Homework 3
Math 128A
Due Wednesday, 11/13/19, 11:59 PM

1. (a) Derive a difference formula to approximate the first derivative at x using function values at $x-2h$, $x-h$, and x. What is the order of accuracy of the approximation?
   (b) Derive a difference formula to approximate the second derivative at x using function values at $x-2h$, $x-h$, and x. What is the order of accuracy of the approximation?
   (c) Apply the difference formulas to $f(x) = \exp(-x)$ at $x = 0$ to approximate $f'(0)$ and $f''(0)$. Use increasingly smaller values of h, and make tables of the approximate derivatives and the absolute errors of each of the approximations.
   (d) Plot the absolute errors vs. h on a log-log graph.
   (e) Explain how the table of errors and the graph of the errors demonstrate the order of convergence.

2. (a) For the data below find the least squares polynomials of degrees 1, 2 3, and 6. You may find the MATLAB commands `polyfit` and `polyval` useful. In Python, the same commands are available in the `numpy` library.
   (b) Make plots of the data and the polynomials on the same graph.
   (c) For each least squares polynomial, $P(x)$, compute the error

$$E_2 = \sum_j \left(y_j - P(x_j)\right)^2 .$$

   (d) Which polynomial do you think best approximates the data? Does increasing the degree of the least squares polynomial always decrease the error? Does increasing the degree of the least squares polynomial give a better approximation? Explain.

x	0.000	0.143	0.286	0.429	0.571	0.714	0.857	1.000
y	0.103	0.121	0.179	0.250	0.444	0.519	0.897	1.269

3. Let $f(x) = (x+1)\exp\left(\dfrac{-3(x+1)^2}{4}\right)$.

   (a) Find the third degree Taylor polynomial, $S(x)$, of $f(x)$ about $x = 0$.
   (b) Find the third degree polynomial, $P(x)$, that minimizes

$$\int_{-1}^{1} \big(f(x) - P(x)\big)^2 \, dx.$$

   Solve this problem by using orthogonal polynomials.

   You could, but you need not, preform the integration by hand. You can perform the necessary integration using symbolic computation using, for example, Maple, Mathematica, or Wolfram Alpha. State what program you used and give the results. Alternately, you may perform the integration numerically. In matlab the routine to use is `integral`. In python, the corresponding routine is `quad` from the library `scipy.integrate`. Examples are given below. We will learn about how these methods work later in the quarter.

(c) Find the third degree polynomial, $Q(x)$, that interpolates f at the Chebyshev points: ± 1, ± 0.5.

(d) Make plots of f, S, P, and Q, and plots of the differences $|f - S|$, $|f - P|$, and $|f - Q|$ on the interval $[-1, 1]$. Comment on the different approximations? What are the advantages of each of the approximations?

The first four Legendre polynomials, $L_j(x)$, are $L_0(x) = 1$, $L_1(x) = x$, $L_2(x) = (3x^2 - 1)/2$, $L_3(x) = (5x^3 - 3x)/2$.

```
% matlab integration example
%
f=@(x)((x+1).*exp(-0.75*(x+1).^2));
Iexact =  2.0/3.0 * (1.0-exp(-3.0));

xmin=-1;
xmax=1;
I0 =integral(f,xmin,xmax);

fprintf('%f %f %e \n',Iexact,I0,abs(Iexact-I0)/I0);
```

```
# python integration example
#
from math import exp
from scipy.integrate import quad

def f(x):
    return (x+1)*exp(-0.75*(x+1)**2)
Iexact =  2.0/3.0 * (1.0-exp(-3.0))

xmin = -1
xmax =  1
I0, err = quad(f,xmin,xmax)

print I0,Iexact, abs(I0-Iexact)/Iexact
```