Math 207C
Homework 5
Due Friday, May 18th

1. Consider the Lagerstrom model for Low Reynolds number flow:

$$U'' + \frac{2}{R}U' + \epsilon U U' = 0$$
$$U(1) = 0$$
$$U(\infty) = 1.$$

   (a) Compute the leading order expansion of $U'(1)$ in the limit of small ϵ. This is the analog to Stokes's original calculation for the force on a translating sphere in 3D.

   (b) Compute the expansion of $U'(1)$ up to order ϵ. You will encounter a problem similar to the "Whitehead paradox" which you will resolve using intermediate scale matching. You may find the following asymptotic expansion useful for small r:

$$\int_r^\infty \frac{\mathrm{e}^{-x}}{x^2}\,dx = \frac{1}{r} + \log(r) + \gamma - 1 + \mathcal{O}(r),$$

   where γ is Euler's constant.

2. For the van der Pol equation

$$\epsilon \dot{u} = v + u - \frac{u^3}{3},$$
$$\dot{v} = -u,$$

   we found equations for the inner and outer solutions in class. On the inner and outer layers, one of the two equations above is in pseudo-steady-state. There is a corner layar at $u = 1$, $v = -2/3$ (and $u = -1$, $v = 2/3$). Find the equations for the corner layer and identify the thickness of the layer. Use a balance argument which involves retaining the time derivatives of both state variables.

3. The FitzHugh-Nagumo equations are a model system from electrophysiology to describe the cross membrane electrical potential in neurons:

$$\epsilon \dot{v} = v(a - v)(v - 1) - w,$$
$$\dot{w} = v - \gamma w.$$

   The variable v is the voltage and w is called a recovery variable. Suppose that $\epsilon > 0$ is small, $0 < a < 1$, and γ is sufficiently small so that $v = 0$, $w = 0$ is the only equilibrium.

   The single equilibrium is a global attractor: all trajectories approach it. For some initial conditions, the solution quickly approaches the rest state. For others the voltage rapidly rises to a large value for some period of time, then overshoots the equilibrium value, and finally approaches the rest state. This second response is called an action potential. See the below figures for examples, and note the difference in scales of the vertical axes. The MATLAB code to solve the equations and plot the solution is given below.

   Analyze the inner and outer layer structures of this system to characterize which initial conditions lead an action potential, and describe the structure of the action potential using a layer analysis.

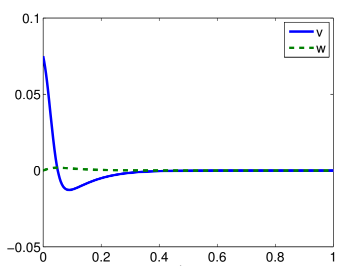
```
% solve the FitzHugh-Nagumo equations
%  ep*v' = v*(a-v)*(v-1) - w
%     w' = v - g*w
% and plot the solution

% parameters
%
a  = 0.1;
g  = 0.5;
ep = 2e-3;

tend = 1;
v0   = 0.2;
w0   = 0.0;

% make ode-rhs function and solve
%
dydt= @(t,y)([(y(1).*(a-y(1)).*(y(1)-1)  -y(2) )/ep; y(1)-g*y(2)]);
[t,y]=ode23s(dydt,[0:tend],[v0,w0]);

% make the plot
%
figure;
hp=plot(t,y,'linewidth',3);
axis([ 0 1 -0.5 1]);
set(hp(2),'linestyle','--');
set(gca,'fontsize',16);
xlabel('time');
legend('v','w');
```



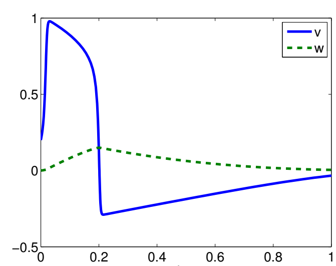


Figure 1: Solution to FitzHugh-Nagumo equations for different initial conditions for $a = 0.1$, $\gamma = 0.5$, and $\epsilon = 0.002$. The left plot shows the response to a subthreshold stimulus, and the right shows that action potential that results from a superthreshold stimulus.