Extra Problems
Math 128A, Fall 2019

- Below are some problems to help study for the midterm. This is **not** a practice exam, but working these problems will help you study for the exam. You should additionally study your homework, textbook, and notes from class.
- For the midterm, you may bring one handwritten $8.5'' \times 11''$ sheet of notes to the exam.

1. Let us look at the format of the IEEE **half-precision floating point number** with 16 bits:

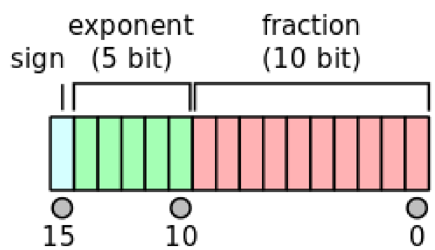


Figure 1: from Wikipedia

(a) Given a similar 16 bits binary machine number:

0		1	0	1	0	0		0	1	0	0	0	0	0	0	0	0

Write down the next largest and smallest machine numbers in binary form.
The next largest machine number:

The next smallest machine number:

(b) What is machine epsilon? What is the rounding unit?

(c) Which of the following is the smallest floating point number in this system which is greater than 10? Explain.

A. $10 + 2^{-7}$
B. $10 + 2^{-10}$
C. $10 + 2^{-23}$
D. $10 + 2^{-24}$

(d) Find smallest value of δ so that for all $r < \delta$

$$\mathrm{fl}(10 + r) = 10.$$

(e) Is δ related to the rounding unit? In particular, what can you say about δ if 10 is replaced by another floating point number?

2. Consider a cubic spline with knots $x_0 < x_1 < \cdots < x_n$ interpolating the values $y_0, y_1, \cdots, y_n$ at the knots. (the term “knot” means the interpolated points and their corresponding values). There is one correct answer. Circle it and explain briefly.

   A. Given the choice of knots and values, there is a unique interpolating cubic spline.

   B. Given the choice of knots and values, there is more than one interpolating cubic spline.

   C. There is a unique interpolating cubic spline only if $n = 3$.

   D. There may be an interpolating cubic spline or there may not be one, depending on the particular points and values.

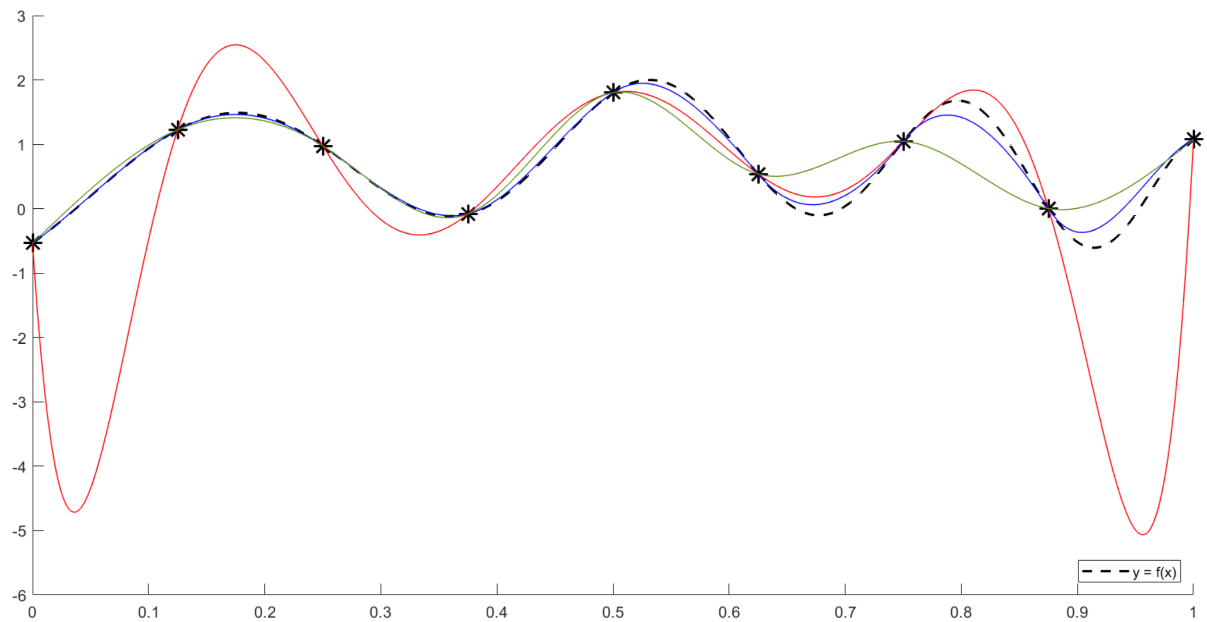
3. In the following picture, the black dash line represents a oscillatory $y = f(x)$. The data points marked with asterisks ($*$) are interpolated using, with three different colors: red, green, and blue,

   (i). Lagrange interpolation (with equally spaced point),

   (ii). spline (with natural boundary condition),

   (iii). piecewise cubic Hermite interpolation

   Determine which color corresponds to which interpolation. Explain your answers briefly.



4. (a). Given data $y(0) = 1, y(1) = 2$, let $p(x)$ be the unique polynomial which interpolates this data. Compute $p(1/2)$.

   (b). Repeat part (a) with Hermite interpolation, if $y(t)$ is a solution to $y' = y$.

   (c). Repeat part (b) with the extra data point $y''(0) = 1$.

5. Given the finite difference formula

$$\frac{-f(x) + 3f(x+h) - 3f(x+2h) + f(x+3h)}{h^3},$$

determine what derivative it approximates. What is the order of accuracy of the approximation?

6. Consider the following code which sums the first ten million terms in the harmonic series in two different ways using single precision.

```
N=10e6;
S1=single(0.0);
S2=single(0.0);

for k=1:N
  S1 = S1 + 1/k;
  S2 = S2 + 1/(N+1-k);
end

fprintf('Sum 1 = %10.5f \n',S1);
fprintf('Sum 2 = %10.5f \n',S2);
fprintf('Rel diff = %e \n',abs(S1-S2)/S1);
```

The output of the code is

```
Sum 1 =   15.40368
Sum 2 =   16.68603
Rel diff = 8.324949e-02
```

So there is a difference of about 8% between the two sums. Which one do you think is more accurate? Explain.

7. In class we stated that relative error is often more meaningful. Explain why. However, there are problems where the absolute error is more meaningful. Give and example and explain.

8. For what range of values of θ will the approximation $\sin\theta \approx \theta$ give relative error $\leq 10^{-4}$?

9. (a) Use divided differences (show the table) to write the Newton interpolating polynomial for these data:

x	4	2	0	3
f(x)	63	11	7	28

   (b) For the same data, write the Lagrange interpolating polynomial.

   (c) Show that these two forms are the same polynomial.

10. When would you use a Hermite interpolating polynomial over a cubic spline? When would you use a cubic spline over a Hermite interpolating polynomial?

11. Let S be a cubic spline on $[0,4]$ with knots at $x=0$, $x=1$, $x=3$, and $x=4$.

$$S(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & \text{if } x \in [0,1] \\ c(x-2)^2, & \text{if } x \in [1,3] \\ d(x-2)^2 + e(x-3)^3, & \text{if } x \in [3,4] \end{cases}$$

(a) Determine **all** the the possible values of a, b, c, d, e so that S is cubic spline.

(b) Determine **all** the the possible values of a, b, c, d, e so that S satisfies the not-a-knot conditions.

(c) Determine **all** the the possible values of a, b, c, d, e so that S is a natural spline.

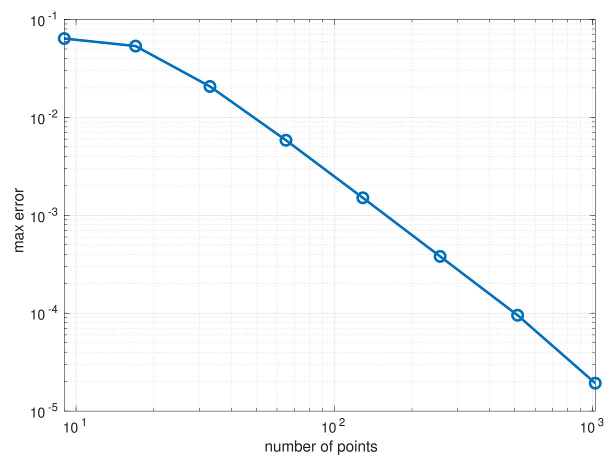
12. Let P be a piecewise polynomial that interpolates the function f on the interval $[a,b]$ at the equally spaced points $x_j = a + jh$ for $j = 0 \ldots n$ and $h = (b-a)/n$.

(a) The plot to the right shows

$$E = \max_{x \in [a,b]} |P(x) - f(x)|$$

as a function of n. At what rate does E appear to converge to zero?

(b) Assuming that f has at least 4 continuous derivatives, is P most likely a piecewise Hermite cubic function, piecewise linear function, a spline, none of the above?



13. Derive the finite difference formula for approximating the second derivative at x using function values at $x-2h$, $x-h$, and x. Give the expression for the error term.