1. Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDEs below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

(a) \( u_{xx} = \exp(x) \), \( u(0) = 0 \), \( u(1) = 1 \)

(b) \( u_{xx} = 2 \cos^2(\pi x) \), \( u_x(0) = 0 \), \( u_x(1) = 1 \)

2. As a general rule, we usually think that an \( O(h^p) \) local truncation error (LTE) leads to an \( O(h^p) \) error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on \([0, 1]\) with homogeneous boundary conditions. The standard discretization of this problem gives an \( O(h^2) \) LTE provided the solution is at least \( C^4 \). The LTE may be lower order because the solution is not \( C^4 \) or because we use a lower order discretization at some points.

(a) Suppose that the LTE is \( O(h^p) \) at the first grid point (\( x_1 = h \)). What effect does this have on the error? What is the smallest value of \( p \) that gives a second order accurate error? Hint: Use equation (2.46) from LeVeque to aid in your argument.

(b) Suppose that the LTE is \( O(h^p) \) at an interior point (i.e. a point that does not limit to the boundary as \( h \to 0 \)). What effect does this have on the error? What is the smallest value of \( p \) that gives a second order accurate error?

(c) Verify the results of your analysis from parts (a) and (b) using numerical tests.

3. Let \( u \) be the solution to \( u_{xx} = f \) on the unit interval with Dirichlet boundary conditions. Suppose that \( f \) has a jump discontinuity in its derivative at some point \( x = a \) for \( 0 < a < 1 \). That is,

\[
\lim_{x \to a^+} f'(x) - \lim_{x \to a^-} f'(x) = C,
\]

for some nonzero \( C \). Suppose the \( f \) has at least two continuous derivatives on the intervals \((0, a)\) and \((a, 1)\). The solution to the Poisson equation, \( u \), will have a jump in the third derivative at \( x = a \).

(a) Given an expression for the local truncation error at the grid points near the discontinuity in \( f \).

(b) What rate does the numerical solution converge in max norm using the standard second-order discretization to this problem?

4. We have typically discretized the interval \([0, 1]\) into equally spaced points \( x_j = jh \) for \( j = 0 \ldots N + 1 \) with \( h = 1/(N + 1) \). Another common discretization is the cell centered mesh, in which \([0, 1]\) is discretized into \( N \) cells. This approach is commonly used with finite-volume methods. The grid points are placed at centers of the cells: \( x_j = (j - 1/2)h \) for \( j = 1 \ldots N \) where \( h = 1/N \). This type of discretization is more natural for some problems, particularly those with Neumann boundary conditions.
(a) We may write $u_{xx} = (-J)_x$, where $J = -u_x$ is the diffusive flux. Suppose we discretize this problem by using a centered difference to compute the flux at the cell edges, $J_{j-1/2}$, followed by another centered difference of the flux. Show that at interior points this gives the standard second-order discretization of $u_{xx}$.

(b) Again using the idea of flux differencing, derive the discrete approximation to $u_{xx}$ at the first interior grid point adjacent to a boundary with Neumann boundary condition $u_x = g$. 