Infinite-dimensional Flag Varieties and other Applications of Well-ordered Filtrations

Nathaniel Gallup

UC Davis

October 22nd, 2020

Conventions

- All rings are commutative and unital.
- If M is an R-module, we define $\mathrm{Sub}(M)$ to be the set of submodules of M.
  - $(\mathrm{Sub}(M), \subseteq)$ is a poset.
- All categories are locally small.

Filtrations

Definition

A **poset filtration** of an k-vector space V consists of

- a partially ordered set $(P, \leq)$
- a function $F : P \to \mathrm{Sub}(V)$

satisfying:

(F1) $F_p = \{0\}$ for some $p \in P$

(F2) $F_q = V$ for some $q \in P$

(F3) $p \leq q \implies F_p \subseteq F_q$ (F is order-preserving)

- (F, P) is **linear** if $(P, \leq)$ is totally ordered.
- (F, P) is **embedded** if $p \leq q \iff F_p \subseteq F_q$ (F is an order embedding). [$\implies F$ is injective.]
- Linear (F, I) is **maximal** if $F(I)$ is a maximal chain of $\mathrm{Sub}(V)$.

Examples of Filtrations

$k^{\oplus \mathbb{N}} := \bigoplus_{n \in \mathbb{N}} k$ ("finitely supported sequence space")

Example (A)

$I = \mathbb{Z}_{\geq 0} \cup \{\infty\}$

$$
\begin{aligned}
E_\infty &= \{(a_1, a_2, \ldots)\} = k^{\oplus \mathbb{N}} \\
&\vdots \\
E_3 &= \{(a_1, a_2, a_3, 0, \ldots)\} \\
E_2 &= \{(a_1, a_2, 0, \ldots)\} \\
E_1 &= \{(a_1, 0, \ldots)\} \\
E_0 &= 0
\end{aligned}
$$

- (E, I) is a maximal embedded linear filtration of $k^{\oplus \mathbb{N}}$.

Examples of Filtrations

Example (B)

$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$

$$\begin{aligned} F_0 &= \{(a_1, a_2, \ldots)\} = k^{\oplus \mathbb{N}} \\ F_{-1} &= \{(0, a_2, \ldots)\} \\ F_{-2} &= \{(0, 0, a_3, \ldots)\} \\ F_{-3} &= \{(0, 0, 0, a_4, \ldots)\} \\ &\vdots \\ F_{-\infty} &= 0 \end{aligned}$$

- (F, J) is a maximal embedded linear filtration of $k^{\oplus \mathbb{N}}$.

Examples of Filtrations

$k^{\mathbb{N}} := \prod_{n \in \mathbb{N}} k$ ("sequence space")

Example (C)

$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$

$$
\begin{aligned}
G_0 &= \{(a_1, a_2, \ldots)\} = k^{\mathbb{N}} \\
G_{-1} &= \{(0, a_2, \ldots)\} \\
G'_{-2} &= \{(0, 0, a_3, \ldots)\} \\
G'_{-3} &= \{(0, 0, 0, a_4, \ldots)\} \\
&\vdots \\
G_{-\infty} &= 0
\end{aligned}
$$

- (G, J) is a maximal embedded linear filtration of $k^{\mathbb{N}}$.

Gradations

Definition

A **gradation** of (F, P) is a function $S : P \to \mathrm{Sub}(V)$ satisfying:

- $F_p = \bigoplus_{q \leq p} S_q$ for all $p \in P$

Definition

An **almost gradation** of (F, P) is a function $S : P \to \mathrm{Sub}(V)$ satisfying:

- $F_p = F_{<p} \oplus S_p$ for all $p \in P$

(where $F_{<p} := \sum_{q<p} F_q$)

- Gradation $\implies$ almost gradation.

Examples of Gradations

Example (A)

$I = \mathbb{Z}_{\geq 0} \cup \{\infty\}$

$$\begin{array}{rcl} E_\infty & = & \{(r_1, r_2, \ldots)\} = k^{\oplus \mathbb{N}} \\ & \vdots & \\ E_3 & = & \{(a_1, a_2, a_3, 0, \ldots)\} \\ E_2 & = & \{(a_1, a_2, 0, \ldots)\} \\ E_1 & = & \{(a_1, 0, \ldots)\} \\ E_0 & = & 0 \end{array} \qquad \begin{array}{rcl} S_\infty & = & 0 \\ & \vdots & \\ S_3 & = & \mathrm{span}\{e_3\} \\ S_2 & = & \mathrm{span}\{e_2\} \\ S_1 & = & \mathrm{span}\{e_1\} \\ S_0 & = & 0 \end{array}$$

- S is a gradation of E.
- We'll see that *every* almost gradation of E is a gradation.

Examples of Gradations

Example (B)

$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$

$$
\begin{array}{ccc}
F_0 & = & \{(a_1, a_2, \ldots)\} = k^{\oplus \mathbb{N}} \\
F_{-1} & = & \{(0, a_2, a_3, \ldots)\} \\
F_{-2} & = & \{(0, 0, a_3, \ldots)\} \\
F_{-3} & = & \{(0, 0, 0, a_4, \ldots)\} \\
 & \vdots & \\
F_{-\infty} & = & 0
\end{array}
\qquad
\begin{array}{ccc}
T_0 & = & \mathrm{span}\{e_1\} \\
T_{-1} & = & \mathrm{span}\{e_2\} \\
T_{-2} & = & \mathrm{span}\{e_3\} \\
T_{-3} & = & \mathrm{span}\{e_4\} \\
 & \vdots & \\
T_{-\infty} & = & 0
\end{array}
$$

- T is a gradation of F.
- Unlike before, not every almost gradation of F is a gradation.

Examples of Gradations

Example (B')

$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$

$$
\begin{array}{rclcrcl}
F_0 & = & \{(a_1, a_2, \dots)\} = k^{\oplus\mathbb{N}} & & T'_0 & = & \text{span}\{e_1 - e_2\} \\
F_{-1} & = & \{(0, a_2, a_3, \dots)\} & & T'_{-1} & = & \text{span}\{e_2 - e_3\} \\
F_{-2} & = & \{(0, 0, a_3, \dots)\} & & T'_{-2} & = & \text{span}\{e_3 - e_4\} \\
F_{-3} & = & \{(0, 0, 0, a_4, \dots)\} & & T'_{-3} & = & \text{span}\{e_4 - e_5\} \\
 & \vdots & & & & \vdots & \\
F_{-\infty} & = & 0 & & T'_{-\infty} & = & 0
\end{array}
$$

- T' is an almost gradation of F.
- T' is *not* a gradation of F because $T'_i \subseteq \ker(1, 1, \dots)$ for all $i \in I'$, so

$$\sum_{i \leq 0} T'_i \subseteq \ker(1, 1, \dots) \subsetneq k^{\oplus\mathbb{N}} = F_0.$$

Examples of Gradations

Example (C)

$J = \{-\infty\} \cup \mathbb{Z}_{\leq 0}$

$$\begin{aligned}
G_0 &= \{(a_1, a_2, \ldots)\} = k^{\mathbb{N}} \\
G_{-1} &= \{(0, a_2, \ldots)\} \\
G'_{-2} &= \{(0, 0, a_3, \ldots)\} \\
G'_{-3} &= \{(0, 0, 0, a_4, \ldots)\} \\
&\vdots \\
G_{-\infty} &= 0
\end{aligned}$$

- S any almost gradation, $S_i \cong G_i/G_{i-1}$ $\forall i \in \mathbb{Z}_{\leq 0}$ which has dimension 1 $\implies$ $\dim(\sum_{i \leq 0} S_i)$ is countable, but $\dim(k^{\mathbb{N}})$ is uncountable.
- $\implies$ There are *NO* gradations of (G, J)!

Spanning and Independent Almost Gradations

Definition

Let S be an almost gradation of a post filtration (F, P).

- S **spans** F if for all $p \in P$ we have that $F_p = \sum_{q \leq p} S_q$.
- S is **independent** if whenever we have a finite subset $Q \subseteq P$ and for each $q \in Q$, a vector $s_q \in S_q$ such that $\sum_{q \in Q} s_q = 0$, then $s_q = 0$ for all $q \in Q$.

- Almost gradation S of (F, P) is a gradation $\iff$ S is independent and spans F.

Guaranteeing Independence

Theorem

Every almost gradation of a *linear* filtration is independent.

Definition

The **intersection** of two poset filtrations (E, P) and (F, Q) of M is the poset filtration consisting of

- the poset $(P \times Q, \leq_\times)$ (the product poset)
- the function $E \cap F : P \times Q \to \mathrm{Sub}(M)$ defined by $[E \cap F]_{(p,q)} = E_p \cap F_q$.

Theorem

Every almost gradation of the *intersection of two linear filtrations* is independent.

- It is *NOT* true that every almost gradation of the intersection of <u>three</u> linear filtrations is independent!

Guaranteeing Spanning

Definition

A poset is **well-ordered** if every nonempty subset has a minimal element.

Theorem

Every almost gradation S of a *well-ordered* poset filtration (F, P) spans F.

Proof.

Transfinite induction. ∎

Guaranteeing Independence and Spanning

Corollary

- Every almost gradation of a linear well-ordered filtration (F, I) is a gradation.
- Every almost gradation of the intersection $(E \cap F, I \times J)$ of two linear well-ordered filtrations (E, I) and (F, J) is a gradation.

Partial converse:

Theorem

Let (E, I) be a maximal embedded linear filtration of V. If every almost gradation of (E, I) is a gradation, then $(I, \leq)$ is well-ordered.

Applications: $A_{\infty,\infty}$ Quivers

$$\cdots \;—\; \bullet \;—\; \bullet \;—\; \bullet \;—\; \bullet \;—\; \bullet \;—\; \cdots$$

Figure 1: $A_{\infty,\infty}$ Graph

Theorem

Let Q be an $A_{\infty,\infty}$ quiver. Every (not necessarily finite-dimensional) representation of Q is a direct sum of connected thin (locally 0 or 1 dimensional) sub-representations $\iff$ Q is eventually outward.

$$k \qquad\qquad k \xrightarrow{1} k$$

$$k \xrightarrow{1} k \xleftarrow{1} k \qquad\qquad k \xleftarrow{1} \cdots \xrightarrow{1} k$$

Figure 2: Thin Representations

Applications: Double Complexes

Theorem

Every (not necessarily finite-dimensional or bounded) double complex supported in the first and third quadrants is a direct sum of "zig-zags" and "squares".

$$\begin{array}{ccccc} k & \xrightarrow{1} & k & & \\ & & \uparrow{\scriptstyle 1} & & \\ & & k & \xrightarrow{1} & \ddots \\ & & & & \uparrow{\scriptstyle 1} \\ & & & & k \xrightarrow{1} k \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{1} & k \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle -1} \\ k & \xrightarrow{1} & k. \end{array}$$

Figure 3: Zigzags and Squares

Applications: Infinite-dimensional Flag Spaces

Definition

Let I be a well-ordered set. An I-**flag** in k-vector space V is a maximal embedded linear filtration (F, I) of V. Denote the set of I-flags of V by $\mathrm{Fl}_I(V)$.

- This will be our analog of the full flag variety.
- $\mathrm{GL}(V)$ acts transitively on $\mathrm{Fl}_I(V)$.
- Fix a reference flag $E \in \mathrm{Fl}_I(V)$, and denote the stabilizer of E by $\mathrm{B}_E \subseteq \mathrm{GL}(V)$.
- B_E acts on $\mathrm{Fl}_I(V)$.
- The Bruhat Decomposition gives a nice description of the set of orbits $\mathrm{B}_E \backslash \mathrm{Fl}_I(V)$.

The Bruhat Decomposition

- $I_s := \{\text{successor elements of } I\}$
- $\mathrm{Sym}(I_s) :=$ group of permutations of I_s (the **Weyl group**).
- For each $w \in \mathrm{Sym}(I_s)$, let $X_w \subseteq \mathrm{Fl}_I(V)$ (the **Schubert cell**) denote the set of I-flags F whose intersection with E satisfies:

$$\dim (F \cap E)_{(i,j)}/(F \cap E)_{<(i,j)} = \begin{cases} 1 & \text{if } i = w(j) \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Bruhat Decomposition)

$$B_E \backslash \mathrm{Fl}_I(V) = \{X_w | w \in \mathrm{Sym}(I_s)\}$$
$$\implies \mathrm{Fl}_I(V) = \bigsqcup_{w \in \mathrm{Sym}(I_s)} X_w$$

The Geometry of Finite-dimensional Flag Varieties

- Let V be finite-dimensional, $k = \overline{k}$.
- To simplify our initial discussion, restrict to the case of Grassmannians: $\mathrm{Gr}_r(V) := \{W \subseteq V \mid \dim W = r\}$.
- $\mathrm{Gr}_r(V)$ can be made into a variety in several different ways:
  1. Explicitly define charts.
  2. Use the Plücker embedding

$$\begin{aligned} \mathrm{Gr}_r(V) &\hookrightarrow \mathbb{P}(\Lambda^r V) \\ \mathrm{span}(v_1, \ldots, v_r) &\mapsto \mathrm{span}(v_1 \wedge \ldots \wedge v_r) \end{aligned}$$

  and note that the image is Zariski-closed.
- $\overline{X_w} = \bigsqcup_{u \leq_b w} X_u$, where $\leq_b$ denotes the **Bruhat order** on $\mathrm{Sym}(I_s) = S_n$ where $n = \dim(V)$.

The Geometry of the Infinite-dimensional Flag Spaces

- Now let V be infinite-dimensional.
- Above approaches to defining a topology on $\mathrm{Fl}_I(V)$ no longer work.
  1. The domains of the charts are no longer affine - (they are products of ind-affine ind-schemes).
  2. The Plücker embedding is no longer well-defined.
- Goal: Find a "natural" topology on $\mathrm{Fl}_I(V)$ and a "Bruhat order" on $\mathrm{Sym}(I_s)$ such that the closure theorem still holds.
- Understanding the geometry of the infinite flag space is a moduli problem - use Grothendieck's "functor of points" approach.

The Functor of Points: An Approach to Moduli Problems

- Moduli space - the points are known, the geometric structure is not.
- Grothendieck [1]: to study a moduli space we study maps into it.
- Work in the category of schemes.
- A morphism of schemes $f : X \to \mathrm{Gr}_r(k^n)$ can be interpreted as algebraically assigning to every point in X, an r-dimensional subspace of k^n.
- In other words if $\mathrm{Gr}_r(k^n)$ were a scheme, $\mathrm{Mor}_{\mathbf{Sch}}(X, \mathrm{Gr}_r(k^n))$ would be in bijection with rank-r subbundles of the rank-n trivial bundle on X.

Representable Functors

Definition

Define a functor $\mathcal{G}r_{r,n} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ by sending

- a scheme X to the set of rank-r subbundles of the rank-n trivial bundle $\mathscr{O}_X^{\oplus n}$.
- a scheme morphism φ to its pullback φ^*.

Definition

For any scheme Y, we define a functor $h_Y : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ by sending

- a scheme X to the set $\mathrm{Mor}_{\mathbf{Sch}}(X, Y)$
- a morphism φ to pre-composition $- \circ \varphi$.

- $h_- : \mathbf{Sch} \to [\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}]$ is a fully faithful (**Yoneda Lemma**)
  - $\implies$ if $\mathcal{G}r_{r,n} \cong h_Y$ then Y is unique.

The Grassmannian in Finite Dimensions

Definition

A functor $\mathcal{X} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** if there exists a scheme Y such that $\mathcal{X} \cong h_Y$.

Theorem

The functor $\mathcal{G}r_{r,n}$ is representable.

- Explicitly, there is a scheme Y such that the set $\mathrm{Mor}_{\mathbf{Sch}}(X, Y)$ is naturally isomorphic to the set of rank r-subbundles of $\mathscr{O}_X^{\oplus n}$.
- In principle, this is what the Grassmannian *should* be.

The Grassmannian: A Purely Algebraic Approach

This is supposed to be an *algebra* talk ...

- Re-frame algebraic geometry $\longrightarrow$ algebra!
- Schemes are locally affine.
- $\mathrm{Spec} : \mathbf{Ring} \to \mathbf{AffSch}^{\mathrm{op}}$ is an equivalence of categories.
- Serre [2]: A finite-rank vector bundle on an affine scheme $\mathrm{Spec}(R)$ is the same as a finitely generated projective R-module.
  - A subbundle corresponds to a module direct summand.

Note

Re-define $\mathcal{G}r_{r,n} : \mathbf{Ring} \to \mathbf{Set}$ by sending

- a ring R to the set $\{N \subseteq R^n | N \oplus M = R^n \text{ for some } M \leq R^n, \mathrm{rk} N = r\}$,
- a ring homomorphism $R \to S$ to the function $N \mapsto N \otimes_R S$.

The Grassmannian in Infinite Dimensions

- Suggests generalization to infinite-dimensions:

Definition

Let I be a set. We define a functor $\mathcal{G}r_I : \mathbf{Ring} \to \mathbf{Set}$ by sending

- a ring R to $\{N \subseteq R^{\oplus I} | N$ has a direct sum complement$\}$,
- a ring homomorphism $R \to S$ to the function $N \mapsto N \otimes_R S$.

- Is $\mathcal{G}r_I$ representable?
  - I.e. is there a scheme Y such that $\mathcal{G}r_I \cong h_Y(\mathrm{Spec}(-))$?

Theorem

$\mathcal{G}r_I$ is not representable.

I-flags in Modules

Let I be a *well-ordered set*.

- Maximality of a filtration is not functorial.
  - Alter notion of "flag" for modules.

Definition

An I-**flag** in a free R-module M is an embedded linear filtration (F, I) of M such that

- F_i is a direct summand of M for each $i \in I$,
- $F_i / F_{<i}$ has constant rank 1 if $i \in I_s$ and is 0 otherwise.

The Full Flag Space

Definition

Define a functor $\mathcal{F}\ell_I : \textbf{Ring} \to \textbf{Set}$ by sending

- a ring R to $\{I\text{-flags in } R^{\oplus I_s}\}$,
- a ring homomorphism $R \to S$ to the function sending $(F, I) \mapsto (F \otimes_R S, I)$.

Theorem

$\mathcal{F}\ell_I$ is not representable.

The Bruhat Decomposition

- $E :=$ the standard flag in $R^{\oplus I_s}$.
- For $w \in \mathrm{Sym}(I_s)$, define the **Schubert subfunctor** $\mathcal{X}_w \subseteq \mathcal{F}\ell_I$ which sends
  - a ring R to the set of $F \in \mathcal{F}\ell_I(R)$ such that

$$\mathrm{rk}(F \cap E)_{(i,j)}/(F \cap E)_{<(i,j)} = \begin{cases} 1 & \text{if } i = w(j) \\ 0 & \text{otherwise} \end{cases}$$

Theorem (The Bruhat Decomposition)

$\mathcal{F}\ell_I = \bigsqcup_{w \in \mathrm{Sym}(I_s)} \mathcal{X}_w$.

- It makes sense to take the closure of a subfunctor!
  - $\overline{\mathcal{X}}$ is the intersection of all closed subfunctors containing $\mathcal{X}$.
- Current goal: Nicely describe the closure of a Schubert subfunctor by a partial order on $\mathrm{Sym}(I_s)$.

Proof of Non-representability: Subfunctors

- Technical definitions for a representability criterion:

Definition

A functor $\mathcal{Y} : \mathbf{C} \to \mathbf{Set}$ is a **subfunctor** of $\mathcal{X} : \mathbf{C} \to \mathbf{Set}$ if:

- For all objects $A \in \mathbf{C}$, $\mathcal{Y}(A) \subseteq \mathcal{X}(A)$.
- For all morphisms $f : A \to B$, $\mathcal{Y}(f) = \mathcal{X}(f)\big|_{\mathcal{Y}(f)}$.

Definition

If $\eta : \mathcal{Z} \to \mathcal{X}$ is a natural transformation and $\mathcal{Y} \subseteq \mathcal{X}$ is a subfunctor, the **inverse image** is the subfunctor of $\mathcal{Z}$ defined by $\eta^{-1}(\mathcal{Y})(A) = \eta_A^{-1}(\mathcal{Y}(A))$.

Proof of Non-representability: Open/Closed Subfunctors

Definition

A subfunctor $\mathcal{Y} \subseteq \mathcal{X} : \mathbf{Ring} \to \mathbf{Set}$ is **open** (resp. **closed**) if for all rings R and all morphisms $\varphi : h^R \to \mathcal{X}$, the inverse image $\varphi^{-1}(\mathcal{Y}) \subseteq h^R$ is of the form

$$S \mapsto \{f : R \to S | S \cdot f(I) = S\}$$
$$(\text{resp. } S \mapsto \{f : R \to S | f(I) = 0\})$$

for some ideal $I \subseteq R$.

Definition

A collection $\{\mathcal{Y}_j : j \in J\}$ of subfunctors of $\mathcal{X}$ **covers** $\mathcal{X}$ if for all *fields k*, $\mathcal{X}(k) = \bigcup_{j \in J} \mathcal{Y}_j(k)$.

Proof of Non-representability: Zariski Sheaves

Definition

$\mathcal{X}$: **Ring** $\to$ **Set** is a **Zariski sheaf** if for any ring R and any elements $r_1, \ldots, r_n \in R$ with $\langle r_1, \ldots, r_n \rangle = R$, the following diagram of sets is an equalizer.

$$\mathcal{X}(R) \to \prod_{i=1}^{n} \mathcal{X}(R_{r_i}) \rightrightarrows \prod_{i,j=1}^{n} \mathcal{X}(R_{r_i r_j}).$$

Theorem

A functor $\mathcal{X}$: **Ring** $\to$ **Set** is representable $\iff$

1. $\mathcal{X}$ is a Zariski sheaf.
2. There is an open cover of $\mathcal{X}$ by representable subfunctors.

Proof of Non-representability: Zariski Descent of $\mathcal{G}r_I$

Theorem
$\mathcal{G}r_I$ and $\mathcal{F}\ell_I$ are Zariski sheaves (in fact fpqc sheaves).

Proof for $\mathcal{G}r_I$.

- $\mathbf{Mod}_R$ is equivalent to $\mathbf{QCo}(\mathrm{Spec}(R))$.
- Projective modules $\leftrightarrow$ locally projective quasi-coherent sheaves on $\mathrm{Spec}(R)$.
  - (*Injectivity of the First Map*). Follows from the locality property of sheaves.
  - (*Exactness in the Middle*). Pairwise compatible locally projective quasi-coherent sheaves on $\mathrm{Spec}(R_{r_i})$ glue to a quasi-coherent sheaf on $\mathrm{Spec}(R)$. The result is locally projective by a *difficult* theorem of Raynaud and Gruson [3], and Perry [4].

∎

Proof of Non-representability of $\mathcal{G}r_I$

Theorem

$\mathcal{G}r_I$ and $\mathcal{F}\ell_I$ are not representable.

Lemma

Any open or closed subfunctor of a representable functor is itself representable.

Proof of Theorem for $\mathcal{G}r_I$.

- $\mathcal{X}(R) := \{N \in \mathcal{G}r_I(R) \mid \mathrm{rk} N \leq 1\}$. (Closed subfunctor)
- $\pi_i :=$ projection onto the ith coordinate in $R^{\oplus I}$.
- $\mathcal{Y}(R) = \{N \in \mathcal{G}r_I(R) \mid \pi_i|_N \text{ is surjective}\}$. (Open subfunctor)
- If $\mathcal{G}r_I$ is representable, so is $\mathcal{X} \cap \mathcal{Y}$.
- $\mathcal{X} \cap \mathcal{Y}$ isomorphic to $R \mapsto R^{\oplus(I \setminus i)}$, which is not representable.

■

Further Descent Properties

Definition

$\mathcal{X}$ is an **fpqc sheaf** if (1) it is a Zariski sheaf and (2) for every faithfully flat ring homomorphism $R \to S$, the following diagram of sets is an equalizer.

$$\mathcal{X}(R) \to \mathcal{X}(S) \rightrightarrows \mathcal{X}(S \otimes_R S).$$

- Note: Representable $\implies$ fqpc sheaf.

Theorem

$\mathcal{G}r_I$ and $\mathcal{F}\ell_I$ are fpqc sheaves.

Proof.

Results of Raynaud, Gruson, and Perry cover the fpqc case too! [3] [4] ■

References

- Alexander Grothendieck, Introduction au langage fonctoriel, course in Algiers in November 1965, lecture notes by Max Karoubi.
- J.-P. Serre. Modules projectifs et espaces fibrés à fibre vectorielle. Séminaire P. Dubreil, M.-L. Dubreil- Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23. Secrétariat mathématique, Paris, 1958.
- M. Raynaud and L. Gruson. Critères de platitude et de projectivité. Inventiones Math., vol. 13 (1971), 1–89.
- A. Perry, Faithfully flat descent for projectivity of modules, arXiv:1011.0038v1.