Topics for Exam 2

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• The second exam will cover Sections 7.5, 7.6, 8.1, 8.2, first order linear differential equations, chemical kinetics, numerical methods for DEs, 9.1, and 9.2. You should review for your homework, class notes, textbook, and handouts.

• Numerical integration: know how to apply midpoint rule and trapezoidal rule (e.g. p. 388 #51). You do not need to memorize the error bounds, but you should know how to use them (e.g. Section 7.5 #24).

• Compute Taylor polynomials and understand their significance.

• Solve separable differential equations and first order linear equations.

• Find equilibria of autonomous differential equations and classify their stability both graphically and analytically.

• Applications: mixing problems, chemical reactions, population dynamics

• Use Forward Euler for generating approximate solutions to initial value problems.

• Linear systems, Gaussian elimination

• Matrix algebra: addition, subtraction, multiplication, transposes, inverses, determinants. Relationship between matrix equations and linear systems of equations.

More Problems

Below are some problems that may be a little different from your previous homework problems. These are not intended to be sample test problems, but working them will help you understand some concepts from the course and prepare you for the test.

1. Assume that $f$ has at least three continuous derivatives. We know that for small values of $h$

$$
\frac{f(a + h) - f(a)}{h}
$$

approximates the derivative of $f$ at $x = a$. Use a Taylor polynomial of degree 2 to approximate $f(a + h)$ for small $h$ (i.e. expand around $x = a$ and evaluate the polynomial at $a + h$). Comment on how accurately this expression approximates the derivative.

2. Explain whether following statements are true or false.

   (a) Suppose that $A$ is a square matrix and that $\det(A) = 0$. The equation $A\vec{x} = \vec{b}$ has no solution.

   (b) Suppose that $A$ and $B$ are square matrices such that $AB = I$. Then it follows that $AB = BA = I$.

3. Assume that the reaction

$$
A + B \xrightleftharpoons[k_-][k_+] Y
$$

follows mass action kinetics. Initially the concentration of $A$ is 10 $\mu M$, the concentration of $B$ is 5 $\mu M$, and the concentration of $Y$ is 2 $\mu M$. Write an initial value problem for the concentration of $Y$. 

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4. Suppose that the size of a population $x$ is determined by the differential equation

$$\frac{dx}{dt} = x(1-x) - \frac{hx}{a+x},$$

where $h$ and $a$ are positive constants. The first term represents logistic growth, and the second term represents harvesting by predators.

(a) Describe the physical meaning of the constants $h$ and $a$, and provide an interpretation for the form of the predation rate.

(b) Note that $x = 0$ is always an equilibrium value. For a fixed value of $a$, determine for which values of $h$ $x = 0$ is stable and for which values it is unstable. Hint: Use the analytic condition for stability.

5. Take some steps of forward Euler to approximate the solution to the logistic equation

$$\frac{dN}{dt} = 5N(1-N); \quad N(0) = 0.2$$

with $\Delta t = 0.5$. Is this a reasonable approximation to the solution? Explain the reason for the behavior of the approximate solution.

6. Suppose that a tank initially contains 1000 liters of pure water. A salt solution flows into the tank at a rate of 4 liters per minute and a well-stirred mixture flows out at the same rate. The concentration of the salt solution flowing in is $c_{in}(t) = 35(1 + \sin(0.01t))$ grams per liter, where $t$ is time in minutes. Find the mass of salt in the tank as a function of time.

7. Find the solution to

$$2x + 3y - z = 1$$

$$x - y + 5z = 7,$$

and give a geometric description of the solution.

8. Let $C$ be a chemical concentration that satisfies the initial value problem

$$\frac{dC}{dt} = A - kC; \quad C(0) = C_0$$

where $A$ and $k$ are positive constants.

(a) Find the equilibrium concentration, $C_{eq}$, and show that it is stable.

(b) Show that $C = C_0 + (C_{eq} - C_0)(1 - e^{-kt})$ by plugging this expression in the differential equation and validating that it satisfies the initial condition.

(c) Suppose that time is measured in seconds. Then $k$ is in units of $s^{-1}$. What is the significance of the time $k^{-1}$?

9. Suppose that an integral is approximated using the midpoint rule with 10 subintervals and the error is $10^{-3}$. Approximately how many subintervals are needed to give an error less than $10^{-7}$? The error bound for the midpoint rule is of the form

$$E_n = \left| M_n - \int_a^b f(x) \, dx \right| \leq \max_{x \in [a,b]} |f''(x)| \frac{(b-a)^3}{24n^2} = C \frac{n^2}{n^2}.$$

10. Consider the differential equation $x' = f(x)$, with an equilibrium at $x = 0$. Give an example of a function $f$ such that $f'(0) = 0$ and the equilibrium at $x = 0$ is stable. Give an example of a function $f$ such that $f'(0) = 0$ and the equilibrium at $x = 0$ is unstable.