Problem 1. Estimating a Fixed Point in a Population Model

Recall our discussion of recursions from a few weeks back. We can model, say, the population of animals at year $i$, $N_i$, using a simple equation like

$$N_{i+1} = 1.04N_i$$

given $N_0 = 25$. This works well for small populations that are not limited by resources – e.g. you saw that it works well for Whooping Cranes, which had nearly been driven to extinction in the 1940s.

However, these exponential growth models quickly become inaccurate, because exponential growth is just too fast. E.g., in the above example, the population doubles every 17.67 years, so that after 176.7 years, there would be $25 \cdot 2^{10} = 25,600$ individuals, and after 706.92 years (17.67 · 40), there would be $25 \cdot 2^{40} = 27.5$ trillion individuals. Clearly, we can’t sustain a population of 27.5 trillion cranes.

A more realistic population model is the following, which includes resource-limited growth

$$N_{i+1} = N_i + aN_i(K - N_i)$$

This model has two fixed points and, after a long time, the population settles down to one of them (i.e. $\lim_{i \to \infty} N_i$ exists, unlike in the simple exponential growth model).

a) Find the two fixed points

b) Using R/R-studio, and the following code as a template, find $N_i$ for $i = 0, 1, 2, \ldots, 49, 50$, and plot $N_i$ vs. $i$. For this part, assume that $a = 0.04$, $K = 15$ and $N_0 = 2$. Has the population settled down to one of the fixed points? Which one?

This code simulates the simple exponential growth model ($N_{i+1} = 1.04N_i$, with $N_0 = 25$) for $i = 0, 1, 2, \ldots, 49, 50$ and then plots the results. **You will have to modify this code** in order to complete parts b and e.

```r
N=25
for(i in 1:50){
  N[i+1]=1.04*N[i]
}```
The previous model describes resource-limited growth, but suppose we want to include an effect like hunting in our model. Hunters remove individuals from the population if the population is sufficiently large. If the population is too small, then hunters seldom encounter their quarry, and hunting has no effect.

\[ N_{i+1} = N_i + aN_i(K - N_i) - h(1 - e^{-N_i/K}) \]

Notice that you can no longer solve for the fixed points. However, if hunting is sustainable, then the last term should be small compared to the others. Thus, we might expect that the fixed points are near the fixed points you found in part a.

Here’s where linearization can be of use. We can linearize the equation about the fixed point \( N_i = K \), and then solve this linearized equation to approximate the fixed point. We’ll do this in two steps:

c) Step 1: Linearize the function

\[ f(N_i) = N_i + aN_i(K - N_i) - h(1 - e^{-N_i/K}) \]

near \( N_i = K \).

(Recall that we can approximate a function near a real number \( N_i = a \) using the linear approximation of \( f \) around \( a \): \( f(N_i) \approx f(a) + f'(a)(N_i - a) \)).

d) Step 2: Now, the recursion is \( N_{i+1} = \) the linear function of \( N_i \) from part c.

Find the fixed point of this new recursion to approximate the fixed point of the original recursion. The algebra is messy. To simplify your work, assume that \( a = 0.04 \), \( K = 15 \), \( h = 0.1 \) and \( N_0 = 2 \).

Note: Show your work! To check your answer, the general formula is

\[ N = K - \frac{h(1 - e^{-1})}{aK + h/(eK)} \]
e) How well does this estimate work? Using R/R-studio, evaluate the exact, non-linear recursion to get $N_i$ for $i = 0, 1, 2, \ldots, 49, 50$, and plot $N_i$ vs. $i$. For this part, assume that $a = 0.04$, $K = 15$, $h = 0.1$ and $N_0 = 2$. What is $N_{50}$? How does it compare to the fixed point you calculated in part c)?

Discussion questions: Suppose you’re managing a population of animals, and you have to set a limit on hunting. According to the general equation for the fixed point,

$$N = K - \frac{h(1 - e^{-1})}{aK + h/(eK)}$$

how much hunting do you think is sustainable (i.e. how big a value of $h$ would you allow)? Using R/R-studio, simulate the population for 50 years and see if it’s sustainable. Was your choice good? Explore a few values and see if anything interesting happens (HINT: our linear approximation assumes that hunting pressure is small).