Topology Vol. 31, No. 3, pp. 493–517, 1992.
Printed in Great Britain

0040-9383/92 $5.00 + 0.00

HOMOTOPY EQUIVALENCE AND HOMEOMORPHISM OF 3-MANIFOLDS

JOEL HASS† and PETER SCOTT‡

(Received 10 September 1990; in revised form 9 August 1991)

§1. INTRODUCTION AND PRELIMINARY RESULTS

IN THIS paper we extend the class of 3-manifolds which are determined up to homeomorphism by their fundamental groups to the class of closed orientable irreducible 3-manifolds containing a singular surface satisfying two properties, the 1-line-intersection property and the 4-plane property.

A basic problem in the classification of 3-dimensional manifolds is to decide to what extent the homotopy type of a closed manifold determines the manifold up to homeomorphism. In the case of 3-manifolds with finite fundamental groups, it is known that there are homotopy equivalent manifolds which are not homeomorphic, but there are no known examples of closed orientable irreducible 3-manifolds with isomorphic infinite fundamental groups which are not homeomorphic. Waldhausen [30] and Heil [12] proved that if M and M' are Haken 3-manifolds which have isomorphic fundamental groups then they are homeomorphic. If M and M' are hyperbolic then the Mostow rigidity theorem implies the same result [19], but if only M is assumed to be hyperbolic then it is unknown. Boehme [3] extended Waldhausen's theorem to certain non-Haken Seifert fiber spaces. Scott [27] showed that a closed orientable irreducible 3-manifold which is homotopy equivalent to a Seifert fiber space with infinite fundamental group is homeomorphic to that Seifert fiber space. Many of these Seifert fiber spaces are non-Haken. To date however, Seifert fiber spaces have provided the only examples of non-Haken 3-manifolds which are known to be determined up to homeomorphism by their fundamental groups.

For manifolds with boundary there are simple examples of non-homemorphic homotopy equivalent Haken manifolds, such as the product with the circle of a thrice-punctured sphere and the product with the circle of a once-punctured torus. Such examples are well understood in terms of the characteristic decomposition of the 3-manifold [17] [16]. Another possible source of troublesome examples comes by taking the connected sum of a 3-manifold with a fake homotopy 3-sphere, resulting in a homotopy equivalent non-homeomorphic 3-manifold. This possibility would be ruled out by a successful solution to the Poincaré conjecture. To get around this potential problem we work with *irreducible* 3-manifolds, in which any 2-sphere bounds a ball. Since non-orientable P^2-irreducible 3-manifolds are always Haken, we restrict our attention to the orientable case.

For simplicity of notation, we sometimes follow the convention of not distinguishing between a map of a surface into a manifold M and the image of the map. When it is

†Partially supported by NSF grant DMS-8823009 and the Sloan Foundation.
‡Partially supported by NSF grants DMS-8702519 and DMS-9003974.

necessary to distingish the domain and the image, we will denote the image by F and the domain by Σ, so that $F = f(\Sigma)$.

Definitions 1.1. A (possibly singular) surface F in a 3-manifold M satisfies the *1-line-intersection property* if its pre-image in the universal cover $\tilde{M}$ consists of a collection of embedded planes, any two of which either intersect transversely in a single line or are disjoint. F satisfies the *k-plane property* if in each set of k distinct planes in the pre-image of F in $\tilde{M}$, there is at least one pair which do not intersect.

Note 1.2. The assumption that the pre-images of $f(\Sigma)$ in the universal cover are planes implies that the homomorphism $f_*: \pi_1(\Sigma) \to \pi_1(M)$ is injective and that Σ is not S^2 or P^2. Least area surfaces need not satisfy the 1-line intersection property, but if $f: \Sigma \to M$ has this property then so does any least area map homotopic to f. See Lemma 2.4 for a proof.

The main result of this paper will be proven in §5.

THEOREM 5.2. *Let M be a closed orientable irreducible 3-manifold containing an immersion $f: \Sigma \to M$ satisfying the 4-plane and 1-line-intersection properties. Let M' be a closed irreducible 3-manifold homotopy equivalent to M. Then M' is homeomorphic to M.*

The idea behind our proof of Theorem 5.2 is as follows. Let $\varphi: M \to M'$ be a homotopy equivalence. We show, using techniques from the theory of least area surfaces, that the map $\varphi f: \Sigma \to M'$ is homotopic to a map f' which, like f, satisfies the 1-line-intersection and 4-plane properties. We further show that after homotopies of f and f', we can arrange that $f(\Sigma)$ and $f'(\Sigma)$ are homeomorphic. Finally we show how to extend this homeomorphism to a homeomorphism of M with M'.

Note 1.3. Theorem 5.2 does not assume that M has a finite cover which is a Haken manifold. It generalizes the results of Waldhausen [30], as a Haken manifold trivially satisfies the above hypothesis. It also generalizes the results of Scott [27], as he showed that any closed orientable irreducible Seifert fiber space with infinite fundamental group contains an immersion of the torus satisfying the 4-plane and 1-line-intersection properties. However, we do not give new proofs of their results. Instead we consider exactly the cases of Theorem 5.2 not included in [30] and [27]. Suppose that M is finitely covered by a Haken manifold M_1. Thurston showed that either M_1 is hyperbolic or it has non-empty characteristic submanifold [20]. If M_1 is a Seifert fiber space then the results of [27] show that M is also a Seifert fiber space. If M_1 is neither hyperbolic nor a Seifert fiber space, the equivariance of the characteristic submanifold shows that M is Haken. As we will not consider the case where M is Haken or Seifert fibered, we see that the only remaining case of interest to us is when M_1 is hyperbolic.

Example 1.4. Let M be an irreducible 3-manifold which is double covered by a Haken manifold. Then the projection of any embedded incompressible surface in the Haken cover gives a surface in M which satisfies the 3-plane property. For the sheets of the pre-image in the universal cover of M of such a surface split into two families, each family consisting of embedded mutually disjoint planes. Two out of any three planes in the universal cover must belong to the same family and thus not intersect. Similarly, the projection of an embedded incompressible surface in a 3-fold Haken cover yields a surface in M with the 4-plane property. If in addition the 1-line-intersection property holds, then Theorem 5.2 implies that M is determined up to homeomorphism by its fundamental group. For an example where the second assumption holds, consider the case where M is a hyperbolic manifold which is 2-fold or 3-fold-covered by a Haken manifold containing an embedded totally geodesic surface. The projection of such a surface into M has the 1-line-intersection property as totally geodesic planes in hyperbolic 3-space have this property. Thus these manifolds satisfy the hypothesis of Theorem 5.2.

Example 1.5. Let M be a hyperbolic 3-manifold contaning an immersed totally geodesic surface whose self intersections are at angle $\pi/2$. Many such manifolds can be constructed by taking an all right polyhedron P in hyperbolic 3-space H^3, and letting M be the quotient of H^3 by a finite index torsion free subgroup of the group generated by reflections in the faces of P. Andreev's theorem guarantees the existence of many such polyhedra [2]. The image of a plane containing a face of P yields an immersed surface satisfying the 1-line-intersection property and having all its self intersections at angle $\pi/2$. Such a surface must also satisfy the 4-plane property, as it is impossible to have four planes in H^3 every pair of which meets at an angle $\pi/2$. In fact, the angle condition can be weakened to require that the dihedral angles be strictly greater than that of a Euclidean regular tetrahedron, as will be shown in Lemma 6.1. This angle τ is approximately $0.39183\pi \approx 70.5288°$.

Example 1.6. Recent work of Aitchison and Rubinstein [1] shows that many 3-manifolds admit non-positive curvature cubings. Such manifolds can easily be shown to contain surfaces satisfying the 1-line-intersection and 4-plane properties. An interesting example is the Seifert–Weber manifold, which is obtained by identifying opposite faces of the dodecahedron with a twist of $3\pi/5$. This hyperbolic manifold is not known to be Haken, but satisfies the hypothesis of Theorem 5.2, and thus any irreducible 3-manifold with the same fundamental group is homeomorphic to it. In [1], other examples of manifolds that contain surfaces satisfying the 1-line-intersection and 4-plane properties are constructed by branched covering and by surgery on hyperbolic links.

For curves on a surface there are conditions analogous to the 1-line-intersection and k-plane properties of surfaces in 3-manifolds.

Definitions 1.7. If a collection of (possibly singular) curves on a surface is such that the pre-image in the universal cover consists of embedded lines, any pair intersecting transversely in at most one point, then the collection of curves satisfies the *1-point intersection property*. It satisfies the *3-line property* if in each set of 3 distinct lines, there is at least one pair which do not intersect.

Note 1.8. Note that each curve of such a collection must be homotopically essential. The 1-point-intersection condition is automatically satisfied if the curves in the collection are shortest geodesics for some metric on the surface [5].

The methods of this paper can be used to show that the two conditions of Definition 1.7 determine a unique configuration for the collection of curves. The precise result is the following.

THEOREM 4.2. *Let $\{c_i\}$ be a collection of essential closed curves in general position on an orientable surface Σ, and satisfying the 1-point-intersection and 3-line properties. Let $\{c'_i\}$ be another collection of the same number of essential closed curves on Σ, also in general position, satisfying the 1-point-intersection and 3-line properties, and such that c'_i is homotopic to c_i for each i. Then there is a permutation σ of the indices and an isotopy of Σ carrying $\{c'_i\}$ to $\{c_{\sigma(i)}\}$.*

A similar result holds for a surface in a 3-manifold which satisfies the 1-line-intersection and 3-plane properties. There is a unique possible configuration that it can assume in the 3-manifold. The 4-plane property is weaker, and does not imply a unique configuration, but nonetheless it allows a well defined simplified configuration which is unique.

This paper is organized as follows. In §2 we discuss some basic properties of least area surfaces. In §3 we prove that if $f:\Sigma \to M$ has the 4-plane property and the 1-line-intersection property then f can be homotoped to a map g with the same properties, such that in addition the double curves of g satisfy the 1-point intersection property. Thus g has the least possible number of triple points. In §4 we apply the results of §3 to show that if $\varphi:M \to M'$ is a homotopy equivalence and $f' = \varphi f:\Sigma \to M'$ then we can homotop f and f' so that $f(\Sigma)$ and $f'(\Sigma)$ are homeomorphic. In §5 we prove our main result. In §6 we examine the relation between the 4-plane property and totally geodesic surfaces in hyperbolic manifolds.

§2. PRELIMINARIES AND LEAST AREA SURFACES

Let M denote a 3-manifold and let Σ denote a closed surface. Given an immersion $f:\Sigma \to M$, *the singularity set of* f, $S(f)$, is defined to be the set: $S(f) = \{x \in \Sigma \mid f(x) = f(y)$ for some $y \in \Sigma$, $y \neq x\}$. If f is a general position immersion and Σ and M are oriented then $S(f)$ consists of immersed curves intersecting transversely, which come in pairs that are identified to one another. Double points of these immersed curves on Σ correspond to triple points of the image of the immersion in M. If Σ and M are oriented, then there is an induced orientation on the curves of $S(f)$.

A map $f:\Sigma \to M$ is π_1*-injective* if it induces an injection of fundamental groups. We say that a smooth map $f:\Sigma \to M$ of a surface into a Riemannian manifold is *least area* if the restriction of f to any compact subsurface of Σ minimizes area in its homotopy class, rel boundary. We say that f *factors through a covering* if there exists a surface Σ_2 and a covering $p:\Sigma \to \Sigma_2$ and a map $f_2:\Sigma_2 \to M$ such that $f_2 p$ equals f. If f has the 1-line-intersection property, is in general position, and can be homotoped to factor through a covering p, then p must be of degree two and f_2 is 1-sided. For otherwise, the lift of f to the covering M_2 correspondng to $f_{2*}(\pi_1(\Sigma_2))$ must have double curves. Now the lift of f to the covering M_1 correspondng to $f_*(\pi_1(\Sigma))$ is embedded as the pre-image in the universal cover $\tilde{M}$ is an embedded plane Π. It follows that there is a second plane $g\Pi$ such that Π and $g\Pi$ cross and both project to the lift of f to M_2. This implies that the stabilizers of Π and $g\Pi$ intersect in a closed surface group. But this is impossible since the 1-line intersection property implies that the stabilizers of Π and $g\Pi$ intersect in a cyclic group.

Results of minimal surface theory show that any homotopy class of π_1-injective maps of a compact surface (other than S^2 and P^2) into a compact Riemannian 3-manifold M with $\pi_2(M)$ trivial admits a least area representative, and this least area map is immersed [6, 9, 23, 25, 26]. If M covers a compact manifold M_2 and if $\pi_2(M)$ is trivial then the same result holds. This can be seen by projecting into M_2 and applying the existence result there. One can also work in the piecewise linear category. The definition of a least area surface in the PL category is due to Jaco and Rubinstein [15]. They also prove appropriate existence results, and it makes no difference to the rest of this paper whether one thinks of smooth or PL surfaces.

For the rest of this section we will consider the following situation. See Fig. 1.

Let M be an orientable irreducible 3-manifold and let $f:\Sigma \to M$ be a general position immersion of a closed surface into M which is π_1-injective and has image F. Let $\varphi: M \to M'$ be a homotopy equivalence of M to an irreducible 3-manifold M' and let $f':\Sigma \to M'$ be

$$\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\tilde{f}"] \arrow[dd] & \tilde{M} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \tilde{M}' \arrow[d] & \tilde{\Sigma} \arrow[l, "\tilde{f}'"'] \arrow[dd] \\
 & M_1 \arrow[r, "\varphi_1"] \arrow[d] & M_1' \arrow[d] & \\
\Sigma \arrow[r, "f"] \arrow[ur, "f_1"] & M \arrow[r, "\varphi"] & M' & \Sigma \arrow[l, "f'"'] \arrow[ul, "f_1'"']
\end{tikzcd}$$

Fig. 1.

a general position immersion of Σ in M' with image F', such that f' is homotopic to φf by a homotopy h. Let $\tilde{\varphi}: \tilde{M} \to \tilde{M}'$ be a lift of φ to the universal covers $\tilde{M}$ and $\tilde{M}'$ of M and M' respectively. Let P and P' be lifts of F and F' to the universal covers $\tilde{M}$ and $\tilde{M}'$ such that P' is homotopic to $\tilde{\varphi}(P)$ by a lift $\tilde{h}$ of h to $\tilde{M}'$. We will assume that the planes P and P' are embedded (this is automatic if f and f' are least area or if f and f' have the 1-line intersection property). Let $\operatorname{stab}(P)$ denote the subgroup of the group of covering transformations of $\tilde{M}$ leaving P invariant. Let $\{P_i\} = \{g_i \cdot P\}$ be the set of planes forming the pre-image of F in $\tilde{M}$, where $\{g_i\}$ is a collection of elements of $\pi_1(M)$ which runs over the cosets of $\operatorname{stab}(P)$ in $\pi_1(M)$, and let $\{P_i'\} = \{\varphi_*(g_i) \cdot P'\}$ be the set of planes forming the pre-image of F' in $\tilde{M}'$. Note that f does not factor through a covering, since we have assumed that it is in general position, so that $\operatorname{stab}(P)$ is equal to $f_*\pi_1(\Sigma)$. If f' also does not factor though a covering, we have an equivariant bijection between the set of translates of P in $\tilde{M}$ and the set of translates of P' in $\tilde{M}'$, where the equivariance is with respect to $\pi_1(M)$ acting on $\tilde{M}$ and $\pi_1(M') = \varphi_*\pi_1(M)$ acting on $\tilde{M}'$. But this is not necessarily associated to an equivariant homeomorphism between the unions of the two sets of planes.

We will assume that $P_1 = P$ is stabilized by $f_*\pi_1(\Sigma)$. Let M_1 be the cover of M with fundamental group corresponding to $f_*(\pi_1(\Sigma))$ and M_1' the corresponding cover of M', so that M_1 and M_1' are homotopy equivalent by the lift φ_1 of φ induced by $\tilde{\varphi}$. Let $f_1: \Sigma \to M_1$ be a lift of f to M_1, with image F_1. The f_1 is a homotopy equivalence and $f_1(\Sigma)$ is covered by a single plane in $\tilde{M}$, which we take to be the plane P_1. Note that as P_1 is assumed to be embedded, F_1 must also be embedded. Correspondingly we define f_1'. We fix an orientation on F, calling one side of its normal bundle the positive side and the other side the negative side. We orient F' compatibly via the homotopy of φf and f'. All the covers of F and F' inherit orientations. Note that as φ is a proper map, i.e. the pre-image of a compact set is compact, it follows that any lift of φ is also proper. In particular, $\varphi_1: M_1 \to M_1'$ is proper and so maps the two ends of M_1 to the two ends of M_1'.

The immersion f' is homotopic in M' to φf by a homotopy moving every point along a path of less than some uniformly bounded length. Thus the homotopy $\tilde{h}$ of the plane P' to $\tilde{\varphi}(P)$ moves any point a uniformly bounded distance. The same assertion is true for the homotopy between any covering of F' and the image of the corresponding cover of F under the appropriate lift of φ. As noted at the start of this section, it also follows that if f has the 1-line-intersection property then it cannot be homotoped to factor through a covering except possibly through a 2-fold covering of a 1-sided surface. Note also that a choice of basepoints is implicit in the construction of the lifts φ_1 and $\tilde{\varphi}$ of φ, as well as in the lifts of f and f'.

A key result is the following which is implicit in [27] but not explicitly stated there. We say that *gP crosses P* if gP meets P but does not equal P.

LEMMA 2.2. *Let $\varphi: M \to M'$ be a homotopy equivalence of closed 3-manifolds, let $f: \Sigma \to M$ and $f': \Sigma \to M'$ be least area maps such that f' is homotopic to φf. Let P and P' be* *planes in $\tilde{M}$ and $\tilde{M}'$ chosen as above. Then, for any g in $\pi_1(M)$, the plane gP crosses P if and only if $\varphi_*(g)P'$ crosses P'.*

Remarks. If f and f' do not factor through coverings, so that the stabilizers of P and P' correspond under φ, then we have an equivariant bijection between the translates of P in $\tilde{M}$ and the translates of P' in $\tilde{M}'$. Lemma 2.2 tells us that the intersection patterns of these two families of planes correspond under φ, where the *intersection pattern* of a family of planes is simply the information about which pairs of planes cross.

Proof. Given g in $\pi_1(M)$, we consider the planes P and gP in $\tilde{M}$ and P' and $\varphi_*(g)P'$ in $\tilde{M}'$. The image in M_1 of P is the embedded surface F_1 and the image in M_1 of gP is a possibly singular surface F_2. The images of P' and $\varphi_*(g)P'$ in M'_1 are F'_1 and a possibly singular surface F'_2. Now F_1 separates M_1 into two non-compact pieces and the intersection of F_2 with the closure of each of these pieces must also be non-compact. For otherwise, an exchange can be made between two compact subsurfaces of a pair of least area surfaces, as shown in the proof of Theorem 5.1 of [4], yielding a contradiction. Thus F_2 contains a path joining the two ends of M_1. As φ_1 is proper, it follows that $\varphi_1(F_2)$ contains a path joining the two ends of M'_1. As F'_2 is homotopic to $\varphi(F_2)$ by a homotopy moving points a bounded distance, it follows that F'_2 also contains such a path so that it must meet F'_1. Hence $\varphi_*(g)P'$ must cross P' as required. The converse follows by symmetry.

LEMMA 2.3. *Let $f:\Sigma \to M$ be least area and suppose that P_1 and P_2 are planes in the pre-image in $\tilde{M}$ of $f(\Sigma)$ which cross. Then $P_1 \cap P_2$ is a single line if and only if $stab(P_1) \cap stab(P_2)$ is infinite cyclic. In this situation, P_1 and P_2 must intersect transversely.*

Proof. If P_1 and P_2 intersect in one line then the intersection of their stabilizers must be infinite cyclic, as it stabilizes this line. If $\text{stab}(P_1) \cap \text{stab}(P_2)$ is infinite cyclic, then Lemma 6.5 of [4] shows that P_1 and P_2 intersect transversely in a single line. This finishes the proof of Lemma 2.3.

The next lemma states that we can homotop a surface to a least area surface while retaining the 1-line-intersection and k-plane properties.

LEMMA 2.4. *Let $\hat{f}:\Sigma \to M$ be an immersion of a surface into a Riemannian 3-manifold M such that $\hat{f}$ satisfies the 1-line-intersection and k-plane properties, with $k \geq 3$, and does not factor through a covering. Let $f:\Sigma \to M$ be a least area surface in the homotopy class of $\hat{f}$ which also does not factor through a covering. Then f intersects itself transversely and satisfies the 1-line-intersection and k-plane properties. If $k = 3$ then f is in general position.*

Proof. The pre-image in $\tilde{M}$ of $f(\Sigma)$ consists of a collection of planes $\{P_i\}$ with P_i correspondng to a plane $\hat{P}_i$ above $\hat{f}(\Sigma)$. Since f is a least area map, the planes $\{P_i\}$ in $\tilde{M}$ are least area and embedded [4], [8].

Suppose first that $\hat{P}_1 \cap \hat{P}_2 \neq \phi$, and thus is a single line. Lemma 2.3 shows that their stabilizers intersect in a cyclic group, so that the same must be true of the stabilizers of P_1 and P_2. As Lemma 2.2 shows that P_1 and P_2 must cross, we can apply Lemma 2.3 to deduce that they intersect transversely in exactly one line.

Suppose now that $\hat{P}_1$ and $\hat{P}_2$ are disjoint. Then P_1 and P_2 are disjoint or coincident by Lemma 2.2. It follows from the above that f has the k-plane property and the 1-line-intersection property.

If f satisfies the 3-plane property, then in particular f has no triple points. The picture near a double point of f in M is the same as the picture near a point of intersection of two embedded least area planes P_1 and P_2 in the universal cover $\tilde{M}$. As this intersection is transverse, it follows that f is in general position. This concludes the proof of Lemma 2.4.

Recall that we are considering a π_1-injective map $f:\Sigma \to M$, a homotopy equivalence $\varphi: M \to M'$ and a map f' homotopic to φf. As a result of Lemmas 2.2 and 2.4, it will suffice to assume in the following lemmas that f and f' are least area. Note that least area surfaces not satisfying the 3-plane property are not known to be triangulable, though Meeks and Yau conjecture that they are [21]. If M has an analytic metric, then a least area surface will be analytic [22], and hence triangulable by [18].

LEMMA 2.5. *Let $f:\Sigma \to M$ and $f':\Sigma \to M'$ be least area immersions such that f' is homotopic to φf. If f satisfies the* 1*-line-intersection and k-plane properties then so does f'. Moreover, if P_1 and P_2 are two planes in $\tilde{M}$ which intersect in a line λ, then the corresponding planes P'_1 and P'_2 in $\tilde{M}'$ also intersect in a single line λ', and $\tilde{\varphi}(\lambda)$ and λ' each lie a bounded distance from the other.*

Proof. Lemmas 2.2 and 2.4 shows that f' satisfies the 1-line-intersection and k-plane properties. Now consider two planes P_1 and P_2 which intersect in a line λ. The first assertion of the lemma shows that P'_1 and P'_2 also intersect in a single line, which we call λ'. The circles of intersection $F'_1 \cap F'_2$ and $\varphi(F_1 \cap F_2)$ are homotopic, and so their lifts to the universal cover are a bounded distance from one another, yielding the second assertion of Lemma 2.5.

§3. REDUCING THE NUMBER OF TRIPLE POINTS

In this section we assume that Σ is an orientable closed surface and consider an immersion $f:\Sigma \to M$ which satisfies the 1-line-intersection property and the 4-plane property. We want to minimize the number of triple points of f. In [27], Scott showed how to do this in the case of a torus mapped into certain Seifert fibre spaces. We will extend these results to other surfaces.

As in §2, the pre-image of $f(\Sigma)$ in $\tilde{M}$ consists of translates by $\pi_1(M)$ of a plane P stabilized by $f_*(\pi_1(\Sigma))$. We will consider the curves of intersection of P and its translates. Given a collection of embedded curves γ on a surface P, a *k-gon* on P is a subdisk of P bounded by k embedded subarcs of γ, with the interiors of the subarcs disjoint. We refer to a simple closed curve as a *circle* and a simple non-compact curve as a *line*.

We prove the following.

THEOREM 3.1. *Let M be a closed orientable irreducible* 3*-manifold and let Σ be a closed orientable surface. Let $f:\Sigma \to M$ be an immersion in general position with the* 1*-line-intersection and* 4*-plane properties. Then f is homotopic to an immersion g also in general position and with the* 1*-line-intersection and* 4*-plane properties such that the double curves of g have the* 1*-point-intersection property, i.e. g has the least possible numbe of triple points.*

Remark. The examples in Gulliver–Scott [7] show that it is possible that in some metrics, no least area map homotopic to f minimizes the number of triple points.

Proof. If the double curves of f fail to have the 1-point-intersection property, then there is a 2-gon in P between two double lines. We will show how to homotop f to reduce the number of triple points by two. The homotopy will preserve the intersection pattern (in pairs) of the planes above $f(\Sigma)$ but it may well destroy the 1-line-intersection property of f. Thus the new map will still have the 4-plane property, but if two planes above $f(\Sigma)$ meet then the corresponding planes for the new map may meet in more than one line. Their intersection can only consist of an odd number of lines, together with some circles. We will also show how to remove circle components of intersection. Finally, we will show how to restore the 1-line intersection property. Many of our arguments are drawn from [20] and [27], but as the context is somewhat different we will give some of those arguments. First, we define the complexity of any general position immersion $f: \Sigma \to M$ to be the pair (t, d) where t is the number of triple points on Σ and d is the number of null-homotopic double curves on Σ. Now we consider a general position immersion $f: \Sigma \to M$ which has the 4-plane property and is homotopic to a map with the 1-line intersection property and the same intersection pattern.

LEMMA 3.2. *If there is a disc or 2-gon in P, then there is an innermost disc or 2-gon in P, i.e. a disc or 2-gon in P whose interior does not meet any of the double curves in P.*

Proof. This is essentially in [27], but we give the argument for completeness. Let γ denote the double curves in P. Let D denote either a 2-disc in P bounded by a circle component of γ or a 2-gon in P bounded by two arcs of γ. We will show that if D is not innermost, then D contains a smaller such 2-disc. By repeating this argument, we can then obtain an innermost disc or innermost 2-gon as required.

If the interior of D contains a circle component S of γ, then S bounds a 2-disc in the interior of D yielding a smaller 2-disc. Otherwise there is a component S of γ which crosses ∂D. If ∂D is a circle component of γ, we let υ be a sub-arc of S which is properly embedded in D. Then D contains a 2-gon bounded by υ and a sub-arc of ∂D and we have again found a smaller disc. Otherwise D is a 2-gon bounded by sub-arcs λ and μ of γ. The 4-plane property implies that υ cannot meet both λ and μ. Thus D contains a smaller 2-gon bounded by υ and by a sub-arc of λ or μ. This completes the proof of Lemma 3.2.

LEMMA 3.3. *If there is an innermost disc in P, there is a homotopy of f which reduces (t, d) and leaves the intersection pattern of P and its translates unchanged.*

Proof. Suppose that gP meets P in a circle C which bounds an innermost 2-disc D in P. Then C bounds a 2-disc D' in gP and $D \cup D'$ forms an embedded 2-sphere in $\tilde{M}$ as D is innermost. Let B denote the 3-ball in $\tilde{M}$ bounded by this sphere. We claim that D' projects injectively into the quotient of gP by its stabilizer. The required homotopy of f is then defined by homotoping D' across B and past D. In order to prove the claim we need to show that if h stabilizes gP and hD' meets D' then h must be the trivial element of $\pi_1(M)$. As ∂D equals $\partial D'$ and has no triple points, $\partial D'$ and $h(\partial D')$ must coincide or be disjoint. Thus D' and hD' are disjoint or coincide or one is contained in the other. Either of the last two cases implies that h fixes a point of D and hence that h is the identity which completes the proof of Lemma 3.3.

A key result is the following, which is based on ideas of Rubinstein [24].

LEMMA 3.4. *If D is an innermost 2-gon in P and g is an element of $\pi_1(M)$ such that gD meets D, then g is trivial.*

Proof. Let λ and μ denote the edges of D and let x and y denote the vertices of D. Let l and m denote the double lines which contain λ and μ. Let P_1 denote the other plane containing l and let P_2 denote the other plane containing m.

Now suppose that g is a non-trivial element of $\pi_1(M)$ such that gD meets D. Then g must send one of the vertices of D to one of the vertices of D. As g is not trivial, it cannot fix a point and so gx equals y or gy equals x. Also, as g has infinite order, it cannot interchange x and y. We deduce that $gD \cap D$ is a single point, and after relabeling we can assume that $gx = y = gD \cap D$. It follows that g must preserve the union of the three planes P, P_1 and P_2. We will consider the permutation of these planes included by g.

If g preserves each of the planes P, P_1 and P_2, it must also preserve each of l amd m, as $gx = y$. As g has no fixed points, it must preserve the orientations of l and m. This implies that the restriction of g to P is orientation reversing which is a contradiction as Σ is orientable and f is in general position and so cannot factor through a covering.

If g preserves one plane and interchanges the other two, it is again easy to derive a contradiction. Suppose that P' and P'' are intersecting planes interchanged by an element g of $\pi_1(M)$. The intersection $P' \cap P''$ consists of an odd number of lines together with some circles, and must be preserved by g. As g induces an involution on this collection of lines we see that g must preserve a line l of $P' \cap P''$. The fact that g preserves l and interchanges P' and P'' implies that g acts on a neighborhood of l by a screw motion whose rotation part has order 4. But then g^2 acts reversing orientation on P' and P'' which again contradicts our assumption that Σ is orientable.

Finally, if g cyclically permutes the three planes, we consider the induced action of Z_3 on the quotient of $\tilde{M}$ by the cyclic group generated by g^3. In this quotient, the images of the three planes are three embedded annuli A, A_1 and A_2 and D projects injectively into A, as g^3D cannot meet D since g^3 fixes each of P, P_1, P_2. We again use g to denote the generator of the action of Z_3 and use D to denote the image of D in A. We are assuming that gD meets D and that g cyclically permutes the annuli A, A_1 and A_2. Thus gD cannot equal D nor can $g\lambda$ equal λ or $g\mu$ equal μ. If $g\lambda$ equals μ then g^2 must fix each of x and y contradicting the fact that we have a free action of Z_3. Similarly we cannot have $g\mu$ equal to λ. If $gD \cap D = \{x, y\}$, then g must fix x and y again contradicting the freeness of our action. It follows that we must have $gD \cap D = \{x\}$ or $\{y\}$. Without loss of generality we can assume that $gD \cap D = \{y\}$ and that $gx = y$. Now we will argue as at the end of the proof of Lemma 6.6 of [20] where consideration of Figs 6.7(a) and (b) leads to a contradiction. This completes the proof of Lemma 3.4.

LEMMA 3.5. *If there is an innermost 2-gon in P, then we can homotop f so as to reduce its complexity (t, d), while preserving the intersection pattern of P and its translates.*

Proof. Move one edge of the 2-gon across the other. This move can be made equivariant by Lemma 3.4 and so corresponds to a homotopy of f.

The previous lemmas imply that if we have a map $f: \Sigma \to M$ with the 1-line intersection property and the 4-plane property then we can homotop f to a map f_2, still with the 4-plane property, and such that there are no trivial double curves and no 2-gons on P. However, f_2 may not have the 1-line-intersection property. If P and P_1 are planes above $f(\Sigma)$ which meet in one line, then the corresponding planes above $f_2(\Sigma)$ may meet in any odd finite number of lines. To cope with this, we need to prove a result which was not needed in [20]. Repeated application of Lemma 3.6 will complete the proof of Theorem 3.1.

LEMMA 3.6. *Suppose that f_2 is as above. Suppose that $P \cap P_1$ consists of more than one line. Then we can homotop f_2 to reduce the total number of double curves while preserving the* *intersection pattern of P and its translates and preserving the absence of trivial double curves and 2-gons.*

Before starting on the proof of Lemma 3.6, we need some definitions.

If P is a plane in $\tilde{M}$ above $f(\Sigma)$ and l and m are disjoint double curves in P with the same stabilizer, we say that l and m are parallel and the region between them in P is called a *strip*. If the union of n strips in planes above $f(\Sigma)$ forms the boundary of a region Y in $\tilde{M}$, we say that Y is an *n-gon prism region.* Such a region is homeomorphic to the product of an n-gon with R where the strips correspond to the products of the edges with R and the boundary lines of the strips correspond to the products of the vertices with R. A prism region is *innermost* if it contains no other prism regions. A prism region Y is $\pi_1(M)$-equivariant if gY equals Y or is disjoint from Y, for all g in $\pi_1(M)$.

LEMMA 3.7. *Suppose that $f_2:\Sigma \to M$ has the 4-plane property and is homotopic to $f:\Sigma \to M$ which has the 1-line-intersection property and the same intersection pattern. Suppose that no planes above $f_2(\Sigma)$ contain 2-gons or discs.*

(a) *If $P \cap P_1$ consists of more than one line, then there is a 2-gon prism region Y between P and P_1 in $\tilde{M}$.*

(b) *If Y is a $\pi_1(M)$-equivariant 2-gon prism region in $\tilde{M}$ between planes P and P_1, there is a homotopy of f_2 which reduces the total number of double curves while preserving the intersection pattern and the absence of trivial double curves and 2-gons.*

Proof. (a) Let α denote a generator of the common stabilizer of P and P_1, and let M_α denote the quotient of $\tilde{M}$ by this common stabilizer. The images of P and P_1 in M_α are embedded annuli A and A_1 meeting in at least two essential circles. It follows that there are compact annuli B and B_1 in A and A_1 such that $B \cap B_1 = \partial B = \partial B_1$ and $B \cup B_1$ bounds a solid torus region X in M_α whose interior does not meet A or A_1. Then a component Y of the pre-image of X in $\tilde{M}$ is a 2-gon prism region between P and P'.

(b) Suppose that Y is a $\pi_1(M)$-equivariant 2-gon prism region in $\tilde{M}$ between planes P and P_1. Then we can define the required homotopy of f_2 as follows. Choose a homotopy of A supported on a small neighborhood of B which homotops B across X and thus removes two circles of intersection of A and A_1. This induces an α-equivariant homotopy of P in $\tilde{M}$, supported on a small neighborhood of $\tilde{B}$, a component of the pre-image of B. We extend to a $\pi_1(M)$-equivariant homotopy of P and its translates which is the identity except on a small neighborhood of all translates of $\tilde{B}$. This can be done because of our assumption that Y is $\pi_1(M)$-equivariant.

Now we need to analyze the situation when we may have a non-equivariant 2-gon prism region.

LEMMA 3.8. *Let X be a n-gon prism region bounded by strips $S_1, \ldots, S_n$ contained in planes $P_1, \ldots, P_n$. Let l_i denote the line of intersection of S_i with S_{i+1}, $1 \leq i \leq n$, where S_{n+1} is defined to equal S_1 and P_{n+1} is defined to equal P_1. If there are no 2-gons in any of the planes above $f(\Sigma)$ then either there are no triple points on the l_i's, or there is a plane P which meets each of $P_1, \ldots, P_n$. If n equals 2 or 3, there are no triple points on the l_i's.*

Proof. Let P be a plane in $\tilde{M}$ above $f(\Sigma)$ which meets some l_i and is not P_i or P_{i+1}. Then $P \cap P_i$ is a finite union of parallel lines. Let α denote a generator of the common stabilizer of these lines, and let M_α denote the quotient of $\tilde{M}$ by the cyclic group generated by α. Then the image of each of P and P_i in M_i is an embedded annulus. We denote these annuli by A and

A_i respectively. Then $A \cap A_i$ consists of the image of $P \cap P_i$, so is a finite union of essential simple closed curves. The image of l_i in A_i must be a line joining the two ends of A_i as l_i crosses one of the lines forming $P \cap P_i$ and intersects it in only one point. It follows that the image of l_i in A_i meets each circle of $A \cap A_i$, so that l_i must meet each of the lines of intersection of P and P_i. As l_{i-1} lies in P_i and is parallel to l_i, we see also that each line of $P \cap P_i$ meets l_{i-1}. By repeating this argument, we see that P meets each l_j. In the case when $n = 3$, this is impossible as P, P_1, P_2, P_3 would all meet each other violating the 4-plane property. In the case when $n = 2$, we see that $P \cap S_1$ and $P \cap S_2$ each consist of a finite number of line segments joining l_1 to l_2. This is because $P \cap P_1$ and $P \cap P_2$ each consist of a finite number of lines and there are no 2-gons between these lines, l_1 and l_2. Thus these segments must bound 2-gons in P, again contradicting our hypothesis.

Remark. It follows from the above that if the planes above $f(\Sigma)$ contain no 2-gons between double lines and X is an innermost n-gon prism region with no triple points on its edges, then X is actually the closure of a component of $\tilde{M} - N(\tilde{F})$. For if a plane P meets X, then the above result shows that P meets none of the l_i's. It follows that P meets ∂X in lines parallel to the l_i's and hence cuts X into two prism regions showing that X was not innermost.

The above arguments show that if Y is a 2-gon prism region which contains no other such region then each plane which meets Y other than P_1 and P_2 must cut Y into two 3-gon prism regions. Any pair of distinct such planes must be disjoint by the 4-plane property so that the planes which meet Y must cut it into two 3-gon prism regions Z and Z' and some 4-gon prism regions. Let $n(Y)$ denote the number of strips across Y, so that if $n(Y)$ is non-zero then there are $n(Y) - 1$ 4-gon prism regions in Y.

LEMMA 3.9. *If a 2-gon prism region Y has $n(Y) = 0$, then Y is $\pi_1(M)$-equivariant.*

Proof. Let P_1 and P_2 be the planes which bound Y with S_i denoting the strip $P_i \cap Y$. Let l_1 and l_2 denote the two lines of $S_1 \cap S_2$. If gY meets Y, we must have gl_1 or gl_2 equal to l_1 or l_2. Thus g preserves $P_1 \cup P_2$. But g cannot interchange P_1 and P_2 as shown in the proof of Lemma 3.4. Thus g preserves P_1 and P_2 and hence preserves any line of $P_1 \cap P_2$. In particular, $gl_1 = l_1$ and $gl_2 = l_2$ and so $gY = Y$ as required.

LEMMA 3.10. *Suppose that there is a 2-gon prism region Y which contains no other such region and that $n(Y)$ is non-zero. Then there is a homotopy of f_2 which*

(a) *preserves the total number of double curves of f_2*
(b) *preserves the intersection pattern of P and its translates*
(c) *preserves the absence of trivial double curves and 2-gons, and*
(d) *produces a 2-gon prism region Y' with $n(Y') < n(Y)$.*

If there is a non innermost 2-gon prism region, there is such a region Y which contains no other 2-gon prism regions. By applying Lemma 3.10 repeatedly, we can eventually obtain an innermot 2-gon prism region and then reduce the total number of double curves by applying Lemmas 3.7 and 3.9. This then completes the proof of Lemma 3.6 and hence of Theorem 3.1.

In order to prove Lemma 3.10, we will need the following result.

LEMMA 3.11. *Let Z be an innermost 3-gon prism region. Then for all g in $\pi_1(M)$, if $gZ \cap Z$ contains a strip then $gZ = Z$.*

Proof. Assume that $gZ \neq Z$. If $gZ \cap Z$ contains a strip, then g must preserve $P_1 \cup P_2 \cup P_3$. As g cannot interchange intersecting planes, g must preserve each P_i or cyclically permute them. Thus g^3 preserves each P_i and hence each double curve, so $g^3Z = Z$. But then g^2Z must have a strip in common with Z and gZ which is impossible, as Z and gZ lie on opposite sides of the plane P_i which contains their common strip. This completes the proof of Lemma 3.11.

Proof of Lemma 3.10. Let P_1 and P_2 be the planes which bound Y, and let S_i denote the strip $P_i \cap Y, i = 1, 2$. Let Z and Z' be the two 3-gon prism regions in Y and let l and l' denote the lines of $S_1 \cap S_2$ with l in Z and l' in Z'. Let $S_3, \ldots, S_r$ denote the strips across Y where $r = n(Y) + 2$, and let P_i be the plane contaning S_i. Note that it is possible that P_i and P_j can coincide for some i and $j \geq 3$. Let m and n denote $S_3 \cap S_2$ and $S_3 \cap S_1$ respectively. Finally, denote the 4-gon prism region in Y between S_i and S_{i+1} by U_i.

Case $n(Y) = 1$. Thus Y equals $Z \cup Z'$. We first show that at least one of Z and Z' is equivariant. Suppose that gZ meets Z and does not equal Z. Then $gZ \cap Z$ equals l, m or n. If both gZ and $g^{-1}Z$ meet Z in l, we obtain a contradiction. For this would imply that gl equals l, so that g preserves each of P_1 and P_2 and each of their intersection lines and hence $gZ = Z$. Thus by replacing g by its inverse if necessary we can suppose that gZ meets Z in m or n. There is no difference in the roles of P_1 and P_2, so we will assume that $gZ \cap Z = m$. This means that gZ meets Z' in a strip and that the boundary of gZ is formed of strips from the planes P_1, P_2 and P_3. See Fig. 2. (Note that, in this and later figures, planes are represented by lines, and n-gon prism regions are represented by n-gons. Thus the true situation consists of the product of the figure with the real line R.) Thus g preserves the union of these three planes. As in the proof of Lemma 3.4 it follows that g cyclically permutes the three planes, that g^3 is a power of α and hence that g^3 preserves each of the lines of intersection of these three planes. In particular, $g^3Z = Z$ and g^2Z must meet Z and gZ in an edge.

Now we consider the permutation of the intersecton lines induced by g. We cannot have $m = gm$ as this would imply that g preserves $P_2 \cup P_3$, contradicting the fact that g permutes the three planes cyclically. If $m = gl$, then g sends P_1 to P_2 to P_3. This implies that gm lies in $P_3 \cap P_1$ and hence must lie as shown in Fig. 3. But, as g^2Z contains gm, this implies that g^2Z cannot meet Z, a contradiction. Thus $m = gn$, and g sends P_2 to P_1 to P_3. This means that gm lies in $P_2 \cap P_1$ and so gm equals l'. It follows that g preserves $m \cup n \cup l'$ as this is the orbit of m under g, and hence gZ' equals Z'. See Fig. 4.

We conclude that if there exists g such that gZ meets Z but does not equal Z, then $gZ' = Z'$ and the double lines on P_3 adjacent to m and n lie on P_1 and P_2 adjacent to l'.

If there exists h such that hZ' meets Z' but does not equal Z', then the same argument shows that the double lines on P_3 adjacent to m and n lie on P_1 and P_2 adjacent to l. It

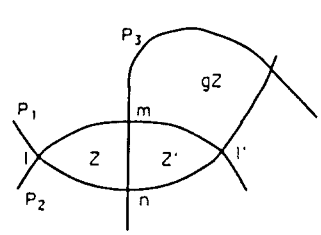

Fig. 2.

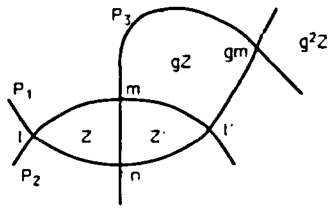

Fig. 3.

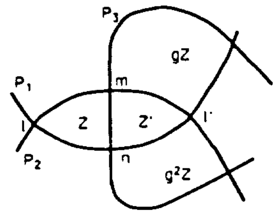

Fig. 4.

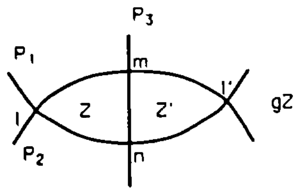

Fig. 5.

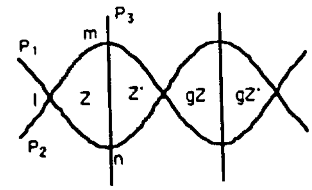

Fig. 6.

follows that at least one of Z and Z' must be equivariant as claimed. We will suppose that Z is equivariant. Now we claim the stronger result that if gZ meets Y then gZ must equal Z.

If gZ meets Y but is not equal to Z then gZ cannot meet S_1 and S_2 except in l' as gZ cannot meet Z. Thus gZ must be as shown in Fig. 5. Lemma 3.11 shows that gZ' cannot meet Z' in a strip. As gZ' meets gZ in a strip, we deduce that gZ' must be as shown in Fig. 6. But this implies that gY is bounded by P_1 and P_2 so that g must preserve $P_1 \cup P_2$ and hence preserve each of these planes and each of their double curves. This contradicts our hypothesis that gZ is not equal to Z and so proves the claim.

Now we can complete the proof of Lemma 3.10 in the case $n(Y) = 1$. We isotop one of the strips forming the boundary of Z across Z and extend to an equivariant homotopy of f_2. Because no translate of Z meets Y the new map has an innermost 2-gon region Y' in place of Y. Note that this homotopy of f_2 will not alter the total number of double curves.

Case $n(Y) > 1$. Suppose that there is g such that gZ meets Y and is not equal to Z. Then gZ equals Z' or gZ does not lie inside Y. In the second case, we see that gZ cannot have a strip in common with any U_i as the two regions U_{i+} and U_{i-} which meet U_i in a strip and do not lie in Y cannot be 3-gon prism regions. For if P_i and P_{i+1} are distinct then the 4-plane property implies that they are disjoint so that U_{i+} and U_{i-} could not be 3-gon prism regions. Lemma 3.11 shows that gZ cannot meet Z in a strip. We conclude that gZ must equal Z' or $gZ \cap Z$ equals l or gZ meets Z' and is not contained in Y.

Suppose that $gZ \cap Z$ equals l. Then $g^{-1}Z$ must also meet Z and the above argument applies to show that $g^{-1}Z \cap Z$ equals l. Thus it follows that gl equals l. As in the first case of this Lemma, this is a contradiction.

If gZ meets Z' in a strip, then $Z \cup Z'$ forms a 2-gon prism region Y' with $n(Y') = 1$, so that the result of Lemma 3.10 is true with the trivial homotopy of f_2.

If gZ meets Z' in l', we obtain a contradiction as follows. We know that gU_3 meets gZ in a strip. If that strip is disjoint from Z', then gY must be bounded by $P_1 \cup P_2$. Thus g must preserve P_1 and P_2 and so gZ equals Z, contradicting our hypothesis. Thus gU_3 must meet Z' in a strip. It follows that gU_4 meets U_{r-1} in a strip and, by induction, that gU_s meets U_{r-s+3} in a strip, for $4 \leq s \leq r-1$. See Fig. 7. Hence gZ' would have to meet U_3 in a strip, which is impossible as U_{3+} and U_{3-} cannot be 3-gon prism regions.

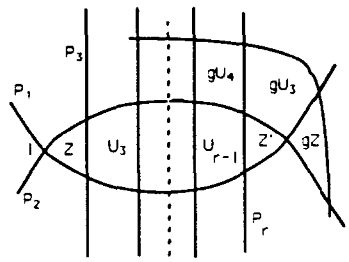

Fig. 7.

We conclude that if gZ meets Y then gZ must equal Z or Z'. If the only translate of Z which meets Y is Z itself, then we can isotop S_3 across Z and extend to an equivariant homotopy of f_2 which will reduce $n(Y)$ by one. Note that it is important that no other translate of Z meets Y as such a translate might yield an increase in $n(Y)$ by one, cancelling out our reduction. If there are two translates of Z which meet Y, namely Z and Z', we perform the same homotopy of f_2 and this will reduce $n(Y)$ by two. This completes the proof of Lemma 3.10 and hence of Theorem 3.1.

§4. SURFACES IN 3-MANIFOLDS

In this section we consider the image of a map $f:\Sigma \to M$ wth the 1-line-intersection property. We show that the image is unique up to homeomorphism if f also satisfies the 3-plane-property. If f satisfies the 4-plane property, its image need not be unique up to homeomorphism, but we show how to homotop f to have canonical image by using the main result of §3.

Throughout this section, let M be an orientable irreducible 3-manifold and let $f:\Sigma \to M$ be a general position immersion which is π_1-injective of a closed orientable surface into M with image F. Let $\varphi: M \to M'$ be a homotopy equivalence of M to an irreducible 3-manifold M' and let $f':\Sigma \to M'$ be a general position immersion of Σ in M' with image F', such that f' is homotopic to φf by a homotopy h. In the following we will use the notation of §2 to label the various manifolds and their covers, as in Fig. 1. We say that two 2-complexes are *isomorphic* if there is a homeomorphism from one to the other preserving the cell structure.

LEMMA 4.1. *Suppose that f and f' are least area maps. Suppose that $f:\Sigma \to M$ satisfies the 1-line-intersection and 3-plane properties. Then the 2-complex formed by the union of the planes $\{P_i\}$ in $\tilde{M}$ is equivariantly isomorphic to the 2-complex formed by the union of the planes $\{P'_i\}$ in $\tilde{M}$.*

Proof. Recall from Lemma 2.2 that f' also satisfies the 1-line-intersection and 3-plane properties. Note that as f and f' are assumed to be in general position, they cannot factor through coverings. A least area map which is homotopic to a map which factors through a covering of an orientable surface must itself factor through such a cover. Thus if f is homotopic to a map which factors through a covering and is in general position, then it can only factor through a double cover of a non-orientable surface [4]. Also recall from the remarks immediately after Lemma 2.2 that there is an equivariant bijection between the planes in $\tilde{M}$ and those in $\tilde{M}'$.

First we consider the case when f is not homotopic to a degree two cover of a 1-sided surface.

If P_1, P_2 are two disjoint planes such that P_2 lies on the positive side of P_1, we will show that P'_2 lies on the positive side of P'_1 in $\tilde{M}'$. In the covering spaces M_1 and M'_1 the planes P_2 and P'_2 project to surfaces F_2 and F'_2, disjoint from F_1 and F'_1 respectively. The lift φ_1 of φ which maps M_1 to M'_1 sends the positive and negative ends of M_1 to the positive and negative ends of M'_1 respectively, by our choice of orientations. Hence if P_2 is a plane in $\tilde{M}$ on the positive side of P_1 such that F_2 is not compact then $\varphi_1(F_2)$ has its ends on the positive side of F'_1. As $\varphi_1(F_2)$ is homotopic to F'_2 by a homotopy moving points a bounded distance, it follows that F'_2 has its ends on the positive side of F'_1. As F'_2 is disjoint from F'_1, we see that P'_2 is on the positive side of P'_1 as required.

If P_2 and P'_2 project to compact surfaces F_2 and F'_2, in M_1 and M'_1, then F_2 and F'_2 are disjoint parallel copies of F_1 and F'_1 respectively. The surfaces F_1 and F_2 are disjoint because P_1 and P_2 are disjoint. The surface F_2 is embedded in M_1 by Theorem 5.4 of [4], which implies that if a least area map is homotopic to a covering of a 2-sided embedded surface, then it is a covering of an embedded surface. Recalling that $P_2 = g_2 P_1$, in this case g_2 lies in the normalizer of stab(P_1) and stab(P'_1), and thus acts on M_1 and M'_1 so as to carry F_1 to F_2 and F'_1 to F'_2. Iterating this action carries F_1 to one of the two ends of M_1 and F'_1 to one of the two ends of M'_1 unless g_2 interchanges the two ends of M_1 or M'_1. The last case can not occur by our assumption that f is not homotopic to a degree two cover of a 1-sided surface. The end in question is determined by a lift of the loop g_2 to M_1 and M'_1, and thus if $g_2 F_1$ lies on the positive side of F_1 then $g_2 F'_1$ lies on the positive side of F'_1. It follows in both cases that if P_2 lies on the positive side of P_1 then P'_2 lies on the positive side of P'_1.

Now let P_1 and P_2 be intersecting planes in $\tilde{M}$. Lemmas 2.2 and 2.3 imply that P'_1 and P'_2 also intersect in a line. We consider all the planes P_i which intersect P_1. Note that the 3-plane property implies that any two planes, each of which meets P_1, are disjoint and hence that the lines of intersection of $P_1 \cap \{\cup P_i\}$ can not cross. Lemma 2.2 implies that the same is true for the planes $\{P'_i\}$. It follows that the pattern of lines of intersection on P_1 of $P_1 \cap \{\cup P_i\}$ is identical to that of $P'_1 \cap \{\cup P'_i\}$ on P'_1.

An equivariant homeomorphism from the 2-complex formed by the union of the planes $\{P_i\}$ in $\tilde{M}$ to the 2-complex formed by the union of the planes $\{P'_i\}$ in $\tilde{M}'$ can now be constructed inductively.

Next we consider the case when f is homotopic to a degree two cover of a 1-sided map $f_2: \Sigma_2 \to M$. The preceding proof needs elaboration, because the normalizer $N(\pi_1(\Sigma))$ of $\pi_1(\Sigma)$ in $\pi_1(M)$ contains elements which interchange the ends of the covering M_1 of M corresponding to $\pi_1(\Sigma)$. When $N(\pi_1(\Sigma))$ is not equal to $\pi_1(\Sigma)$, there is no canonical equivariant bijection between the planes above $f(\Sigma)$ in $\tilde{M}$ and those above $f'(\Sigma)$ in $\tilde{M}'$, but given a plane P in $\tilde{M}$ above $f(\Sigma)$ the bijection is uniquely determined by a choice of a corresponding plane P' in $\tilde{M}'$. We need to specify how the choice of P' is to be made. Note that if the quotient group $N(\pi_1(\Sigma))/\pi_1(\Sigma)$ is Z then there are infinitely many planes parallel to P in $\tilde{M}$, i.e. having the same stabilizer as P. There is correspondingly an infinite number of choices for P'. However, in this case no element of $N(\pi_1(\Sigma))$ interchanges the ends of M_1 so any choice will do as in the preceding argument. Standard results in 3-manifold theory show that the quotient group $N(\pi_1(\Sigma))/\pi_1(\Sigma)$ is one of $1, Z, Z_2$, or $Z_2 * Z_2$ [13] [28]. In the last case the quotient W of $\tilde{M}$ by $N(\pi_1(\Sigma))$ is the union of two twisted 1-bundles over closed surfaces, glued along Σ. This again presents the problem that in $\tilde{M}$ there are infinitely many parallel planes above Σ. We handle this by noting that the corresponding quotient W' of $\tilde{M}'$ is homeomorphic to W by Waldhausen [30] as they are Haken manifolds. This homeomorphism lifts to a $N(\pi_1(\Sigma))$-equivariant homeomorphism of $\tilde{M}$ wth $\tilde{M}'$, giving a bijection between the translates of P by $N(\pi_1(\Sigma))$ and some of the planes in $\tilde{M}'$ above $f'(\Sigma)$. Let P' correspond to P under this bijection, and choose corresponding orientations. If g lies in $N(\pi_1(\Sigma))$ and gP is on the positive side of P, then gP' lies on the positive side of P'. Now the bijection between all translates of P and P' yields an equivariant homeomorphism as in the first part of the proof.

If $N(\pi_1(\Sigma))/\pi_1(\Sigma)$ is Z_2, we need a slight modification of this argument as the quotients of $\tilde{M}$ and $\tilde{M}'$ by $N(\pi_1(\Sigma))$ are not known to be homeomorphic. Let P and gP denote the two planes in $\tilde{M}$ stabilized by $\pi_1(\Sigma)$. Note that $g^2 P = P$. There are two corresponding planes in $\tilde{M}'$ and we need to choose which one corresponds to P. To do this consider the quotient M_1 of $\tilde{M}$ by $\pi_1(\Sigma)$ and the corresponding quotient M'_1 of $\tilde{M}'$. Our homotopy equivalence of M with M' determines a correspondence between the two ends of M_1 and the two ends of M'_1. P and gP project to disjoint embedded surfaces F and gF in M_1. Let e denote the end of $M_1 - gF$ which contains F, i.e. the end closer to F. We choose P' so that the corresponding end of $M'_1 - gF'$ contains F'. We also choose the orientation on P', which is not determined by the homotopy equivalence in this case, to ensure that gP is on the positive side of P if and only if gP' is on the positive side of P'. This allows us to complete the proof of Lemma 4.1.

THEOREM 4.2. *Let $\{c_i\}$ be a collection of essential closed curves in general position on an orientable closed surface Σ, satisfying the 1-line-intersection and 3-line properties. Let $\{c'_i\}$ be another collection of the same number of essential closed curves on Σ, also in general position, satisfying the 1-line-intersection and 3-line properties, and such that c'_i is homotopic to c_i for each i. Then there is a permutation σ of the indices and an isotopy of Σ carrying $\{c'_i\}$ to $\{c_{\sigma(i)}\}$.*

Remark. The same result also holds in the non-orientable case. The proof needs much the same idea as the second part of the proof of Lemma 4.1.

Proof. Let C denote the union of $c_1, \ldots c_n$ and C' the union of $c'_1, \ldots, c'_n$ and let $N(C)$, $N(C')$ denote regular neighborhoods of C and C'. First consider the case when $n = 1$. We argue as in the proof of Lemma 4.1. Let f and f' be homotopic immersions of S^1 into Σ. If f were homotopic to a map which factors through a covering of circles and $f_2: S^1 \to \Sigma$, then the covering would be of degree 2 and f_2 would be one-sided. As Σ is orientable, f cannot be homotopic to such a map. Thus the first part of the proof of Lemma 4.1 shows that there is a homeomorphism h of $N(C)$ to $N(C')$ which induces the identity on the images of their fundamental groups in $\pi_1(\Sigma)$. This homeomorphism extends to a homeomorphism of Σ which induces the identity on $\pi_1(\Sigma)$. For if a boundary curve S of $N(C)$ is inessential in Σ the corresponding curve S' in $\partial N(C')$ will also be inessential, so that h extends to the union of $N(C)$ with the 2-disk bounded by S. After repeating, we can assume that all components of $\partial N(C)$ are essential. This implies that $\partial N(C)$ is homotopic to $\partial N(C')$ and hence isotopic to $\partial N(C')$, so the rest of the extension problem is clear. The resulting homeomorphism of Σ is homotopic to the identity and hence isotopic to the identity. This provides the required ambient isotopy of Σ.

For $n > 1$, the proof that disjoint lines in the universal cover which correspond to covers of the curves in C and C' lie on determined sides of one another may fail. But it fails only when two of these lines project to distinct but homotopic curves. In this case we may have to do some relabeling. If Σ is not the torus, a given line in the universal cover can only be parallel to finitely many other lines. Thus the required re-ordering of the labels is clear. If Σ is the torus, we will have infinitely many lines all parallel in the universal cover even in the case $n = 1$. In this case we can argue as follows. As each c_i is not homotopic to a proper power in $\pi_1(\Sigma)$, we see that each c_i is a simple closed curve. The 3-line property implies that there are at most two isotopy classes of the c_i's. Thus either the c_i's are all disjoint or they divide Σ into quadrilaterals. The same comments apply to the c_i' family, so that the required result is clear.

THEOREM 4.3. *Suppose that f and f' are least area maps in general position and that $f:\Sigma \to M$ satisfies the 1-line-intersection and 4-plane properties. Then f and f' can be homotoped so that the 2-complex formed by the union of the planes $\{P_i\}$ in $\tilde{M}$ is equivariantly isomorphic to the 2-complex formed by the union of the planes $\{P_i'\}$ in $\tilde{M}'$.*

Proof. As in the proof of Lemma 4.1, we consider the case when f is not homotopic to a degree two cover of a 1-sided surface. We can extend to the general case by using the same arguments as in the second part of the proof of Lemma 4.1.

Theorem 3.1 tells us that we can homotop f and f' preserving the 1-line-intersection and 4-plane properties so that the double curves have the 1-point-intersection property. We claim that for a surface in general position in M satisfying all these conditions, the existence of a 3-gon prism region in $\tilde{M}$ is equivalent to the existence of three planes which meet pairwise and whose stabilizers have a common infinite cyclic subgroup. This is immediate from the definitions if $\tilde{M}$ contains a 3-gon prism region. Conversely, suppose that $\tilde{M}$ contains planes P_1, P_2 and P_3 such that P_i meets P_j in a line l_{ij} with a common stabilizer generated by α. Then each pair of these lines must be disjoint or meet in infinitely many points. As we are assuming that the double curves of f have the 1-point-intersection property, these lines must be disjoint or coincide. As f is in general position they must be disjoint, so that there is a 3-gon prism region in $\tilde{M}$ bounded by P_1, P_2 and P_3. Now it follows that the absence of 3-gon prism regions in $\tilde{M}$ implies the absence of 3-gon prism regions in $\tilde{M}'$.

For clarity of presentation, we will assume temporarily that there are no 3-gon prism regions in $\tilde{M}$. This assumption can often be shown to hold in specific situations, such as those of Examples 1.5 and 1.6, and makes the completion of the proof much simpler. For we will see that this implies that the unions of the families of planes in $\tilde{M}$ and $\tilde{M}'$ are equivariantly homeomorphic without further homotopies of f and f'. After handling this case we will return to the case where there are 3-gon prism regions and show how to deal with the additional complications.

We begin with a plane P_1 and consider a plane P_2 which crosses it in a line λ. The corresponding planes P_1' and P_2' also cross in a line λ'. If any other planes cross λ transversely, then the order in which they do so is the same as that with which the corresponding planes of $\{P_i'\}$ cross λ', as any two planes crossing λ are disjoint by the 4-plane property, and we saw in Lemma 4.1 that the order of disjoint planes is the same in $\tilde{M}$ and $\tilde{M}'$. Thus we can construct a homeomorphism of a neighborhood of λ taking planes P_i in $\tilde{M}$ to the corresponding planes P_i' in $\tilde{M}'$. This can then be extended, line by line, to a neighborhood of the component of $P_1 \cap \{\cup P_i\}$ containing λ.

Next suppose that P_3 is a plane intersecting P_1 in a line μ disjoint from λ. If P_3 is disjoint from P_2 then it lies either on the positive or negative side of P_2 and the corresponding statement holds for P_3' and P_2' in M'. If P_3 does meet P_2 then it does so in a line ν disjoint from P_1. The planes P_1', P_2' and P_3' meet similarly in disjoint lines $\lambda', \mu', \upsilon'$. Consider the quotient M_2 of M by the stabilizer of P_2. Each of P_1 and P_3 project to cylinders running from one end of M_2 to the other. The intersection line μ of P_1 and P_3 projects to either a circle or a line in M_2, according to whether the stabilizers of the three planes have cyclic or trivial intersection. If μ projects to a circle, then there is a 3-gon prism region in M, because P_1, P_2 and P_3 intersect pairwise and have stabilizers which intersect in an infinite cyclic group. This contradicts our assumption that there are no 3-gon prism regions. If the intersection line μ projects to a line in M_2, then this line is contained in one end of M_2. It does not lie within a finite distance of the projection of P_2 to M_2, and thus μ does not lie within a finite distance of λ. By Lemma 2.5, μ' lies on the positive side of λ', since λ' lies within a finite distance of $\varphi(\lambda)$. Thus we can extend the homeomorphism to the component of $P_1 \cap \{\cup P_i\}$ containing μ. Continuing, we get a homeomorphism of P_1 to P'_1 taking the intersections of planes P_i and P_1 in M to the corresponding intersections of planes P'_i and P'_1 in M'. We can now extend to planes $P_2, P_3, \ldots$ obtaining an equivariant homeomorphism of the two 2-complexes as desired.

Finally we consider the situation where 3-gon prism regions may exist in $\tilde{M}$. If a 3-gon prism region exists, we aim to collapse this region to a line. This move is rather different from our previous moves in that repetition of it may destroy the 4-plane property and 1-line-intersection properties by causing two planes in $\tilde{M}$ to intersect which were previously disjoint. This could occur if one of the edge lines of the prism region in $\tilde{M}$ has a curve of triple points on its boundary from a previous collapse, leadng to a curve of quadruple points when the prism region is collapsed. Two previously disjoint planes now meet non-transversely along this line. As this collapsing process is repeated, circles of k-tuple points may be created in M.

Our collapsing process has to be defined carefully, as it is not clear that even one collapse can be made. For technical reasons, it seems simplest to collapse n-gon prism regions for all values of n. We restrict our attention to those prism regions whose edges contain no triple points. We call such a prism region a *good prism region*. Any plane which meets the interior of a good prism region divides it into two good prism regions. Note that under the equivariant bijection between the planes in $\tilde{M}$ and in $\tilde{M}'$, n-gon prism regions correspond and good n-gon prism regions correspond. This is by the same reasoning by which we showed that 3-gon prism regions in $\tilde{M}$ correspond to 3-gon prism regions in $\tilde{M}'$.

Recall that a prism region is innermost if it contains no other prism region. Let X be an innermost good prism region in $\tilde{M}$. Then X is the closure of a component of $\tilde{M}$ minus the planes in $\tilde{M}$. Thus the image in M of X is the closure of a component of $M - f(\Sigma)$. It will be called a *compact prism region* in M. As translates of X can only meet X in some union of strips and edges in ∂X, it follows that a regular neighborhood of a compact prism region in M is a Seifert fiber space in which the images of the double lines are fibers. If we let N denote a regular neighborhood of the union of all the compact prism regions in M, it follows that N is a Seifert fiber space, not necessarily connected. Note that the pre-image in $\tilde{M}$ of N is a regular neighborhood of the union of all the good prism regions, not just the good innermost regions.

As M is not a Seifert fiber space, N cannot equal M. As M is not Haken, each component of ∂N is compressible in M. Each component of ∂N carries a loop essential in M, namely a fiber of the Seifert fibration of N. Thus no component of ∂N can lie in a ball. Hence each component of ∂N bounds a solid torus in M. Let N_0 denote a component of N and let T be a component of ∂N_0. Let V be the solid torus in M bounded by T. We will show that V must equal N_0. Note that it is conceivable that N_0 is contained in and not equal to V, or that $V \cap N_0 = T$. Let V_1 denote the union of all components of $M - f(\Sigma)$ which meet V and let U_1 denote a component of the pre-image in $\tilde{M}$ of V_1. As the stabilizer of U_1 is infinite cyclic and the quotient by this stabilizer is compact, it follows that U_1 is a prism region. Now

U_1 must be a good prism region as it meets the good prism regions in the pre-image of N_0. This implies that V lies in N and hence in N_0. We conclude that each component of N is a solid torus.

We will collapse each component of N to a circle by a homotopy of f, chosen so as to induce an isotopy of all of the P_i's in $\tilde{M}$. This can be done as the Seifert fibration on N is such that $f(\Sigma)$ meets N in a union of fibers. This isotopy will have the effect of simultaneously collapsing every good prism region in $\tilde{M}$ to a line.

In order to keep track of the collapsing that occurs, we will define an equivalence relation on the double lines l_{ij} formed by transversely intersecting planes P_i and P_j. If the planes $P_1, P_2, \ldots, P_n$ together bound a good n-gon prism region, then we define the intersection lines of P_i and P_{i+1} to be equivalent. We consider the family of equivalence classes generated by this relation. We consider a similarly defined equivalence class of lines l'_{ij} in M'. The lines in an equivalence class are precisely the lines which collapse to a common line in $\tilde{M}$ when all good prism regions are collapsed. It follows that an equivalence class of lines in $\tilde{M}$ corresponds to one in $\tilde{M}'$.

Before we can define the required equivariant homeomorphism, we need to prove some technical facts about the results of our collapse. These are contained in the following results.

LEMMA 4.4. *Let P_1 and P_2 be two planes in $\tilde{M}$.*

(a) *If P_1 and P_2 were disjoint before the collapse, then after the collapse their intersection consists of disjoint lines of non-transverse intersection.*

(b) *If P_1 and P_2 crossed before the collapse, then after the collapse they still meet in only one line and the intersection is still transverse.*

Proof. (a) This is clear as the collapse of each component of N produces at most one line of intersection between P_1 and P_2 and this intersection is non-transverse.

(b) Suppose that after the collapse P_1 and P_2 meet transversely in a line l and non-transversely in a line λ. These lines must be disjoint. Then before the collapse, P_1 and P_2 met transversely in l and both also met some component U of the union of all good prism regions in $\tilde{M}$. Now $P_1 \cap U$ and $P_2 \cap U$ must each consist of a finite union of strips in U. In particular, some element α which stabilizes U must also stabilize both P_1 and P_2. As the intersection of their stabilizers is infinite cyclic by Lemma 2.3, it follows that α stabilizes l. But this implies that there is a prism region W in $\tilde{M}$ with l as an edge which meets U. As W meets U, it is a good prism region. We conclude that l lies in U and that l equals λ, a contradiction. This completes the proof of Lemma 4.4.

Next we consider a line λ in $\tilde{M}$ which is the intersection of planes $P_1, \ldots, P_n$. We know that the corresponding planes $P'_1, \ldots, P'_n$ in $\tilde{M}'$ intersect in a line λ' corresponding to λ. We need to know that these families of planes are arranged in the same way around this common line. Let N be a regular neighborhood of λ in $\tilde{M}$ which is invariant under the stabilizer of λ. Choose an orientation of λ and let P_{i+} and P_{i-} denote the components of $P_i - \lambda$, with notation chosen according to some orientation convention. A choice of orientation for λ is equivalent to a choice of a generator for stab(λ). Thus we can make the corresponding choice of orientation of λ' and choose P'_{i+} and P'_{i-} according to the same orientation convention as before. This means that when we homotop $\tilde{\varphi}(P_i)$ to P'_i by our homotopy which moves points a bounded distance, we can arrange to homotop $\tilde{\varphi}(\lambda)$ to λ', $\tilde{\varphi}(P_{i+})$ to P'_{i+} and $\tilde{\varphi}(P_{i-})$ to P'_{i-}. The precise result we need can be stated as follows.

LEMMA 4.5. *There is a homeomorphism of N wth N' sending $P_i \cap N$ to $P'_i \cap N'$, $1 \le i \le n$, which is equivariant with respect to the actions of the stabilizers of λ and λ', sends P_{i+} to P'_{i+} and sends P_{i-} to P'_{i-}, $1 \le i \le n$.*

Proof. We consider first any two planes P_1 and P_2. Lemma 4.4, shows that P_1 crosses P_2 at λ if and only if P'_1 crosses P'_2 at λ', and orientations are preserved, so we can construct a homeomorphism from N to N' carrying P_{i+} to P'_{i+} and P_{i-} to P'_{i-}, $i = 1, 2$. If P_1 and P_2 don't cross, then we have shown in Lemma 4.4 that P_1 lies on the positive side of P_2 if and only if P'_1 lies on the positive side of P'_2.

Since the planes meeting λ resulted from the collapse of prism regions, given any pair of planes Q, Q' meeting λ there is a sequence of planes $Q = Q_1, Q_2, \ldots, Q' = Q_k$ such that Q_i crosses Q_{i+1} for $1 \le i < k$. As a result we can order the planes $P_1, P_2, \ldots, P_n$ so that each P_j crosses at least one of the planes $P_1, P_2, \ldots, P_{j-1}$ for $1 < j \le n$.

We proceed by induction. Consider the case of adding an additional plane P_k to the planes $P_1, \ldots, P_{k-1}$. We assume that there is a homeomorphism of N to N' sending $P_i \cap N$ to $P'_i \cap N'$ for $i = 1, \ldots, k-1$. We will show that there is a homeomorphism of N to N' which in addition sends P_k to P'_k. The planes $P_1, \ldots, P_{k-1}$ split N into $2k-2$ sectors which we label $S_1, \ldots, S_{2k-2}$, and by induction there are sectors $S'_1, \ldots, S'_{2k-2}$ in N' corresponding by the homeomorphism of N to N'. To carry out the induction step it suffices to show that if a half-plane P_{k+} is contained in a sector S_j, the half-plane P'_{k+} must lie in S'_j. Note by the $k = 2$ case that P'_{k+} lies on the positive side of a plane P'_i if and only if P_{k+} lies on the positive side of P_i. Suppose that P'_{k+} lies in a distinct sector S'_i. Then we can form a plane P by joining the two possible placements of the image of P_{k+} in N'. P lies in $S'_j \cup S'_i$ and does not cross any of $P'_1, \ldots, P'_{k-1}$, since each of its two halves lie on the same side of each of $P'_1, \ldots, P'_{k-1}$. It follows that there is a plane P'_j, $1 < j < k$ which does not cross $P'_1, \ldots, P'_{j-1}$, a contradiction. This establishes Lemma 4.5.

We now continue the proof of Theorem 4.3. We proceed to define an equivariant isomorphism of the 2-complex formed by the union of the planes $\{P_i\}$ in $\tilde{M}$ with the 2-complex formed by the union of the planes $\{P'_i\}$ in $\tilde{M}'$. We begin again with a plane P_1 and consider a plane P_2 which meets it in a line $\lambda = l_{12}$. Lemma 4.5 shows that there is a homeomorphism of a neighborhood of λ to a neighborhood of λ' which carries the planes P_i meeting λ to the planes P'_i meeting λ'. If λ was not obtained by collapsing, then only P_2 crosses P_1 along λ and there may be other components of $P_1 \cap \{\cup P_i\}$ crossing λ. As before, the order of these lines corresponds in $\tilde{M}$ and $\tilde{M}'$. If λ was obtained by collapsing then there are three planes mutually crossing along λ so that there are no other components of $P_1 \cap \{\cup P_i\}$ crossing λ by the four plane property. In either case this homeomorphism can then be extended, line by line, to a neighborhood of the component of $P_1 \cap \{\cup P_i\}$ containing λ.

Suppose that μ is a line of $P_1 \cap \{\cup P_i\}$ disjoint from the component of $P_1 \cap \{\cup P_i\}$ containing λ. There is at least one plane P_3 crossing P_1 transversely at μ. We will show that μ lies on the same side of λ as μ' does of λ'. Note that P_3 meets P_1 along exactly one line. We consider two possibilities as to how the three planes meet.

Case 1. If P_3 does not cross P_2 transversely then it lies either on the positive or negative side of P_2, and the corresponding statement holds for P'_3 and P'_2 in M', so that μ lies on the same side of λ as μ' of λ'.

Case 2. If P_3 does cross P_2 transversely, then it crosses P_2 in a line v distinct from λ and μ. The planes P'_1, P'_2 and P'_3 meet similarly in distinct lines λ', μ', v'. Consider the quotient

M_2 of M by the stabilizer of P_2. By Lemma 2.3 each of P_1 and P_3 project to cylinders running from one end of M_2 to the other. The intersection line μ of P_1 and P_3 projects to either a circle or a line in M_2, according to whether the stabilizers of the three planes have cyclic or trivial intersection. If μ projects to a circle, then P_1, P_2 and P_3 intersect transversely pairwise and have stabilizers which intersect in an infinite cyclic group. Since we have collapsed all 3-gon prism regions, the three lines of intersection of these planes in $\tilde{M}$ are already coincident, contradicting the assumption that λ and μ ae disjoint. So the intersection line μ projects to a line in M_2. This line is contained in one end of M_2, so it does not lie within a finite distance of the projection of P_2 to M_2. Thus μ does not lie within a finite distance of λ. It follows from Lemma 2.5 that μ' lies on the positive side of λ' if and only if μ lies on the positive side of λ.

Thus the side of λ on which μ lies in determined, and we can extend the homeomorphism to the component of $P_1 \cap \{\cup P_i\}$ containing μ. Continuing, we get a homeomorphism of P_1 to P'_1 taking the intersections of planes P_i and P_1 in M to the corresponding intersections of planes P'_i and P'_1 in M'. We can now extend to planes $P_2, P_3, \ldots$ obtaining an isomorphism of the two 2-complexes. Moreover, the isomorphism thus obtained extends to a regular neighborhood of these 2-complexes in $\tilde{M}$ and $\tilde{M}'$. This completes the proof of Theorem 4.3.

Remark. Instead of collapsing the solid torus components of N to circles, we could add N to a regular neighborhood of $f(\Sigma)$. Essentially the same arguments will then produce a homeomorphism from this union to the corresponding union in M'.

§5. HOMEOMORPHISMS OF NON-HAKEN 3-MANIFOLDS

In §3 and §4, we assumed that all our maps of surfaces were in general position. A least area immersion f need not be in general position, see [4]. It can be perturbed to a general position immersion f', but f' may not be least area. However, we can choose f' so as to have the same key combinatorial properties as f. These properties can be summarized as follows. Let P_1 and P_2 be planes in $\tilde{M}$ in the pre-image of $f(\Sigma)$ which cross and let G denote the intersection of the stabilizers of P_1 and P_2. If S_1 and S_2 denote the images of P_1 and P_2 in M_G, the quotient of $\tilde{M}$ by G, then there are no compact product regions in M_G between S_1 and S_2. As f has the 1-line-intersection property, all the intersections of f are transverse. Thus a small perturbation f' of f must have the same combinatorial properties as f. Note that f' need not have the same number of triple points as f. Now these combinatorial properties of a least area surface are all that is needed to prove the results of Sections 3 and 4. It follows that the results in those sections can be applied to a map f' which is a small perturbation of a least area map with the 1-line-intersection property.

The following is proved in Lemma 1.4 of [11].

LEMMA 5.1. *Let M be a non-Haken irreducible orientable 3-manifold and let $f: \Sigma \to M$ be a least area surface immersed in M. Let $N(F)$ be a regular neighborhood of $f(F)$. Then the closures of the components of $M - N(F)$ are handlebodies whose fundamental groups inject into $\pi_1(M)$.*

The proof of Lemma 1.4 in [11] does not require that f be in general position, nor that f be least area. It goes through as long as the planes in the universal cover of M have no closed curves of intersection. In particular, it applies to the 2-complex resulting from collapsing of prism regions which may have lines of k-tuple points and lines of non-transverse intersection of planes.

We now restate and prove the main theorem.

THEOREM 5.2. *Let M be a closed orientable irreducible 3-manifold containing an immersed surface F satisfying the 4-plane and 1-line-intersection properties. Let M' be a closed irreducible 3-manifold homotopy equivalent to M. Then M' is homeomorphic to M.*

Proof. By passing to a double cover of F if necessary, we can assume that F is orientable. We fix a metric on M and take $f: F \to M$ to be a least area surface in its homotopy class. f still satisfies the 1-line-intersection and 4-plane properties by Lemma 2.4. We then perturb f so that it lies in general position. By the dicussion at the start of this section, we can continue to assume that f has the key combinatorial properties of a least area surface which are used in Sections 3 and 4.

We first modify f to put it in a canonical position. We eliminate all 2-gons between double curves as in Theorem 3.1 and then collapse all good prism regions as in Theorem 4.3.

Now Theorem 4.3 yields a homeomorphism from a regular neighborhood N of $f(\Sigma)$ to a regular neighborhood N' of $f'(\Sigma)$. We need to extend this homeomorphism to a homeomorphism from M to M'. Lemma 5.1 shows that M and M' are obtained from the regular neighborhoods N and N' by gluing in handlebodies. We need to specify how the handlebodies are glued in. This is determined by specifying which curves of a component S of ∂N are homotopically trivial in the handlebody attached to it. We determine if a curve α is trivial by attempting to lift it to the cover $\tilde{N}$ of N which is the lift of N to $\tilde{M}$. If α lifts to $\tilde{M}$ then the curve is trivial in the handlebody, and if it does not lift, it is non-trivial. Theorem 4.3 shows that N and $\tilde{N}$ are homeomorphic to N' and $\tilde{N}'$ respectively, so that the same set of curves lifts in each case. Thus we see that M and M' are homeomorphic.

§6. HYPERBOLIC MANIFOLDS

In this section we examine the relation between the 4-plane property and totally geodesic surfaces in hyperbolic manifolds.

LEMMA 6.1. *If four hyperbolic planes in hyperbolic 3-space or four Euclidean planes in Euclidean 3-space all intersect pairwise, then one of their dihedral angles is smaller than or equal to τ, the dihedral angle of a regular Euclidean tetrahedron.*

Remark. $\tau \approx 0.39183\pi \approx 70.5288°$.

Proof. First note that if three planes in Euclidean or hyperbolic space intersect pairwise but have no common point of intersection, then there exists a mutually perpendicular plane which intersects them in a triangle. This triangle has angles equal to the corresponding dihedral angles, thus one is no greater than $\pi/3$, which is less than τ, and we are done. Thus we can assume that any three of the four planes meet in a common point. It follows that they bound a tetrahedral region. In hyperbolic space we can scale down the tetrahedron, increasing all dihedral angles. In the limit the 4 planes intersect in a single point, reducing us to the Euclidean case.

We claim that if a Euclidean tetrahedron T has all dihedral angles $\geq \tau$, then T is regular. In particular, any Euclidean tetrahedron has at least one dihedral angle which is $\leq \tau$.

Although this result is probably known, we have not found an explicit statement in the literature, so a proof is presented.

We represent the situation at a vertex v of T by a geodesic triangle on the unit sphere with angles equal to the dihedral angles of T at v. The edge lengths of this triangle are the face angles at v. Hence an equilateral spherical triangle with all angles equal to τ has edge lengths equal to $\pi/3$. The following fact from spherical geometry will be needed.

LEMMA 6.2. *Let Δ be a triangle on the sphere of radius one with all angles $\geq \tau$. Then either all the edge lengths of Δ are $\geq \pi/3$, or two of the edge lengths are $\geq \pi/2$.*

Remark. The case of two edge lengths $> \pi/2$ and one $< \pi/3$ can occur.

Proof. Let ABC be an equilateral triangle on the sphere of radius one with all angles equal to τ, and with A at the North pole. We let A', B' and C' denote the vertices of the given triangle Δ and we assume that A' equals A, that C' lies on AC or its geodesic extension and that the angle $B'A'C'$ contains BAC. We will assume that at most one edge of Δ has length $\geq \pi/2$, or the lemma is already proved.

Case 1. *At least two edges of Δ have length $< \pi/3$.* By relabelling vertices, we can assume that $A'C'$ and $A'B'$ are $< \pi/3$. See Fig. 8. We claim that one of the angles $A'B'C'$, $A'C'B'$ is $< \tau$. For if $AB' = AC' < \pi/3$ and $B'A'C' \geq \tau$, then $A'B'C'$ and $A'C'B'$ are equal and $< \tau$. Now moving B' towards A reduces the angle $A'C'B'$ and moving C' towards A reduces the angle $A'B'C'$. Thus at least one of the angles $A'B'C'$, $A'C'B'$ is $< \tau$, as claimed.

Case 2. *Only one edge of Δ has length $< \pi/3$.* Relabel Δ so that $AC' < \pi/3$ and $AB' < \pi/2$. Choose B'' on AB' so that $AB'' = \pi/3$. See Fig. 9. Then angle $A'B'C' <$ angle $AB''C'$ as we are assuming that $AB' < \pi/2$. Now angle $AB''C' <$ angle $AB''C =$ angle $ACB'' <$ angle $ACB = \tau$. Thus we have a contradiction to our hypothesis that all angles of Δ are $\geq \tau$.

This completes the proof of Lemma 6.2.

Let T be a Euclidean tetrahedron with all dihedral angles $\geq \tau$. Lemma 6.2 shows that for the three face angles at a vertex of T either all three angles are $\geq \pi/3$, or two of them are $\geq \pi/2$. Consideration of the possible cases shows that all the face angles must equal $\pi/3$, so that T is regular, as claimed.

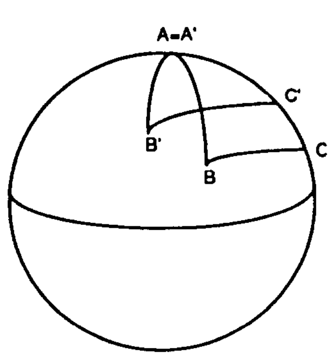

Fig. 8.

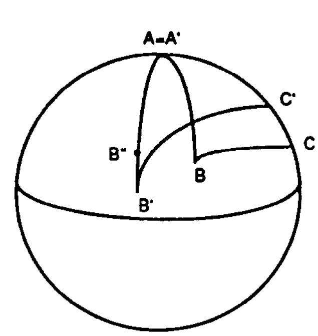

Fig. 9.

COROLLARY 6.5. *Let M be a hyperbolic 3-manifold containing an immersed totally geodesic surface F whose self-intersections at each point form an angle larger than τ. If M' is a closed irreducible 3-manifold homotopy equivalent to M, then M' is homeomorphic to M.*

Proof. It suffices to check that the 4-plane and 1-line-intersection conditions hold for F in order to apply Theorem 5.2. The 1-line-intersection condition is clear, and the 4-plane condition follows from Lemma 6.1.

QUESTION 6.6. *Does every non-Haken irreducible 3-manifold with infinite fundamental group contain a surface satisfying the 1-line-intersection and 4-plane properties? In particular, do hyperbolic 3-manifolds contain such surfaces?*

Acknowledgements—The authors would like to thank the Mathematical Sciences Research Institute, Berkeley and the Hebrew University of Jerusalem for their hospitality during the course of this work.

REFERENCES

1. I. R. AITCHISON and J. H. RUBINSTEIN: An introduction to polyhedral metrics of non-positive curvature on 3-manifolds, preprint.
2. E. M. ANDREEV: On convex polyhedra in Lobacevskii spaces, *Math. USSR Sbornik* **10** (1970), 413–446 (English translation).
3. H. BOEHME: Fast genugend gross irreduzible 3-dimensionale Mannigfaltigkeiten, *Invent. Math.* **17** (1972), 303–316.
4. M. FREEDMAN, J. HASS and P. SCOTT: Least area incompressible surfaces in 3-manifolds, *Invent. Math.* **71** (1983), 609–642.
5. M. FREEDMAN, J. HASS and P. SCOTT: Closed geodesics on surfaces, *Bull. London Math Soc.* **14** (1982), 385–391.
6. R. D. GULLIVER: Regularity of minimizing surfaces of precribed mean curvature, *Ann. Math.* **97** (1973), 275–305.
7. R. D. GULLIVER and P. SCOTT: Least area surfaces can have excess triple points, *Topology* **26** (1987), 345–359.
8. J. HASS: Complete area minimizing surfaces which are not totally geodesic, *Pacific J. Math.* **111** (1984), 35–38.
9. J. HASS and P. SCOTT: The existence of minimal surfaces in 3-manifolds, *Trans. Amer. Math. Soc.* **310**, (1988), 87–114.
10. J. HASS, J. H. RUBINSTEIN and P. SCOTT: Covering spaces of 3-manifolds, *Bulletin of the A.M.S.* **16** (1987), 117–119.
11. J. HASS, J. H. RUBINSTEIN and P. SCOTT: Compactifying coverings of closed 3-manifolds, *J. Diff. Geom.* **30** (1989), 817–832.
12. W. HEIL: On P^2-irreducible 3-manifolds, *Bull. Amer. Math. Soc.* **75** (1973), 772–775.
13. J. HEMPEL: 3-manifolds, *Ann. Math. Stud.* No. 86, Princeton University Press, Princeton, N.J.
14. W. JACO: Lectures on 3-dimensional topology, *CBMS Lecture Notes* **43**, American Mathematical Society, Providence, (1980).
15. W. JACO and J. H. RUBINSTEIN: PL minimal surfaces in 3-manifolds, *J. Diff. Geom.* **27** (1988), 493–524.
16. W. JACO and P. B. SHALEN: Seifert fibered spaces in 3-manifolds, *Mem. Amer. Math. Soc* **220**, (1980).
17. K. JOHANNSON: Homotopy equivalences of 3-manifolds with boundary, Springer Lecture Notes # **761**, 1979.
18. S. LOJASIEWICZ: Triangulation of semi analytic sets, *Ann. Scuola Norm. Sup. di Pisa* **18** (1964), 449–474.
19. G. D. MOSTOW: Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, *Publ. I.H.E.S.* **34** (1968), 53–104.
20. W. H. MEEKS III and P. SCOTT: Finite group actions on 3-manifolds, *Invent. Math.* **86** (1986), 287–346.
21. W. H. MEEKS III and S. T. YAU: Topology of three dimensional manifolds and the embedding theorems in minimal surface theory, *Ann. Math.* **112** (1980), 441–484.
22. C. B. MORREY: *Multiple integrals in the calculus of variations*, Springer-Verlag, Berlin (1966).
23. R. OSSERMAN: A proof of the regularity everywhere of the classical solution to Plateau's problem, *Ann. Math.* **91** (1970), 550–569.
24. J. H. RUBINSTEIN: Free actions of some finite groups on S^3, *Math. Ann.* **240** (1979), 165–175.
25. J. SACKS and K. UHLENBECK: The existence of minimal immersions of 2-spheres, *Ann. Math.* **113** (1981), 1–24.
26. R. SCHOEN and S. T. YAU: Existence of incompressible minimal surfaces and the topology of 3-dimensional manifolds with non-negative scalar curvature, *Ann. Math.* **110** (1979), 127–142.

27. P. SCOTT: There are no fake Seifert fiber spaces, *Ann. Math.* **117** (1983), 35–70.
28. P. SCOTT: Normal subgroups in 3-manifold groups, *J. London Math. Soc.* **13** (1976), 5–12.
29. W. THURSTON: *The geometry and topology of* 3-*manifolds*, Princeton University lecture notes.
30. F. WALDHAUSEN: On irreducible 3-manifolds which are sufficiently large, *Ann. Math.* **87** (1968), 56–88.

Depatment of Mathematics
University of California,
Davis, CA 95616
U.S.A.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
U.S.A.