Math 226A
Homework 2
Due Monday, October 30th

1. A matrix A is strictly column diagonally dominant if for each k,
$$|a_{kk}| > \sum_{i \neq k} |a_{ik}|.$$
Show that if Gaussian elimination with partial pivoting is applied to a strictly column diagonally dominant matrix, no row interchanges occur.

2. Perform a computational experiment in which in a single trial you generate a random square matrix of a given size. Generate a random vector, $\vec{z}$, and compute the $\vec{b} = A\vec{z}$. Now solve the linear system $A\vec{x} = \vec{b}$ for $\vec{x}$. Compute the relative error as $\|\vec{x} - \vec{z}\|/\|\vec{z}\|$ and the condition number of the matrix. Perform a number of trials and try a few different sized problems. Look for a scaling between the condition number and the relative error. Hint: scatter plot the error vs. condition number on a log-log plot. Does the scaling depend on the matrix size? Do the result match your expectation for the relationship between error, condition number, and machine epsilon? Discuss.

3. (a) Show that for Gaussian elimination with partial pivoting applied to any $n \times n$ matrix, the growth factor, ρ, satisfies
$$\rho = \frac{\max_{i,j} |u_{i,j}|}{\max_{i,j} |a_{ij}|} \leq 2^{n-1}.$$
   (b) Pick $n \times n$ matrices for range of n spanning several orders of magnitude with random entries sampled uniformly from $[-1, 1]$ and compute the growth factor. Scatter plot the growth factors against the size on a log-log plot. How does the growth factor appear to scale with size?
   (c) Extrapolating your results from the previous part to a $10^6 \times 10^6$, what growth factor do you expect?

4. (a) Let A be an $m \times n$ matrix. Find constants C_1 and C_2 (which depend on the dimensions) such that
$$C_1 \|A\|_\infty \leq \|A\|_2 \leq C_2 \|A\|_\infty.$$
   (b) Show that your bounds in the previous problem are tight. That is, find examples where equality is obtained for each inequality.
   (c) Based on the above inequalities, bound the condition number of a square matrix using 2-norm above and below in terms of the condition number in infinity norm. Explain why these bounds may not be tight (i.e. equality may not be achieved) even though the bounds on the norms are tight.

5. Below is excerpt of MATLAB code to solve the tridiagonal system of equations
$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \\ \vdots \\ \\ f_n \end{bmatrix} \tag{1}$$

Note that using the code to solve tridiagonal systems in MATLAB will likely be slower than forming the sparse matrix (e.g. using `spdiags`) and using the backslash operator to solve the system. The built-in solver would recognize the structure and call a tridiagonal solver which has been compiled. This avoids the slow down of the interpreter.

(a) Let $A = LU$ be the LU factorization of the matrix above. The code below does not store L, but it could be modified to do so. What are the entries of L and U?

(b) Modify the algorithm to solve problems of the form

$$\begin{bmatrix} b_1 & c_1 & & & a_1 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ c_n & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} f1 \\ \\ \vdots \\ \\ f_n \end{bmatrix} \tag{2}$$

```
x=zeros(n,1);
w=zeros(n,1);

d    = b(1);
x(1) = f(1)/d;
for i = 2:n
    w(i-1) = c(i-1) / d;
    d = b(i) - a(i) * w(i-1);
    x(i) = (f(i) - a(i) * x(i-1))/d;
end

for i = n-1:-1:1
  x(i) = x(i) - w(i) * x(i + 1);
end
```