Math 145: How not to prove theorems in mathematics

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> I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of a mathematician, where half a proof is zero, and it is demanded for proof that every doubt becomes impossible. — Gauss

Although I don't want to spend too much time on how to do things wrong, here are some useful examples of what to avoid.

Proof by pretending

Maybe the worst kind of proof is the kind where the author pretends and guesses through the hard part. This includes the court spectacle that Gauss had in mind: A lawyer might make half of an argument that doesn't work, then half of a different argument that doesn't work, and then leap to, "Therefore my client is innocent!"

To make things worse, partial credit can encourage faking it. (And I do grant partial credit.) If you want higher scores, you might sometimes want to handwave, especially on tests. But if you want to be a good thinker, it is better if you can't bring yourself to write things that you know are wrong, and to admit to what you can and can't prove.

There is a way to get partial credit without writing anything false. If you are supposed to prove X and you can't prove X, it's fine to prove a related Y instead, if you first admit that you changed the question.

Proof by fuzzy logic

Another disappointing kind of proof is a proof by vague sentiment or fuzzy logic. This includes associations such as that A kind-of goes with B, that C generally follows a pattern, that D has certain tendencies, that E seems to work as a solution, etc. This is another part of what Gauss had in mind, that a sort-of proof can go with another sort-of proof to make "enough proof". This is not the way that modern mathematics works. A proof is a description (admittedly an incomplete description that depends on the audience) of an iron chain of reasoning from the hypothesis and from the axioms.

Proof by just showing your work

In a math course, it's easy to cling to the idea that the heart of every problem is to calculate something, and proving it means simply showing your work. In a proof-based course, it's not true. The heart of a proof problem is reasoning, not calculation. Any proof of anything should have at least some explanation in English. You also never need to show any scratch work that led to your explanation; you only need the final draft of your argument to be convincing.

But it is true that some proofs are mostly calculation.

Proof by rote algorithm

Non-proof courses in mathematics generally teach algorithms to do calculations. It would be nice if there was an algorithm to find proofs of theorems. No such luck. There are algorithms to look for proofs of theorems, but it is a theorem (!) that no algorithm can find a proof of every provable theorem. You deserve to see plenty of examples of proof strategies, but you will also need creative thinking.

Proof by undefined symbols

A particularly frustrating way to prove by pretending is to write down symbols and expressions without defining them, or at best without explaining their relevance. This is a tempting thing to do with an abstract-looking problem. Unlike some dubious proof strategies, this one is not particularly useful in a courtroom: “Male A and female B were in car C. Therefore defendant Ellen McCormick is guilty!” Any jury would react with, “Huh?”

Proving the hypothesis from the conclusion

In most walks of life, “If A, then B” suggests a general association between in both directions. For example, “If dark clouds, then rain.” In mathematics, “If A, then B” is often a very different assertion from its converse, “If B, then A”, even when they are both true. You also can’t prove either one with “Assume both A and B; discuss.”

There are some calculations and some arguments in which every step is bidirectional. For example, when you solve linear equations, every step follows from the next step, and implies the next step. In an argument like this, you might well prove that A implies B and and that B implies A at the same time. Please just say what follows from what!

Proof by describing the conclusion

Since the conclusion is the goal of a theorem, it is tempting to simply describe the conclusion in detail as an argument for it. You may have in mind any of several reasons: To show that you understand it, to make it believable, to show “how it works”, etc. This is also a common strategy in court: A prosecutor might describe a guilt scenario in great detail, in order to instill the idea that the defendant is guilty. But in mathematics, a proof is in argument that the conclusion is true, not how it would work if it were true. (It shouldn’t be enough in a trial either, although unfortunately it sometimes is.) It is not enough to just describe the conclusion, and it is not enough to prove the conclusion from the conclusion. You have to prove the conclusion from the hypothesis.

Proving and disproving the same thing

Mathematics is based on objective, logical truth. In some cases you can’t know if a statement is true or false, but no statement can be both true and false at the same time. Political arguments sometimes sound like this. “The other party proved to you that the government should do X. We will prove to you that the government should not do X, and you can decide which proof you like better.” In mathematics, this is completely unreasonable. When two proofs contradict each other, it is essential to understand which one is wrong. (Or maybe they are both wrong.)

The only circumstance where something like this is okay is if a proof has a gap, and you want to decide whether to try to fill the gap or abandon the proof entirely. Then you can reason, “We will never be able to fill this gap, because here is a counterexample.”

Proof by example

Many theorems in mathematics begin with one of the many ways to say “For all”. At the beginning of a theorem, “Let x be” means “For all x”; “If x is” means “For all x”; “Suppose that x is” means “For all x”; and “Given any x” means “For all x”. That means, don’t just verify examples.

Again, if you can’t prove all cases of the theorem, it could be okay to just do a special case or an example, if you first admit that you changed the question.

It can be fine, if sometimes dangerous, to use an example to illustrate an argument that always works.

If a theorem happens to begin with “There exists”, or if you’re trying to disprove a “For all”, then you only need one example. In fact, for some sophisticated questions, you don’t even need to find an example; sometimes there is an indirect reason that an example must exist.

Proof by asking too much from the reader

Any human proof in mathematics has little gaps that are left for the audience. With experience, the audience changes and you earn the right to bigger gaps. In research papers in mathematics, the gaps can be pretty big sometimes. However, it is all too easy to leave gaps that are just too big. There are no rigorous rules for how big is too big. One important principle is that you shouldn't leave more work for the audience than you did yourself.

It is usually fine to fill a gap by citing theorems in the book and previous homework. Sometimes this is not reasonable in homework or test problems, if the whole question is a special case of an established result. If your homework is to bake a cake, you shouldn't just cite a supermarket that sells them.

You should never cite a theorem that uses the thing that you're trying to prove, because that's circular.