Math 207C
Homework 1
Due Friday, April 13th

1. The Exponential integral function is defined as

$$Ei(x) = \int_x^\infty \frac{\exp(-s)}{s} ds.$$

Derive an asymptotic expansion for $Ei(x)$ for large x. Use a computer (e.g. expint(x) in MATLAB) to check the accuracy of your expansion for different values of x and for different numbers of terms. Discuss your results.

2. Find the leading term approximation to

$$I(x) = \int_{-\pi/2}^{\pi/4} \exp\left(-x \sin^4(t)\right) \, dt$$

as $x \to \infty$. Your answer may involve the Gamma function, which is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \, dt.$$

Compare your asymptotic approximation with a numerical approximation for different values of x and discuss your comparison.

3. Suppose a particle is moving by Brownian motion in a potential

$$U(x) = U_0 x^2$$

with a reflecting boundary at $x = 0$ and an absorbing boundary at $x = 1$. The Langevin equation for the particle position is

$$\frac{dX}{dt} = -\frac{1}{\gamma} U'(x) + \xi,$$

where ξ is the random thermal force with $\langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2)$, $D = k_B T/\gamma$ is the diffusion coeffient, γ is the drag coefficient, k_B is the Boltzmann constant, and T is the temperature.

Let $\tau(x)$ denote the mean first passage time to cross the boundary at $x = 1$ for a particle which begins at x. The mean first passage time satisfies the equation

$$-\frac{U'(x)}{\gamma} \frac{d\tau}{dx} + D \frac{d^2\tau}{dx^2} = -1,$$

with boundary conditions $\tau'(0) = 0$ and $\tau(1) = 0$. Give the leading order approixmation to the mean first passage time for a particle which starts at $x = 0$ in the limit that $U_0/(k_B T)$ is large.