**Math 228B**
**Homework 1**
**Due Wednesday, 1/23/19**

1. Let $L$ be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.

   (a) Find the eigenfunctions and corresponding eigenvalues of $L$.

   (b) Show that the eigenfunctions are orthogonal in the $L^2[0,1]$ inner product:
$$\langle u, v\rangle = \int_0^1 uv\, dx.$$

   (c) It can be shown that the eigenfunctions, $\phi_j(x)$, form a complete set in $L^2[0,1]$. This means that for any $f \in L^2[0,1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to
$$u_{xx} = f, \quad u_x(0) = u_x(1) = 0, \tag{1}$$
as a series solution of the eigenfunctions.

   (d) Note that equation (1) does not have a solution for all $f$. Express the condition for existence of a solution in terms of the eigenfunctions of $L$.

   (e) A second-order accurate discretization of the problem using a uniform mesh $x_j = j\Delta x$ for $j = 0 \dots N+1$ and $\Delta x = 1/(N+1)$ involves the matrix
$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{pmatrix}.$$
Note that, unlike the Dirichlet problem, the values of the function are also unknown at the boundary points, and so the matrix is $(N+2)\times(N+2)$. Show the $N+2$ eigenvectors are
$$v_j = \cos(k\pi x_j),$$
for $k = 0 \dots N+1$, and compute the eigenvalues. Plot the eigenvalues of the matrix and the eigenvalues from part (a) on the same graph.

2. Define the functional $F : X \to \mathbb{R}$ by
$$F(u) = \int_0^1 \frac{1}{2}(u_x)^2 + fu\, dx,$$
where $X$ is the space of real valued functions on $[0,1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of $F$ at a point $u$ is defined to be the linear operator $F'(u)$ for which
$$F(u+v) = F(u) + F'(u)v + R(v),$$
where
$$\lim_{||v||\to 0} \frac{||R(v)||}{||v||} = 0.$$

One way to compute the derivative is

$$F'(u)v = \lim_{\epsilon\to 0} \frac{F(u+\epsilon v) - F(u)}{\epsilon}.$$

Note that this looks just like a directional derivative.

(a) Compute the Frechet derivative of $F$.

(b) $u \in X$ is a critical point of $F$ if $F'(u)v = 0$ for all $v \in X$. Show that if $u$ is a solution to the Poisson equation

$$u_{xx} = f, \quad u(0) = u(1) = 0,$$

then it is a critical point of $F$.

(c) Let $X_h$ be a finite dimensional subspace of $X$, and let $\{\varphi_i(x)\}$ be a basis for $X_h$. This means that all $u_h \in X_h$ can be expressed as $u_h(x) = \sum_i u_i\varphi_i(x)$ for some constants $u_i$. Thus we can identify the elements of $X_h$ with vectors $\vec{u}$ that have components $u_i$. Let $G(\vec{u}) = F(u_h)$. Show that the gradient of $G$ (whose components are $(\nabla G)_j = \frac{\partial G}{\partial u_j}$) is of the form $\nabla G(\vec{u}) = A\vec{u} + \vec{b}$, and write expressions for the elements of the matrix $A$ and the vector $\vec{b}$.

(d) Divide the unit interval into a set of $N+1$ equal length intervals $I_i = (x_i, x_{i+1})$ for $i = 0, \ldots, N$. The endpoints of the intervals are $x_i = ih$, where $h = 1/(N+1)$. Let $X_h$ be the subspace of $X$ such that the elements $u_h$ of $X_h$ are linear on each interval, continuous on [0,1], and satisfy $u_h(0) = u_h(1) = 0$. $X_h$ is an $N$ dimensional space with basis elements

$$\varphi_i(x) = \begin{cases} 1 - h^{-1}|x - x_i| & \text{if } |x - x_i| < h \\ 0 & \text{otherwise} \end{cases},$$

for $i = 1, \ldots, N$. Compute the matrix $A$ from the previous problem that appears in the gradient.

Finite element methods are based on these "weak formulations" of the problem. The Ritz method is based on minimizing $F$ and the Galerkin method is based on finding the critical points of $F'(u)$.

3. (a) Using a Taylor expansion, derive the finite difference formula to approximate the second derivative at $x$ using function values at $x - h/2$, $x$, and $x + h$. How accurate is the finite difference approximation?

   (b) Perform a refinement study to verity the accuracy of the difference formula you derived.

   (c) Derive an expression for the quadratic polynomial that interpolates the data $(x - h/2, u(x - h/2))$, $(x, u(x))$, $(x + h, u(x + h))$. How is the finite difference formula your derived in problem 3a related to the interpolating polynomial?