **Set** by sending **a** ring R to the set $\{N \subseteq R^n | N \oplus M = R^n \text{ for some } M \leq R^n, rkN = r\},\$ **a** ring homomorphism $R \rightarrow S$ to the function $N \mapsto N \otimes_R S.$

The Grassmannian in Infinite Dimensions

Suggests generalization to infinite-dimensions:



ls $\mathcal{G}r_I$ representable?

▶ I.e. is there a scheme Y such that $Gr_I \cong h_Y(\text{Spec}(-))$?

Theorem \mathcal{G}_{r_l} is not representable.

I-flags in Modules

Let I be a well-ordered set.

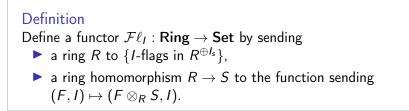
- Maximality of a filtration is not functorial.
 - Alter notion of "flag" for modules.

Definition

An *I*-flag in a free *R*-module *M* is an embedded linear filtration (F, I) of *M* such that

- F_i is a direct summand of M for each $i \in I$,
- ▶ $F_i/F_{<i}$ has constant rank 1 if $i \in I_s$ and is 0 otherwise.

The Full Flag Space



Theorem $\mathcal{F}\ell_I$ is not representable.

The Bruhat Decomposition

• E := the standard flag in $R^{\oplus I_s}$.

▶ For $w \in \text{Sym}(I_s)$, define the Schubert subfunctor $\mathcal{X}_w \subseteq \mathcal{F}\ell_I$ which sends

• a ring R to the set of $F \in \mathcal{F}\ell_I(R)$ such that

$$\mathsf{rk}(F \cap E)_{(i,j)} / (F \cap E)_{<(i,j)} = \begin{cases} 1 & \text{if } i = w(j) \\ 0 & \text{otherwise} \end{cases}$$

Theorem (The Bruhat Decomposition) $\mathcal{T}_{\ell_1} = 11$ \mathcal{V}

$$\mathcal{F}\ell_I = \bigsqcup_{w \in \operatorname{Sym}(I_s)} \mathcal{X}_w.$$

It makes sense to take the closure of a subfunctor!

• $\overline{\mathcal{X}}$ is the intersection of all closed subfunctors containing \mathcal{X} .

 Current goal: Nicely describe the closure of a Schubert subfunctor by a partial order on Sym(*I_s*).

Proof of Non-representability: Subfunctors

Technical definitions for a representability criterion:

Definition A functor $\mathcal{Y} : \mathbf{C} \to \mathbf{Set}$ is a subfunctor of $\mathcal{X} : \mathbf{C} \to \mathbf{Set}$ if: For all objects $A \in \mathbf{C}$, $\mathcal{Y}(A) \subseteq \mathcal{X}(A)$. For all morphisms $f : A \to B$, $\mathcal{Y}(f) = \mathcal{X}(f)|_{\mathcal{V}(f)}$.

Definition

If $\eta : \mathbb{Z} \to \mathcal{X}$ is a natural transformation and $\mathcal{Y} \subseteq \mathcal{X}$ is a subfunctor, the **inverse image** is the subfunctor of \mathcal{Z} defined by $\eta^{-1}(\mathcal{Y})(\mathcal{A}) = \eta^{-1}_{\mathcal{A}}(\mathcal{Y}(\mathcal{A})).$

Proof of Non-representability: Open/Closed Subfunctors

Definition

A subfunctor $\mathcal{Y} \subseteq \mathcal{X}$: **Ring** \rightarrow **Set** is **open** (resp. **closed**) if for all rings R and all morphisms $\varphi : h^R \rightarrow \mathcal{X}$, the inverse image $\varphi^{-1}(\mathcal{Y}) \subseteq h^R$ is of the form

$$S\mapsto \{f:R o S|S\cdot f(I)=S\}$$

(resp. $S\mapsto \{f:R o S|f(I)=0\}$)

for some ideal $I \subseteq R$.

Definition

A collection $\{\mathcal{Y}_j : j \in J\}$ of subfunctors of \mathcal{X} covers \mathcal{X} if for all fields k, $\mathcal{X}(k) = \bigcup_{j \in J} \mathcal{Y}_j(k)$.

Proof of Non-representability: Zariski Sheaves

Definition

 \mathcal{X} : Ring \rightarrow Set is a Zariski sheaf if for any ring R and any elements $r_1, \ldots, r_n \in R$ with $\langle r_1, \ldots, r_n \rangle = R$, the following diagram of sets is an equalizer.

$$\mathcal{X}(R) \to \prod_{i=1}^n \mathcal{X}(R_{r_i}) \Longrightarrow \prod_{i,j=1}^n \mathcal{X}(R_{r_ir_j}).$$

Theorem

A functor $\mathcal{X} : \mathbf{Ring} \to \mathbf{Set}$ is representable \iff

1. \mathcal{X} is a Zariski sheaf.

2. There is an open cover of \mathcal{X} by representable subfunctors.

Proof of Non-representability: Zariski Descent of Gr_I

Theorem $\mathcal{G}r_l$ and $\mathcal{F}\ell_l$ are Zariski sheaves (in fact fpqc sheaves).

Proof for $\mathcal{G}r_l$.

- Mod_R is equivalent to QCo(Spec(R)).
- Projective modules ↔ locally projective quasi-coherent sheaves on Spec(R).
 - (*Injectivity of the First Map*). Follows from the locality property of sheaves.
 - (*Exactness in the Middle*). Pairwise compatible locally projective quasi-coherent sheaves on Spec(*R_{ri}*) glue to a quasi-coherent sheaf on Spec(*R*). The result is locally projective by a *difficult* theorem of Raynaud and Gruson [3], and Perry [4].

Proof of Non-representability of $\mathcal{G}r_I$

Theorem $\mathcal{G}r_l$ and $\mathcal{F}\ell_l$ are not representable.

Lemma

Any open or closed subfunctor of a representable functor is itself representable.

Proof of Theorem for Gr_I .

- ▶ $\mathcal{X}(R) := \{N \in \mathcal{G}r_l(R) \mid \mathsf{rk}N \leq 1\}.$ (<u>Closed</u> subfunctor)
- $\pi_i :=$ projection onto the *i*th coordinate in $R^{\oplus I}$.
- ▶ $\mathcal{Y}(R) = \{N \in \mathcal{G}r_I(R) \mid \pi_i|_N \text{ is surjective}\}.$ (Open subfunctor)
- If $\mathcal{G}r_I$ is representable, so is $\mathcal{X} \cap \mathcal{Y}$.
- $\mathcal{X} \cap \mathcal{Y}$ isomorphic to $R \mapsto R^{\oplus (I \smallsetminus i)}$, which is not representable.

Further Descent Properties

Definition

 \mathcal{X} is an **fpqc sheaf** if (1) it is a Zariski sheaf and (2) for every faithfully flat ring homomorphism $R \to S$, the following diagram of sets is an equalizer.

$$\mathcal{X}(R) \to \mathcal{X}(S) \rightrightarrows \mathcal{X}(S \otimes_R S).$$

• Note: Representable
$$\implies$$
 fqpc sheaf.

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Theorem \mathcal{G}r_I and \mathcal{F}\ell_I are fpqc sheaves.
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Proof.

Results of Raynaud, Gruson, and Perry cover the fpqc case too! [3] [4]

References

- Alexander Grothendieck, Introduction au langage fonctoriel, course in Algiers in November 1965, lecture notes by Max Karoubi.
- J.-P. Serre. Modules projectifs et espaces fibrés à fibre vectorielle. Séminaire P. Dubreil, M.-L. Dubreil- Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23. Secrétariat mathématique, Paris, 1958.
- M. Raynaud and L. Gruson. Critères de platitude et de projectivité. Inventiones Math., vol. 13 (1971), 1–89.
- A. Perry, Faithfully flat descent for projectivity of modules, arXiv:1011.0038v1.