Math 228B
Homework 2
Due Wednesday 2/6/19

1. Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDEs below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

   (a) $u_{xx} = \exp(x), \quad u(0) = 0, \quad u(1) = 1$

   (b) $u_{xx} = 2\cos^2(\pi x), \quad u_x(0) = 0, \quad u_x(1) = 1$

2. As a general rule, we usually think that an $O(h^p)$ local truncation error (LTE) leads to an $O(h^p)$ error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on $[0, 1]$ with homogeneous boundary conditions. The standard discretization of this problem gives an $O(h^2)$ LTE provided the the solution is at least C^4. The LTE may be lower order because the solution is not C^4 or because we use a lower order discretization at some points.

   (a) Suppose that the LTE is $O(h^p)$ at the first grid point ($x_1 = h$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error? Hint: Use equation (2.46) from LeVeque to aid in your argument.

   (b) Suppose that the LTE is $O(h^p)$ at an interior point (i.e. a point that does not limit to the boundary as $h \to 0$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error?

   (c) Verify the results of your analysis from parts (a) and (b) using numerical tests.

3. Let u be the solution to $u_{xx} = f$ on the unit interval with Dirichlet boundary conditions. Suppose that f has a jump discontinuity in its derivative at some point $x = a$ for $0 < a < 1$. That is,

$$\lim_{x \to a^+} f'(x) - \lim_{x \to a^-} f'(x) = C,$$

   for some nonzero C. Suppose the f has at least two continuous derivatives on the intervals $(0, a)$ and $(a, 1)$. The solution to the Poisson equation, u, will have a jump in the third derivative at $x = a$.

   (a) Given an expression for the local truncation error at the grid points near the discontinuity in f.

   (b) What rate does the numerical solution converge in max norm using the standard second-order discretization to this problem?

4. We have typically discretized the interval $[0, 1]$ into equally spaced points $x_j = jh$ for $j = 0 \ldots N+1$ with $h = 1/(N+1)$. Another common discretization is the *cell centered* mesh, in which $[0, 1]$ is discretized into N cells. This approach is commonly used with finite-volume methods. The grid points are placed at centers of the cells: $x_j = (j - 1/2)h$ for $j = 1 \ldots N$ where $h = 1/N$. This type of discretization is more natural for some problems, particularly those with Neumann boundary conditions.

(a) We may write $u_{xx} = (-J)_x$, where $J = -u_x$ is the diffusive flux. Suppose we discretize this problem by using a centered difference to compute the flux at the cell edges, $J_{j-1/2}$, followed by another centered difference of the flux. Show that at interior points this gives the standard second-order discretization of u_{xx}.

(b) Again using the idea of flux differencing, derive the discrete approximation to u_{xx} at the first interior grid point adjacent to a boundary with Neumann boundary condition $u_x = g$.