MAT 17A - DISCUSSION #6

Metabolic Rate for a California Condor

In many biological situations, we directly measure one quantity, but we really want to know a function of that measured quantity. For example, we might measure the partial pressure of oxygen in the blood (p) but want to know the saturation of hemoglobin, S, which is given by the hemoglobin saturation equation $S = f(p) = 1/(1 + e^{-ap})$, where a is a positive constant that depends on temperature, pH and other properties of the blood).

When using real data to predict the output of a function, measurement error for the independent variable (say, x) leads to an error in the estimate of the dependent variable (say, y). This phenomenon, known as **error propagation**, can be estimated using linear approximation.

(a) We begin with a linear model. Suppose that $y = f(x)$ is a linear function of x. Thus we know that

$$f(x) = mx + b$$

for some constants m and b. Suppose x is measured in an experiment with some error of Δx. We sometimes write the measured value as $x \pm \Delta x$, which means that the actual value of x (without measurement error) is contained somewhere within the interval $(x - \Delta x, x + \Delta x)$.

If we use the measured valued of x to compute y, the error in x propagates to the computed value of y. We want to know how Δy depends on Δx. Use the expression for f above to compute the error in y as

$$\Delta y = f(x + \Delta x) - f(x).$$

You should have Δy in terms of Δx.

(b) For a nonlinear function, we can use linear approximation to relate the error in a computed quantity to the error in the measured quantity. Let $L(x)$ be the linear function which approximates f around $x = a$ given by

$$L(x) = f(a) + f'(a)(x - a),$$

Notice $L(a) = f(a)$, and for x close to a, $L(x) \approx f(x)$.

(i) Use the linear approximation of $f(x)$ at $x = a$ to fill in the following lines:

$$\begin{aligned} \Delta y &= f(a + \Delta x) - f(a) \\ &\approx \underline{\qquad\qquad} - f(a) \\ &= \underline{\qquad\qquad} \end{aligned}$$

Note that in this particular context, the term $f'(a)$ in the final expression is referred to as the **sensitivity** of y to x at $x = a$.

ii) Conceptually, why do you think we refer this term as **sensitivity**? What would a larger (or smaller) sensitivity value change about our linear approximation at a particular $x = a$?

(c) To examine error propagation in more detail, let's consider a model for the metabolic rate of animals. The following curve models how the metabolic rate R (in kilocalories/day) depends on body mass M (in kilograms):

$$R = e^{4.2} M^{0.75}$$

(i) Use this equation to predict the metabolic rate of a California Condor weighing 10 kg.

(ii) Compute a linear approximation to the function $R(M)$ about the value $M = 10$ kg.

(iii) Plot $R(M)$ and its linear approximation about $M = 10$ on the same graph for $0 \leq M \leq 20$.

(iv) Naturally, not every California Condor weighs exactly 10 kg, some weigh more and some weigh less. What is the sensitivity of our model to this measurement? That is, give an expression for how small error errors, ΔM, in M propagate to errors in R when $M = 10$. Include units in your expression for ΔR.

(v) Note that above, we were discussing ΔR, which contains units. Is ΔR small? The value of ΔR and whether it looks big or small depends on the units we use for it. To better understand the size of ΔR, we consider the relative error, which is the size of ΔR relative to R. That is, suppose that we know that most adult Condors weigh within 10% of 10kg. Then, we would know that $\Delta M/M = 0.1$. Given a 10% error in the estimate of the condor's weight, what (approximate) percent error will we have in the metabolic rate?