HOMOTOPY EQUIVALENCE AND HOMEOMORPHISM OF 3-MANIFOLDS

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§1. INTRODUCTION AND PRELIMINARY RESULTS

In this paper we extend the class of 3-manifolds which are determined up to homeomorphism by their fundamental groups to the class of closed orientable irreducible 3-manifolds containing a singular surface satisfying two properties, the 1-line-intersection property and the 4-plane property.

A basic problem in the classification of 3-dimensional manifolds is to decide to what extent the homotopy type of a closed manifold determines the manifold up to homeomorphism. In the case of 3-manifolds with finite fundamental groups, it is known that there are homotopy equivalent manifolds which are not homeomorphic, but there are no known examples of closed orientable irreducible 3-manifolds with isomorphic infinite fundamental groups which are not homeomorphic. Waldhausen [30] and Heil [12] proved that if $M$ and $M'$ are Haken 3-manifolds which have isomorphic fundamental groups then they are homeomorphic. If $M$ and $M'$ are hyperbolic then the Mostow rigidity theorem implies the same result [19], but if only $M$ is assumed to be hyperbolic then it is unknown. Boehme [3] extended Waldhausen’s theorem to certain non-Haken Seifert fiber spaces. Scott [27] showed that a closed orientable irreducible 3-manifold which is homotopy equivalent to a Seifert fiber space with infinite fundamental group is homeomorphic to that Seifert fiber space. Many of these Seifert fiber spaces are non-Haken. To date however, Seifert fiber spaces have provided the only examples of non-Haken 3-manifolds which are known to be determined up to homeomorphism by their fundamental groups.

For manifolds with boundary there are simple examples of non-homeomorphic homotopy equivalent Haken manifolds, such as the product with the circle of a thrice-punctured sphere and the product with the circle of a once-punctured torus. Such examples are well understood in terms of the characteristic decomposition of the 3-manifold [17] [16]. Another possible source of troublesome examples comes by taking the connected sum of a 3-manifold with a fake homotopy 3-sphere, resulting in a homotopy equivalent non-homeomorphic 3-manifold. This possibility would be ruled out by a successful solution to the Poincaré conjecture. To get around this potential problem we work with irreducible 3-manifolds, in which any 2-sphere bounds a ball. Since non-orientable $P^2$-irreducible 3-manifolds are always Haken, we restrict our attention to the orientable case.

For simplicity of notation, we sometimes follow the convention of not distinguishing between a map of a surface into a manifold $M$ and the image of the map. When it is

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necessary to distinguish the domain and the image, we will denote the image by $F$ and the domain by $\Sigma$, so that $F = f(\Sigma)$.

Definitions

1. A (possibly singular) surface $F$ in a 3-manifold $M$ satisfies the 1-line-intersection property if its pre-image in the universal cover $\tilde{M}$ consists of a collection of embedded planes, any two of which either intersect transversely in a single line or are disjoint. $F$ satisfies the $k$-plane property if in each set of $k$ distinct planes in the pre-image of $F$ in $\tilde{M}$, there is at least one pair which do not intersect.

Note 1.2. The assumption that the pre-images of $f(\Sigma)$ in the universal cover are planes implies that the homomorphism $f_*: \pi_1(\Sigma) \to \pi_1(M)$ is injective and that $\Sigma$ is not $S^2$ or $P^2$. Least area surfaces need not satisfy the 1-line intersection property, but if $f: \Sigma \to M$ has this property then so does any least area map homotopic to $f$. See Lemma 2.4 for a proof.

The main result of this paper will be proven in §5.

Theorem 5.2. Let $M$ be a closed orientable irreducible 3-manifold containing an immersion $f: \Sigma \to M$ satisfying the 4-plane and 1-line-intersection properties. Let $M'$ be a closed irreducible 3-manifold homotopy equivalent to $M$. Then $M'$ is homeomorphic to $M$.

The idea behind our proof of Theorem 5.2 is as follows. Let $\varphi: M \to M'$ be a homotopy equivalence. We show, using techniques from the theory of least area surfaces, that the map $\varphi f: \Sigma \to M'$ is homotopic to a map $f'$ which, like $f$, satisfies the 1-line-intersection and 4-plane properties. We further show that after homotopies of $f$ and $f'$, we can arrange that $f(\Sigma)$ and $f'(\Sigma)$ are homeomorphic. Finally we show how to extend this homeomorphism to a homeomorphism of $M$ with $M'$.

Note 1.3. Theorem 5.2 does not assume that $M$ has a finite cover which is a Haken manifold. It generalizes the results of Waldhausen [30], as a Haken manifold trivially satisfies the above hypothesis. It also generalizes the results of Scott [27], as he showed that any closed orientable irreducible Seifert fiber space with infinite fundamental group contains an immersion of the torus satisfying the 4-plane and 1-line-intersection properties. However, we do not give new proofs of their results. Instead we consider exactly the cases of Theorem 5.2 not included in [30] and [27]. Suppose that $M$ is finitely covered by a Haken manifold $M_1$. Thurston showed that either $M_1$ is hyperbolic or it has non-empty characteristic submanifold [20]. If $M_1$ is a Seifert fiber space then the results of [27] show that $M$ is also a Seifert fiber space. If $M_1$ is neither hyperbolic nor a Seifert fiber space, the equivariance of the characteristic submanifold shows that $M$ is Haken. As we will not consider the case where $M$ is Haken or Seifert fibered, we see that the only remaining case of interest to us is when $M_1$ is hyperbolic.

Example 1.4. Let $M$ be an irreducible 3-manifold which is double covered by a Haken manifold. Then the projection of any embedded incompressible surface in the Haken cover gives a surface in $M$ which satisfies the 3-plane property. For the sheets of the pre-image in the universal cover of $M$ of such a surface split into two families, each family consisting of embedded mutually disjoint planes. Two out of any three planes in the universal cover must belong to the same family and thus not intersect. Similarly, the projection of an embedded incompressible surface in a 3-fold Haken cover yields a surface in $M$ with the 4-plane property. If in addition the 1-line-intersection property holds, then Theorem 5.2 implies that $M$ is determined up to homeomorphism by its fundamental group. For an example where
the second assumption holds, consider the case where \( M \) is a hyperbolic manifold which is 2-fold or 3-fold-covered by a Haken manifold containing an embedded totally geodesic surface. The projection of such a surface into \( M \) has the 1-line-intersection property as totally geodesic planes in hyperbolic 3-space have this property. Thus these manifolds satisfy the hypothesis of Theorem 5.2.

**Example 1.5.** Let \( M \) be a hyperbolic 3-manifold containing an immersed totally geodesic surface whose self intersections are at angle \( \pi/2 \). Many such manifolds can be constructed by taking an all right polyhedron \( P \) in hyperbolic 3-space \( H^3 \), and letting \( M \) be the quotient of \( H^3 \) by a finite index torsion free subgroup of the group generated by reflections in the faces of \( P \). Andreev's theorem guarantees the existence of many such polyhedra \([Z]\). The image of a plane containing a face of \( P \) yields an immersed surface satisfying the 1-line-intersection property and having all its self intersections at angle \( \pi/2 \). Such a surface must also satisfy the 4-plane property, as it is impossible to have four planes in \( H^3 \) every pair of which meets at an angle \( \pi/2 \). In fact, the angle condition can be weakened to require that the dihedral angles be strictly greater than that of a Euclidean regular tetrahedron, as will be shown in Lemma 6.1. This angle \( \tau \) is approximately \( 0.39183 \approx 70.5288^\circ \).

**Example 1.6.** Recent work of Aitchison and Rubinstein \([1]\) shows that many 3-manifolds admit non-positive curvature cubings. Such manifolds can easily be shown to contain surfaces satisfying the 1-line-intersection and 4-plane properties. An interesting example is the Seifert-Weber manifold, which is obtained by identifying opposite faces of the dodecahedron with a twist of \( 3\pi/5 \). This hyperbolic manifold is not known to be Haken, but satisfies the hypothesis of Theorem 5.2, and thus any irreducible 3-manifold with the same fundamental group is homeomorphic to it. In \([1]\), other examples of manifolds that contain surfaces satisfying the 1-line-intersection and 4-plane properties are constructed by branched covering and by surgery on hyperbolic links.

For curves on a surface there are conditions analogous to the 1-line-intersection and \( k \)-plane properties of surfaces in 3-manifolds.

**Definitions 1.7.** If a collection of (possibly singular) curves on a surface is such that the pre-image in the universal cover consists of embedded lines, any pair intersecting transversely in at most one point, then the collection of curves satisfies the \( l \)-point intersection property. It satisfies the \( 3 \)-line property if in each set of 3 distinct lines, there is at least one pair which do not intersect.

**Note 1.8.** Note that each curve of such a collection must be homotopically essential. The \( l \)-point-intersection condition is automatically satisfied if the curves in the collection are shortest geodesics for some metric on the surface \([5]\).

The methods of this paper can be used to show that the two conditions of Definition 1.7 determine a unique configuration for the collection of curves. The precise result is the following.

**Theorem 4.2.** Let \( \{c_i\} \) be a collection of essential closed curves in general position on an orientable surface \( \Sigma \), and satisfying the \( l \)-point-intersection and \( 3 \)-line properties. Let \( \{c'_i\} \) be another collection of the same number of essential closed curves on \( \Sigma \), also in general position, satisfying the \( l \)-point-intersection and \( 3 \)-line properties, and such that \( c'_i \) is homotopic to \( c_i \) for each \( i \). Then there is a permutation \( \sigma \) of the indices and an isotopy of \( \Sigma \) carrying \( \{c'_i\} \) to \( \{c_{\sigma(i)}\} \).
A similar result holds for a surface in a 3-manifold which satisfies the 1-line-intersection and 3-plane properties. There is a unique possible configuration that it can assume in the 3-manifold. The 4-plane property is weaker, and does not imply a unique configuration, but nonetheless it allows a well defined simplified configuration which is unique.

This paper is organized as follows. In §2 we discuss some basic properties of least area surfaces. In §3 we prove that if $f: \Sigma \to M$ has the 4-plane property and the 1-line-intersection property then $f$ can be homotoped to a map $g$ with the same properties, such that in addition the double curves of $g$ satisfy the 1-point intersection property. Thus $g$ has the least possible number of triple points. In §4 we apply the results of §3 to show that if $\varphi: M \to M'$ is a homotopy equivalence and $f' = \varphi f: \Sigma \to M'$ then we can homotop $f$ and $f'$ so that $f(\Sigma)$ and $f'(\Sigma)$ are homeomorphic. In §5 we prove our main result. In §6 we examine the relation between the 4-plane property and totally geodesic surfaces in hyperbolic manifolds.

§2. PRELIMINARIES AND LEAST AREA SURFACES

Let $M$ denote a 3-manifold and let $\Sigma$ denote a closed surface. Given an immersion $f: \Sigma \to M$, the singularity set of $f$, $S(f)$, is defined to be the set: $S(f) = \{x \in \Sigma | f(x) = f(y) \text{ for some } y \in \Sigma, y \neq x\}$. If $f$ is a general position immersion and $\Sigma$ and $M$ are oriented then $S(f)$ consists of immersed curves intersecting transversely, which come in pairs that are identified to one another. Double points of these immersed curves on $\Sigma$ correspond to triple points of the image of the immersion in $M$. If $\Sigma$ and $M$ are oriented, then there is an induced orientation on the curves of $S(f)$.

A map $f: \Sigma \to M$ is $\pi_1$-injective if it induces an injection of fundamental groups. We say that a smooth map $f: \Sigma \to M$ of a surface into a Riemannian manifold is least area if the restriction of $f$ to any compact subsurface of $\Sigma$ minimizes area in its homotopy class, rel boundary. We say that $f$ factors through a covering if there exists a surface $\Sigma_2$ and a covering $\rho: \Sigma \to \Sigma_2$ and a map $f_2: \Sigma_2 \to M$ such that $f_2 \rho$ equals $f$. If $f$ has the 1-line-intersection property, is in general position, and can be homotoped to factor through a covering $\rho$, then $\rho$ must be of degree two and $f_2$ is 1-sided. For otherwise, the lift of $f$ to the covering $M_2$ corresponding to $f_2* (\pi_1(\Sigma_2))$ must have double curves. Now the lift of $f$ to the covering $M_1$ corresponding to $f_1(\pi_1(\Sigma))$ is embedded as the pre-image in the universal cover $\tilde{M}$ is an embedded plane $\Pi$. It follows that there is a second plane $g\Pi$ such that $\Pi$ and $g\Pi$ cross and both project to the lift of $f$ to $M_2$. This implies that the stabilizers of $\Pi$ and $g\Pi$ intersect in a closed surface group. But this is impossible since the 1-line intersection property implies that the stabilizers of $\Pi$ and $g\Pi$ intersect in a cyclic group.

Results of minimal surface theory show that any homotopy class of $\pi_1$-injective maps of a compact surface (other than $S^2$ and $P^2$) into a compact Riemannian 3-manifold $M$ with $\pi_2(M)$ trivial admits a least area representative, and this least area map is immersed [6, 9, 23, 25, 26]. If $M$ covers a compact manifold $M_2$ and if $\pi_4(M)$ is trivial then the same result holds. This can be seen by projecting into $M_2$ and applying the existence result there. One can also work in the piecewise linear category. The definition of a least area surface in the PL category is due to Jaco and Rubinstein [15]. They also prove appropriate existence results, and it makes no difference to the rest of this paper whether one thinks of smooth or PL surfaces.

For the rest of this section we will consider the following situation. See Fig. 1.

Let $M$ be an orientable irreducible 3 manifold and let $f: \Sigma \to M$ be a general position immersion of a closed surface into $M$ which is $\pi_1$-injective and has image $F$. Let $\varphi: M \to M'$ be a homotopy equivalence of $M$ to an irreducible 3-manifold $M'$ and let $f': \Sigma \to M'$ be
a general position immersion of $\Sigma$ in $M'$ with image $F'$, such that $f'$ is homotopic to $\varphi f$ by a homotopy $h$. Let $\tilde{\varphi}: \tilde{M} \rightarrow \tilde{M}'$ be a lift of $\varphi$ to the universal covers $\tilde{M}$ and $\tilde{M}'$ of $M$ and $M'$ respectively. Let $P$ and $P'$ be lifts of $F$ and $F'$ to the universal covers $\tilde{M}$ and $\tilde{M}'$ such that $P'$ is homotopic to $\tilde{\varphi}(P)$ by a lift $\tilde{h}$ of $h$ to $\tilde{M}'$. We will assume that the planes $P$ and $P'$ are embedded (this is automatic if $f$ and $f'$ are least area or if $f$ and $f'$ have the 1-line intersection property). Let $\text{stab}(P)$ denote the subgroup of the group of covering transformations of $\tilde{M}$ leaving $P$ invariant. Let $\{P_i\} = \{g_i \cdot P\}$ be the set of planes forming the pre-image of $F$ in $\tilde{M}$, where $\{g_i\}$ is a collection of elements of $\pi_1(M)$ which runs over the cosets of $\text{stab}(P)$ in $\pi_1(M)$, and let $\{P'_i\} = \{\varphi_n(g_i) \cdot P'\}$ be the set of planes forming the pre-image of $F'$ in $\tilde{M}'$. Note that $f$ does not factor through a covering, since we have assumed that it is in general position, so that $\text{stab}(P)$ is equal to $f_\ast \pi_1(\Sigma)$. If $f'$ also does not factor though a covering, we have an equivariant bijection between the set of translates of $P$ in $\tilde{M}$ and the set of translates of $P'$ in $\tilde{M}'$, where the equivariance is with respect to $\pi_1(M)$ acting on $\tilde{M}$ and $\pi_1(M') = \varphi_\ast \pi_1(\Sigma)$ acting on $\tilde{M}'$. But this is not necessarily associated to an equivariant homeomorphism between the unions of the two sets of planes.

We will assume that $P_1 = P$ is stabilized by $f_\ast \pi_1(\Sigma)$. Let $M_1$ be the cover of $M$ with fundamental group corresponding to $f_\ast \pi_1(\Sigma)$ and $M'_1$ the corresponding cover of $M'$, so that $M_1$ and $M'_1$ are homotopy equivalent by the lift $f_1$ of $\varphi$ induced by $\tilde{\varphi}$. Let $f_1: \Sigma \rightarrow M_1$ be a lift of $f$ to $M_1$, with image $F_1$. The $f_1$ is a homotopy equivalence and $f_1(\Sigma)$ is covered by a single plane in $\tilde{M}$, which we take to be the plane $P_1$. Note that as $P_1$ is assumed to be embedded, $F_1$ must also be embedded. Correspondingly we define $f'_1$. We fix an orientation on $F$, calling one side of its normal bundle the positive side and the other side the negative side. We orient $F'$ compatibly via the homotopy of $\varphi f$ and $f'$. All the covers of $F$ and $F'$ inherit orientations. Note that as $\varphi$ is a proper map, i.e. the pre-image of a compact set is compact, it follows that any lift of $\varphi$ is also proper. In particular, $f_1: M_1 \rightarrow \tilde{M}_1$ is proper and so maps the two ends of $M_1$ to the two ends of $M'_1$.

The immersion $f'$ is homotopic in $M'$ to $\varphi f$ by a homotopy moving every point along a path of less than some uniformly bounded length. Thus the homotopy $h$ of the plane $P'$ to $\varphi(P)$ moves any point a uniformly bounded distance. The same assertion is true for the homotopy between any covering of $F'$ and the image of the corresponding cover of $F$ under the appropriate lift of $\varphi$. As noted at the start of this section, it also follows that if $f$ has the 1-line-intersection property then it cannot be homotoped to factor through a covering except possibly through a 2-fold covering of a 1-sided surface. Note also that a choice of basepoints is implicit in the construction of the lifts $f_1$ and $\tilde{\varphi}$ of $\varphi$, as well as in the lifts of $f$ and $f'$.

A key result is the following which is implicit in [27] but not explicitly stated there. We say that $gP$ crosses $P$ if $gP$ meets $P$ but does not equal $P$.

**Lemma 2.2.** Let $\varphi: M \rightarrow M'$ be a homotopy equivalence of closed 3-manifolds, let $f: \Sigma \rightarrow M$ and $f': \Sigma \rightarrow M'$ be least area maps such that $f'$ is homotopic to $\varphi f$. Let $P$ and $P'$ be
planes in $\tilde{M}$ and $\tilde{M}'$ chosen as above. Then, for any $g$ in $\pi_1(M)$, the plane $gP$ crosses $P$ if and only if $\phi_*(g)P'$ crosses $P'$.

Remarks. If $f$ and $f'$ do not factor through coverings, so that the stabilizers of $P$ and $P'$ correspond under $\phi$, then we have an equivariant bijection between the translates of $P$ in $\tilde{M}$ and the translates of $P'$ in $\tilde{M}'$. Lemma 2.2 tells us that the intersection patterns of these two families of planes correspond under $\phi$, where the intersection pattern of a family of planes is simply the information about which pairs of planes cross.

Proof. Given $g$ in $\pi_1(M)$, we consider the planes $P$ and $gP$ in $\tilde{M}$ and $P'$ and $\phi_*(g)P'$ in $\tilde{M}'$. The image in $M_1$ of $P$ is the embedded surface $F_1$, and the image in $M_1$ of $gP$ is a possibly singular surface $F_2$. The images of $P'$ and $\phi_*(g)P'$ in $M_1'$ are $F_1'$ and a possibly singular surface $F_2'$. Now $F_1$ separates $M_1$ into two non-compact pieces and the intersection of $F_2$ with the closure of each of these pieces must also be non-compact. For otherwise, an exchange can be made between two compact subsurfaces of a pair of least area surfaces, as shown in the proof of Theorem 5.1 of [4], yielding a contradiction. Thus $F_2$ contains a path joining the two ends of $M_1$. As $\phi_1$ is proper, it follows that $\phi_1(F_2)$ contains a path joining the two ends of $M_1$. As $F_2$ is homotopic to $\phi(F_2')$ by a homotopy moving points a bounded distance, it follows that $F_2'$ also contains such a path so that it must meet $F_1'$. Hence $\phi_*(g)P'$ must cross $P'$ as required. The converse follows by symmetry.

Lemma 2.3. Let $f: \Sigma \to M$ be least area and suppose that $P_1$ and $P_2$ are planes in the pre-image in $\tilde{M}$ of $f(\Sigma)$ which cross. Then $P_1 \cap P_2$ is a single line if and only if $\text{stab}(P_1) \cap \text{stab}(P_2)$ is infinite cyclic. In this situation, $P_1$ and $P_2$ must intersect transversely.

Proof. If $P_1$ and $P_2$ intersect in one line then the intersection of their stabilizers must be infinite cyclic, as it stabilizes this line. If $\text{stab}(P_1) \cap \text{stab}(P_2)$ is infinite cyclic, then Lemma 6.5 of [4] shows that $P_1$ and $P_2$ intersect transversely in a single line. This finishes the proof of Lemma 2.3.

The next lemma states that we can homotop a surface to a least area surface while retaining the 1-line-intersection and $k$-plane properties.

Lemma 2.4. Let $\tilde{f}: \Sigma \to M$ be an immersion of a surface into a Riemannian 3-manifold $M$ such that $\tilde{f}$ satisfies the 1-line-intersection and $k$-plane properties, with $k \geq 3$, and does not factor through a covering. Let $f: \Sigma \to M$ be a least area surface in the homotopy class of $\tilde{f}$ which also does not factor through a covering. Then $f$ intersects itself transversely and satisfies the 1-line-intersection and $k$-plane properties. If $k = 3$ then $f$ is in general position.

Proof. The pre-image in $\tilde{M}$ of $f(\Sigma)$ consists of a collection of planes $\{P_i\}$ with $P_i$ corresponding to a plane $\tilde{P}_i$ above $f(\Sigma)$. Since $f$ is a least area map, the planes $\{P_i\}$ in $\tilde{M}$ are least area and embedded [4], [8].

Suppose first that $\tilde{P}_1 \cap \tilde{P}_2 \neq \emptyset$, and thus is a single line. Lemma 2.3 shows that their stabilizers intersect in a cyclic group, so that the same must be true of the stabilizers of $P_1$ and $P_2$. As Lemma 2.2 shows that $P_1$ and $P_2$ must cross, we can apply Lemma 2.3 to deduce that they intersect transversely in exactly one line.

Suppose now that $\tilde{P}_1$ and $\tilde{P}_2$ are disjoint. Then $P_1$ and $P_2$ are disjoint or coincident by Lemma 2.2. It follows from the above that $f$ has the $k$-plane property and the 1-line-intersection property.
If \( f \) satisfies the 3-plane property, then in particular \( f \) has no triple points. The picture near a double point of \( f \) in \( M \) is the same as the picture near a point of intersection of two embedded least area planes \( P_1 \) and \( P_2 \) in the universal cover \( \tilde{M} \). As this intersection is transverse, it follows that \( f \) is in general position. This concludes the proof of Lemma 2.4.

Recall that we are considering a \( \pi_1 \)-injective map \( f: \Sigma \to M \), a homotopy equivalence \( \varphi: M \to M' \) and a map \( f' \) homotopic to \( \varphi f \). As a result of Lemmas 2.2 and 2.4, it will suffice to assume in the following lemmas that \( f \) and \( f' \) are least area. Note that least area surfaces not satisfying the 3-plane property are not known to be triangulable, though Meeks and Yau conjecture that they are [21]. If \( M \) has an analytic metric, then a least area surface will be analytic [22], and hence triangulable by [18].

**Lemma 2.5.** Let \( f: \Sigma \to M \) and \( f': \Sigma \to M' \) be least area immersions such that \( f' \) is homotopic to \( \varphi f \). If \( f \) satisfies the 1-line-intersection and \( k \)-plane properties then so does \( f' \).

Moreover, if \( P_1 \) and \( P_2 \) are two planes in \( \tilde{M} \) which intersect in a line \( \lambda \), then the corresponding planes \( P'_1 \) and \( P'_2 \) in \( M' \) also intersect in a single line \( \lambda' \), and \( \varphi(\lambda) \) and \( \lambda' \) each lie a bounded distance from each other.

**Proof.** Lemmas 2.2 and 2.4 shows that \( f' \) satisfies the 1-line-intersection and \( k \)-plane properties. Now consider two planes \( P_1 \) and \( P_2 \) which intersect in a line \( \lambda \). The first assertion of the lemma shows that \( P'_1 \) and \( P'_2 \) also intersect in a single line, which we call \( \lambda' \). The circles of intersection \( F_1 \cap F_2 \) and \( \varphi(F_1 \cap F_2) \) are homotopic, and so their lifts to the universal cover are a bounded distance from one another, yielding the second assertion of Lemma 2.5.

### §3. REDUCING THE NUMBER OF TRIPLE POINTS

In this section we assume that \( \Sigma \) is an orientable closed surface and consider an immersion \( f: \Sigma \to M \) which satisfies the 1-line-intersection property and the 4-plane property. We want to minimize the number of triple points of \( f \). In [27], Scott showed how to do this in the case of a torus mapped into certain Seifert fibre spaces. We will extend these results to other surfaces.

As in §2, the pre-image of \( f(\Sigma) \) in \( \tilde{M} \) consists of translates by \( \pi_1(M) \) of a plane \( P \) stabilized by \( f_\#(\pi_1(\Sigma)) \). We will consider the curves of intersection of \( P \) and its translates. Given a collection of embedded curves \( \gamma \) on a surface \( P \), a \( k \)-gon on \( P \) is a subdisk of \( P \) bounded by \( k \) embedded subarcs of \( \gamma \), with the interiors of the subarcs disjoint. We refer to a simple closed curve as a *circle* and a simple non-compact curve as a *line*.

We prove the following.

**Theorem 3.1.** Let \( M \) be a closed orientable irreducible 3-manifold and let \( \Sigma \) be a closed orientable surface. Let \( f: \Sigma \to M \) be an immersion in general position with the 1-line-intersection and 4-plane properties. Then \( f \) is homotopic to an immersion \( g \) also in general position and with the 1-line-intersection and 4-plane properties such that the double curves of \( g \) have the 1-point-intersection property, i.e. \( g \) has the least possible number of triple points.

**Remark.** The examples in Gulliver–Scott [7] show that it is possible that in some metrics, no least area map homotopic to \( f \) minimizes the number of triple points.

**Proof.** If the double curves of \( f \) fail to have the 1-point-intersection property, then there is a 2-gon in \( P \) between two double lines. We will show how to homotop \( f \) to reduce the
number of triple points by two. The homotopy will preserve the intersection pattern (in pairs) of the planes above \( f(\Sigma) \) but it may well destroy the 1-line-intersection property of \( f \). Thus the new map will still have the 4-plane property, but if two planes above \( f(\Sigma) \) meet then the corresponding planes for the new map may meet in more than one line. Their intersection can only consist of an odd number of lines, together with some circles. We will also show how to remove circle components of intersection. Finally, we will show how to restore the 1-line intersection property. Many of our arguments are drawn from \([20]\) and \([27]\), but as the context is somewhat different we will give some of those arguments. First, we define the complexity of any general position immersion \( f: \Sigma \to M \) to be the pair \((t, d)\) where \( t \) is the number of triple points on \( \Sigma \) and \( d \) is the number of null-homotopic double curves on \( \Sigma \). Now we consider a general position immersion \( f: \Sigma \to M \) which has the 4-plane property and is homotopic to a map with the 1-line intersection property and the same intersection pattern.

**Lemma 3.2.** If there is a disc or 2-gon in \( P \), then there is an innermost disc or 2-gon in \( P \), i.e. a disc or 2-gon in \( P \) whose interior does not meet any of the double curves in \( P \).

**Proof.** This is essentially in \([27]\), but we give the argument for completeness. Let \( \gamma \) denote the double curves in \( P \). Let \( D \) denote either a 2-disc in \( P \) bounded by a circle component of \( \gamma \) or a 2-gon in \( P \) bounded by two arcs of \( \gamma \). We will show that if \( D \) is not innermost, then \( D \) contains a smaller such 2-disc. By repeating this argument, we can then obtain an innermost disc or innermost 2-gon as required.

If the interior of \( D \) contains a circle component \( S \) of \( \gamma \), then \( S \) bounds a 2-disc in the interior of \( D \) yielding a smaller 2-disc. Otherwise there is a component \( S \) of \( \gamma \) which crosses \( \partial D \). If \( \partial D \) is a circle component of \( \gamma \), we let \( u \) be a sub-arc of \( S \) which is properly embedded in \( D \). Then \( D \) contains a 2-gon bounded by \( u \) and a sub-arc of \( \partial D \) and we have again found a smaller disc. Otherwise \( D \) is a 2-gon bounded by sub-arcs \( \lambda \) and \( \mu \) of \( \gamma \). The 4-plane property implies that \( u \) cannot meet both \( \lambda \) and \( \mu \). Thus \( D \) contains a smaller 2-gon bounded by \( u \) and by a sub-arc of \( \lambda \) or \( \mu \). This completes the proof of Lemma 3.2.

**Lemma 3.3.** If there is an innermost disc in \( P \), there is a homotopy of \( f \) which reduces \((t, d)\) and leaves the intersection pattern of \( P \) and its translates unchanged.

**Proof.** Suppose that \( gP \) meets \( P \) in a circle \( C \) which bounds an innermost 2-disc \( D \) in \( P \). Then \( C \) bounds a 2-disc \( D' \) in \( gP \) and \( D \cup D' \) forms an embedded 2-sphere in \( \bar{M} \) as \( D \) is innermost. Let \( B \) denote the 3-ball in \( \bar{M} \) bounded by this sphere. We claim that \( D' \) projects injectively into the quotient of \( gP \) by its stabilizer. The required homotopy of \( f \) is then defined by homotoping \( D' \) across \( B \) and past \( D \). In order to prove the claim we need to show that if \( h \) stabilizes \( gP \) and \( hD' \) meets \( D' \) then \( h \) must be the trivial element of \( \pi_1(M) \). As \( \partial D \) equals \( \partial D' \) and has no triple points, \( \partial D' \) and \( h(\partial D') \) must coincide or be disjoint. Thus \( D' \) and \( hD' \) are disjoint or coincide or one is contained in the other. Either of the last two cases implies that \( h \) fixes a point of \( D \) and hence that \( h \) is the identity which completes the proof of Lemma 3.3.

A key result is the following, which is based on ideas of Rubinstein \([24]\).

**Lemma 3.4.** If \( D \) is an innermost 2-gon in \( P \) and \( g \) is an element of \( \pi_1(M) \) such that \( gD \) meets \( D \), then \( g \) is trivial.
Proof. Let $\lambda$ and $\mu$ denote the edges of $D$ and let $x$ and $y$ denote the vertices of $D$. Let $l$ and $m$ denote the double lines which contain $\lambda$ and $\mu$. Let $P_1$ denote the other plane containing $l$ and let $P_2$ denote the other plane containing $m$.

Now suppose that $g$ is a non-trivial element of $\pi_1(M)$ such that $gD$ meets $D$. Then $g$ must send one of the vertices of $D$ to one of the vertices of $D$. As $g$ is not trivial, it cannot fix a point and so $g\lambda = \lambda$ or $g\mu = \mu$. After re-labeling we can assume that $g\lambda = \lambda = gD \cap D$. It follows that $g$ must preserve the union of the three planes $P$, $P_1$, and $P_2$. We will consider the permutation of these planes induced by $g$.

If $g$ preserves each of the planes $P$, $P_1$, and $P_2$, it must also preserve each of $l$ and $m$, as $g\lambda = \lambda$. As $g$ has no fixed points, it must preserve the orientations of $l$ and $m$. This implies that the restriction of $g$ to $P$ is orientation reversing which is a contradiction as $\Sigma$ is orientable and $f$ is in general position and so cannot factor through a covering.

If $g$ preserves one plane and interchanges the other two, it is again easy to derive a contradiction. Suppose that $P'$ and $P''$ are intersecting planes interchanged by an element $g$ of $\pi_1(M)$. The intersection $P' \cap P''$ consists of an odd number of lines together with some circles, and must be preserved by $g$. As $g$ induces an involution on this collection of lines we see that $g$ must preserve a line $l$ of $P' \cap P''$. The fact that $g$ preserves $l$ and interchanges $P'$ and $P''$ implies that $g$ acts on a neighborhood of $l$ by a screw motion whose rotation part has order 4. But then $g^2$ acts reversing orientation on $P'$ and $P''$ which again contradicts our assumption that $\Sigma$ is orientable.

Finally, if $g$ cyclically permutes the three planes, we consider the induced action of $Z_3$ on the quotient of $M$ by the cyclic group generated by $g^3$. In this quotient, the images of the three planes are three embedded annuli $A$, $A_1$, and $A_2$ and $D$ projects injectively into $A$, as $g^3D$ cannot meet $D$ since $g^3$ fixes each of $P$, $P_1$, $P_2$. We again use $g$ to denote the generator of the action of $Z_3$ and use $D$ to denote the image of $D$ in $A$. We are assuming that $gD$ meets $D$ and that $g$ cyclically permutes the annuli $A$, $A_1$, and $A_2$. Thus $gD$ cannot equal $D$ nor can $g\lambda = \lambda$ or $g\mu = \mu$. If $g\lambda = \lambda$ then $g^2$ must fix each of $x$ and $y$ contradicting the fact that $g^3$ acts freely on $Z_3$. Similarly we cannot have $g\mu = \lambda$. If $gD \cap D = \{x, y\}$, then $g$ must fix $x$ and $y$ again contradicting the freeness of our action. It follows that we must have $gD \cap D = \{x\}$ or $\{y\}$. Without loss of generality we can assume that $gD \cap D = \{y\}$ and that $g\lambda = \lambda$. Now we will argue as at the end of the proof of Lemma 6.6 of [20] where consideration of Figs 6.7(a) and (b) leads to a contradiction. This completes the proof of Lemma 3.4.

Lemma 3.5. If there is an innermost 2-gon in $P$, then we can homotop $f$ so as to reduce its complexity $(t, d)$, while preserving the intersection pattern of $P$ and its translates.

Proof. Move one edge of the 2-gon across the other. This move can be made equivariant by Lemma 3.4 and so corresponds to a homotopy of $f$.

The previous lemmas imply that if we have a map $f : \Sigma \to M$ with the 1-line intersection property and the 4-plane property then we can homotop $f$ to a map $f_2$, still with the 4-plane property, and such that there are no trivial double curves and no 2-gons on $P$. However, $f_2$ may not have the 1-line intersection property. If $P$ and $P_1$ are planes above $f(\Sigma)$ which meet in one line, then the corresponding planes above $f_2(\Sigma)$ may meet in any odd finite number of lines. To cope with this, we need to prove a result which was not needed in [20]. Repeated application of Lemma 3.6 will complete the proof of Theorem 3.1.

Lemma 3.6. Suppose that $f_2$ is as above. Suppose that $P \cap P_1$ consists of more than one line. Then we can homotop $f_2$ to reduce the total number of double curves while preserving the
intersection pattern of $P$ and its translates and preserving the absence of trivial double curves and 2-gons.

Before starting on the proof of Lemma 3.6, we need some definitions.

If $P$ is a plane in $\tilde{M}$ above $f(\Sigma)$ and $l$ and $m$ are disjoint double curves in $P$ with the same stabilizer, we say that $l$ and $m$ are parallel double curves in $P$ if the union of $n$ strips in planes above $f(\Sigma)$ forms the boundary of a region $Y$ in $\tilde{M}$, we say that $Y$ is an $n$-gon prism region. Such a region is homeomorphic to the product of an $n$-gon with $R$ where the strips correspond to the products of the edges with $R$ and the boundary lines of the strips correspond to the products of the vertices with $R$. A prism region is innermost if it contains no other prism regions. A prism region $Y$ is $\pi_1(M)$-equivariant if $gY$ equals $Y$ or is disjoint from $Y$, for all $g$ in $\pi_1(M)$.

**Lemma 3.7.** Suppose that $f_2: \Sigma \to M$ has the 4-plane property and is homotopic to $f: \Sigma \to M$ which has the 1-line-intersection property and the same intersection pattern. Suppose that no planes above $f_2(\Sigma)$ contain 2-gons or discs.

(a) If $P \cap P_1$ consists of more than one line, then there is a 2-gon prism region $Y$ between $P$ and $P_1$ in $\tilde{M}$.

(b) If $Y$ is a $\pi_1(M)$-equivariant 2-gon prism region in $\tilde{M}$ between planes $P$ and $P_1$, there is a homotopy of $f_2$ which reduces the total number of double curves while preserving the intersection pattern and the absence of trivial double curves and 2-gons.

**Proof.** (a) Let $\alpha$ denote a generator of the common stabilizer of $P$ and $P_1$, and let $M_\alpha$ denote the quotient of $M$ by this common stabilizer. The images of $P$ and $P_1$ in $M_\alpha$ are embedded annuli $A$ and $A_1$ meeting in at least two essential circles. It follows that there are compact annuli $B$ and $B_1$ in $A$ and $A_1$ such that $B \cap B_1 = \partial B = \partial B_1$ and $B \cup B_1$ bounds a solid torus region $X$ in $M_\alpha$ whose interior does not meet $A$ or $A_1$. Then a component $Y$ of the pre-image of $X$ in $\tilde{M}$ is a 2-gon prism region between $P$ and $P_1$.

(b) Suppose that $Y$ is a $\pi_1(M)$-equivariant 2-gon prism region in $\tilde{M}$ between planes $P$ and $P_1$. Then we can define the required homotopy of $f_2$ as follows. Choose a homotopy of $A$ supported on a small neighborhood of $B$ which homotops $B$ across $X$ and thus removes two circles of intersection of $A$ and $A_1$. This induces an $\alpha$-equivariant homotopy of $P$ in $\tilde{M}$, supported on a small neighborhood of $B$, a component of the pre-image of $B$. We extend to a $\pi_1(M)$-equivariant homotopy of $P$ and its translates which is the identity except on a small neighborhood of all translates of $B$. This can be done because of our assumption that $Y$ is $\pi_1(M)$-equivariant.

Now we need to analyze the situation when we may have a non-equivariant 2-gon prism region.

**Lemma 3.8.** Let $X$ be a $n$-gon prism region bounded by strips $S_1, \ldots, S_n$ contained in planes $P_1, \ldots, P_n$. Let $l_i$ denote the line of intersection of $S_i$ with $S_{i+1}$, $1 \leq i \leq n$, where $S_{n+1}$ is defined to equal $S_1$ and $P_{n+1}$ is defined to equal $P_1$. If there are no 2-gons in any of the planes above $f(\Sigma)$ then either there are no triple points on the $l_i$'s, or there is a plane $P$ which meets each of $P_1, \ldots, P_n$. If $n$ equals 2 or 3, there are no triple points on the $l_i$'s.

**Proof.** Let $P$ be a plane in $\tilde{M}$ above $f(\Sigma)$ which meets some $l_i$ and is not $P_i$ or $P_{i+1}$. Then $P \cap P_i$ is a finite union of parallel lines. Let $\alpha$ denote a generator of the common stabilizer of these lines, and let $M_\alpha$ denote the quotient of $\tilde{M}$ by the cyclic group generated by $\alpha$. Then the image of each of $P$ and $P_i$ in $M_\alpha$ is an embedded annulus. We denote these annuli by $A$ and
$A_i$ respectively. Then $A \cap A_i$ consists of the image of $P \cap P_i$, so is a finite union of essential simple closed curves. The image of $l_i$ in $A_i$ must be a line joining the two ends of $A_i$ as $l_i$ crosses one of the lines forming $P \cap P_i$ and intersects it in only one point. It follows that the image of $l_i$ in $A_i$ meets each circle of $A \cap A_i$, so that $l_i$ must meet each of the lines of intersection of $P$ and $P_i$. As $l_{i-1}$ lies in $P_i$ and is parallel to $l_i$, we see also that each line of $P \cap P_i$ meets $l_{i-1}$. By repeating this argument, we see that $P$ meets each $l_i$. In the case when $n = 3$, this is impossible as $P, P_1, P_2, P_3$ would all meet each other violating the 4-plane property. In the case when $n = 2$, we see that $P \cap S_1$ and $P \cap S_2$ each consist of a finite number of line segments joining $l_1$ to $l_2$. This is because $P \cap P_1$ and $P \cap P_2$ each consist of a finite number of lines and there are no 2-gons between these lines, $l_1$ and $l_2$. Thus these segments must bound 2-gons in $P$, again contradicting our hypothesis.

**Remark.** It follows from the above that if the planes above $f(\Sigma)$ contain no 2-gons between double lines and $X$ is an innermost $n$-gon prism region with no triple points on its edges, then $X$ is actually the closure of a component of $\tilde{M} - N(\tilde{F})$. For if a plane $P$ meets $X$, then the above result shows that $P$ meets none of the $l_i$'s. It follows that $P$ meets $\partial X$ in lines parallel to the $l_i$'s and hence cuts $X$ into two prism regions showing that $X$ was not innermost.

The above arguments show that if $Y$ is a 2-gon prism region which contains no other such region then each plane which meets $Y$ other than $P_1$ and $P_2$ must cut $Y$ into two 3-gon prism regions. Any pair of distinct such planes must be disjoint by the 4-plane property so that the planes which meet $Y$ must cut it into two 3-gon prism regions $Z$ and $Z'$ and some 4-gon prism regions. Let $n(Y)$ denote the number of strips across $Y$, so that if $n(Y)$ is non-zero then there are $n(Y) - 1$ 4-gon prism regions in $Y$.

**Lemma 3.9.** If a 2-gon prism region $Y$ has $n(Y) = 0$, then $Y$ is $\pi_1(M)$-equivariant.

**Proof.** Let $P_1$ and $P_2$ be the planes which bound $Y$ with $S_i$ denoting the strip $P_i \cap Y$. Let $l_1$ and $l_2$ denote the two lines of $S_1 \cap S_2$. If $gY$ meets $Y$, we must have $gl_1$ or $gl_2$ equal to $l_1$ or $l_2$. Thus $g$ preserves $P_1 \cup P_2$. But $g$ cannot interchange $P_1$ and $P_2$ as shown in the proof of Lemma 3.4. Thus $g$ preserves $P_1$ and $P_2$ and hence preserves any line of $P_1 \cap P_2$. In particular, $gl_1 = l_1$ and $gl_2 = l_2$ and so $gY = Y$ as required.

**Lemma 3.10.** Suppose that there is a 2-gon prism region $Y$ which contains no other such region and that $n(Y)$ is non-zero. Then there is a homotopy of $f_2$ which

(a) preserves the total number of double curves of $f_2$
(b) preserves the intersection pattern of $P$ and its translates
(c) preserves the absence of trivial double curves and 2-gons, and
(d) produces a 2-gon prism region $Y'$ with $n(Y') < n(Y)$.

If there is a non innermost 2-gon prism region, there is such a region $Y$ which contains no other 2-gon prism regions. By applying Lemma 3.10 repeatedly, we can eventually obtain an innermost 2-gon prism region and then reduce the total number of double curves by applying Lemmas 3.7 and 3.9. This then completes the proof of Lemma 3.6 and hence of Theorem 3.1.

In order to prove Lemma 3.10, we will need the following result.

**Lemma 3.11.** Let $Z$ be an innermost 3-gon prism region. Then for all $g$ in $\pi_1(M)$, if $gZ \cap Z$ contains a strip then $gZ = Z$. 
Proof. Assume that $gZ \neq Z$. If $gZ \cap Z$ contains a strip, then $g$ must preserve $P_1 \cup P_2 \cup P_3$. As $g$ cannot interchange intersecting planes, $g$ must preserve each $P_i$ or cyclically permute them. Thus $g^3$ preserves each $P_i$ and hence each double curve, so $g^2Z = Z$. But then $g^2Z$ must have a strip in common with $Z$ and $gZ$ which is impossible, as $Z$ and $gZ$ lie on opposite sides of the plane $P_i$ which contains their common strip. This completes the proof of Lemma 3.11.

Proof of Lemma 3.10. Let $P_1$ and $P_2$ be the planes which bound $Y$, and let $S_i$ denote the strip $P_i \cap Y$, $i = 1, 2$. Let $Z$ and $Z'$ be the two 3-gon prism regions in $Y$ and let $l$ and $l'$ denote the lines of $S_1 \cap S_2$ with $l$ in $Z$ and $l'$ in $Z'$. Let $S_3, \ldots, S_r$ denote the strips across $Y$ where $r = n(Y) + 2$, and let $P_i$ be the plane containing $S_i$. Note that it is possible that $P_i$ and $P_j$ can coincide for some $i$ and $j \geq 3$. Let $m$ and $n$ denote $S_3 \cap S_2$ and $S_3 \cap S_1$, respectively. Finally, denote the 4-gon prism region in $Y$ between $S_i$ and $S_{i+1}$ by $U_i$.

Case $n(Y) = 1$. Thus $Y$ equals $Z \cup Z'$. We first show that at least one of $Z$ and $Z'$ is equivariant. Suppose that $gZ$ meets $Z$ and does not equal $Z$. Then $gZ \cap Z$ equals $l$, $m$ or $n$. If both $gZ$ and $g^{-1}Z$ meet $Z$ in $l$, we obtain a contradiction. For this would imply that $gl$ equals $l$, so that $g$ preserves each of $P_1$ and $P_2$ and each of their intersection lines and hence $gZ = Z$. Thus by replacing $g$ by its inverse if necessary we can suppose that $gZ$ meets $Z$ in $m$ or $n$. There is no difference in the roles of $P_1$ and $P_2$, so we will assume that $gZ \cap Z = m$. This means that $gZ$ meets $Z'$ in a strip and that the boundary of $gZ$ is formed of strips from the planes $P_1, P_2$ and $P_3$. See Fig. 2. (Note that, in this and later figures, planes are represented by lines, and $n$-gon prism regions are represented by $n$-gons. Thus the true situation consists of the product of the figure with the real line $R$.) Thus $g$ preserves the union of these three planes. As in the proof of Lemma 3.4 it follows that $g$ cyclically permutes the three planes, that $g^3$ is a power of $a$ and hence that $g^3$ preserves each of the lines of intersection of these three planes. In particular, $g^3Z = Z$ and $g^2Z$ must meet $Z$ and $gZ$ in an edge.

Now we consider the permutation of the intersection lines induced by $g$. We cannot have $m = gm$ as this would imply that $g$ preserves $P_1 \cup P_3$, contradicting the fact that $g$ permutes the three planes cyclically. If $m = gl$, then $g$ sends $P_1$ to $P_2$ to $P_3$. This implies that $gm$ lies in $P_3 \cap P_1$ and hence must lie as shown in Fig. 3. But, as $g^2Z$ contains $gm$, this implies that $g^2Z$ cannot meet $Z$, a contradiction. Thus $m = gn$, and $g$ sends $P_2$ to $P_1$ to $P_3$. This means that $gm$ lies in $P_2 \cap P_1$ and so $gm$ equals $l'$. It follows that $g$ preserves $m \cup n \cup l'$ as this is the orbit of $m$ under $g$, and hence $g^2Z$ equals $Z'$. See Fig. 4.

We conclude that if there exists $g$ such that $gZ$ meets $Z$ but does not equal $Z$, then $gZ' = Z'$ and the double lines on $P_3$ adjacent to $m$ and $n$ lie on $P_1$ and $P_2$ adjacent to $l'$.

If there exists $h$ such that $hZ'$ meets $Z'$ but does not equal $Z'$, then the same argument shows that the double lines on $P_3$ adjacent to $m$ and $n$ lie on $P_1$ and $P_2$ adjacent to $l$. It
follows that at least one of $Z$ and $Z'$ must be equivariant as claimed. We will suppose that $Z$ is equivariant. Now we claim the stronger result that if $gZ$ meets $Y$ then $gZ$ must equal $Z$.

If $gZ$ meets $Y$ but is not equal to $Z$ then $gZ$ cannot meet $S_1$ and $S_2$ except in $I'$ as $gZ$ cannot meet $Z$. Thus $gZ$ must be as shown in Fig. 5. Lemma 3.11 shows that $gZ'$ cannot meet $Z'$ in a strip. As $gZ'$ meets $gZ$ in a strip, we deduce that $gZ'$ must be as shown in Fig. 6. But this implies that $gY$ is bounded by $P_1$ and $P_2$ so that $g$ must preserve $P_1 \cup P_2$ and hence preserve each of these planes and each of their double curves. This contradicts our hypothesis that $gZ$ is not equal to $Z$ and so proves the claim.

Now we can complete the proof of Lemma 3.10 in the case $n(Y) = 1$. We isotop one of the strips forming the boundary of $Z$ across $Z$ and extend to an equivariant homotopy off $f_2$. Because no translate of $Z$ meets $Y$ the new map has an innermost 2-gon region $Y'$ in place of $Y$. Note that this homotopy of $f_2$ will not alter the total number of double curves.

Case $n(Y) > 1$. Suppose that there is $g$ such that $gZ$ meets $Y$ and is not equal to $Z$. Then $gZ$ equals $Z'$ or $gZ$ does not lie inside $Y$. In the second case, we see that $gZ$ cannot have a strip in common with any $U_1$ as the two regions $U_{1+}$ and $U_{1-}$ which meet $U_1$ in a strip and do not lie in $Y$ cannot be 3-gon prism regions. For if $P_1$ and $P_{1+1}$ are distinct then the 4-plane property implies that they are disjoint so that $U_{1+}$ and $U_{1-}$ cannot be 3-gon prism regions. Lemma 3.11 shows that $gZ$ cannot meet $Z$ in a strip. We conclude that $gZ$ must equal $Z'$ or $gZ \cap Z$ equals $I$ or $gZ$ meets $Z'$ and is not contained in $Y$.

Suppose that $gZ \cap Z$ equals $I$. Then $g^{-1}Z$ must also meet $Z$ and the above argument applies to show that $g^{-1}Z \cap Z$ equals $I$. Thus it follows that $gI$ equals $I$. As in the first case of this Lemma, this is a contradiction.

If $gZ$ meets $Z'$ in a strip, then $Z \cup Z'$ forms a 2-gon prism region $Y'$ with $n(Y') = 1$, so that the result of Lemma 3.10 is true with the trivial homotopy of $f_2$.

If $gZ$ meets $Z'$ in $I'$, we obtain a contradiction as follows. We know that $gU_3$ meets $gZ$ in a strip. If that strip is disjoint from $Z'$, then $gY$ must be bounded by $P_1 \cup P_2$. Thus $g$ must preserve $P_1$ and $P_2$ and so $gZ$ equals $Z$, contradicting our hypothesis. Thus $gU_3$ must meet $Z'$ in a strip. It follows that $gU_3$ meets $U_{r-1}$ in a strip and, by induction, that $gU_s$ meets $U_{r-s+3}$ in a strip, for $4 \leq s \leq r - 1$. See Fig. 7. Hence $gZ'$ would have to meet $U_3$ in a strip, which is impossible as $U_{1+}$ and $U_{1-}$ cannot be 3-gon prism regions.
We conclude that if $gZ$ meets $Y$ then $gZ$ must equal $Z$ or $Z'$. If the only translate of $Z$ which meets $Y$ is $Z$ itself, then we can isotop $S_j$ across $Z$ and extend to an equivariant homotopy of $f_2$ which will reduce $n(Y)$ by one. Note that it is important that no other translate of $Z$ meets $Y$ as such a translate might yield an increase in $n(Y)$ by one, cancelling out our reduction. If there are two translates of $Z$ which meet $Y$, namely $Z$ and $Z'$, we perform the same homotopy of $f_2$ and this will reduce $n(Y)$ by two. This completes the proof of Lemma 3.10 and hence of Theorem 3.1.

§4. SURFACES IN 3-MANIFOLDS

In this section we consider the image of a map $f: \Sigma \to M$ with the 1-line-intersection property. We show that the image is unique up to homeomorphism if $f$ also satisfies the 3-plane-property. If $f$ satisfies the 4-plane property, its image need not be unique up to homeomorphism, but we show how to homotop $f$ to have canonical image by using the main result of §3.

Throughout this section, let $M$ be an orientable irreducible 3-manifold and let $f: \Sigma \to M$ be a general position immersion which is $\pi_1$-injective of a closed orientable surface into $M$ with image $F$. Let $\varphi: M \to M'$ be a homotopy equivalence of $M$ to an irreducible 3-manifold $M'$ and let $f': \Sigma \to M'$ be a general position immersion of $\Sigma$ in $M'$ with image $F'$, such that $f'$ is homotopic to $\varphi f$ by a homotopy $h$. In the following we will use the notation of §2 to label the various manifolds and their covers, as in Fig. 1. We say that two 2-complexes are isomorphic if there is a homeomorphism from one to the other preserving the cell structure.

Lemma 4.1. Suppose that $f$ and $f'$ are least area maps. Suppose that $f: \Sigma \to M$ satisfies the 1-line-intersection and 3-plane properties. Then the 2-complex formed by the union of the planes $\{P_i\}$ in $\tilde{M}$ is equivariantly isomorphic to the 2-complex formed by the union of the planes $\{P_i'\}$ in $\tilde{M}'$.

Proof. Recall from Lemma 2.2 that $f'$ also satisfies the 1-line-intersection and 3-plane properties. Note that as $f$ and $f'$ are assumed to be in general position, they cannot factor through coverings. A least area map which is homotopic to a map which factors through a covering of an orientable surface must itself factor through such a cover. Thus if $f$ is homotopic to a map which factors through a covering and is in general position, then it can only factor through a double cover of a non-orientable surface [4]. Also recall from the remarks immediately after Lemma 2.2 that there is an equivariant bijection between the planes in $\tilde{M}$ and those in $\tilde{M}'$. 
First we consider the case when \( f \) is not homotopic to a degree two cover of a 1-sided surface.

If \( P_1, P_2 \) are two disjoint planes such that \( P_2 \) lies on the positive side of \( P_1 \), we will show that \( P_2 \) lies on the positive side of \( P_1' \) in \( \tilde{M'} \). In the covering spaces \( M_1 \) and \( M_1' \), the planes \( P_2 \) and \( P_2' \) project to surfaces \( F_2 \) and \( F_2' \), disjoint from \( F_1 \) and \( F_1' \) respectively. The lift \( \varphi \) of \( \varphi \) which maps \( M_1 \) to \( M_1' \) sends the positive and negative ends of \( M_1 \) to the positive and negative ends of \( M_1' \) respectively, by our choice of orientations. Hence if \( P_2 \) is a plane in \( \tilde{M} \) on the positive side of \( P_1 \) such that \( F_2 \) is not compact then \( \varphi_1(F_2) \) has its ends on the positive side of \( F_1' \). As \( \varphi_1(F_2) \) is homotopic to \( F_1' \) by a homotopy moving points a bounded distance, it follows that \( F_2' \) has its ends on the positive side of \( F_1' \). As \( F_2' \) is disjoint from \( F_1' \), we see that \( P_2' \) is on the positive side of \( P_1' \) as required.

If \( P_2 \) and \( P_2' \) project to compact surfaces \( F_2 \) and \( F_2' \), in \( M_1 \) and \( M_1' \), then \( F_2 \) and \( F_2' \) are disjoint parallel copies of \( F \) and \( F' \) respectively. The surfaces \( F_1 \) and \( F_2 \) are disjoint because \( P_1 \) and \( P_2 \) are disjoint. The surface \( F_2 \) is embedded in \( M_1 \) by Theorem 5.4 of [4], which implies that if a least area map is homotopic to a covering of a 2-sided embedded surface, then it is a covering of an embedded surface. Recalling that \( P_2 = g\bar{P}_1 \), in this case \( g \) lies in the normalizer of \( \text{stab}(P_1) \) and \( \text{stab}(P_1') \), and thus acts on \( M_1 \) and \( M_1' \) so as to carry \( F_1 \) to \( F_2 \) and \( F_1' \) to \( F_2' \). Iterating this action carries \( F_1 \) to one of the two ends of \( M_1 \) and \( F_1' \) to one of the two ends of \( M_1' \) unless \( g \) interchanges the two ends of \( M_1 \) or \( M_1' \). The last case can not occur by our assumption that \( f \) is not homotopic to a degree two cover of a 1-sided surface. The end in question is determined by a lift of the loop \( g \) to \( M_1 \) and \( M_1' \), and thus if \( g \) lies on the positive side of \( F_1 \) then \( g \) lies on the positive side of \( F_1' \). It follows in both cases that if \( P_2 \) lies on the positive side of \( P_1 \) then \( P_2' \) lies on the positive side of \( P_1' \).

Now let \( P_1 \) and \( P_2 \) be intersecting planes in \( \tilde{M} \). Lemmas 2.2 and 2.3 imply that \( P_1 \) and \( P_2 \) also intersect in a line. We consider all the planes \( P_1 \) which intersect \( P_1 \). Note that the 3-plane property implies that any two planes, each of which meets \( P_1 \), are disjoint and hence that the lines of intersection of \( P_1 \cap \{ \cup P_1 \} \) can not cross. Lemma 2.2 implies that the same is true for the planes \( \{ P_1' \} \). It follows that the pattern of lines of intersection on \( P_1 \cap \{ \cup P_1 \} \) is identical to that of \( P_1' \cap \{ \cup P_1 \} \) on \( P_1' \).

An equivariant homeomorphism from the 2-complex formed by the union of the planes \( \{ P_1 \} \) in \( \tilde{M} \) to the 2-complex formed by the union of the planes \( \{ P_1' \} \) in \( \tilde{M'} \) can now be constructed inductively.

Next we consider the case when \( f \) is homotopic to a degree two cover of a 1-sided map \( f_2: \Sigma_2 \rightarrow M \). The preceding proof needs elaboration, because the normalizer \( N(\pi_1(\Sigma)) \) of \( \pi_1(\Sigma) \) in \( \pi_1(M) \) contains elements which interchange the ends of the covering \( M_1 \) of \( M \) corresponding to \( \pi_1(\Sigma) \). When \( N(\pi_1(\Sigma)) \) is not equal to \( \pi_1(\Sigma) \), there is no canonical equivariant bijection between the planes above \( f(\Sigma) \) in \( \tilde{M} \) and those above \( f'(\Sigma) \) in \( \tilde{M'} \), but given a plane \( P \) in \( \tilde{M} \) above \( f(\Sigma) \) the bijection is uniquely determined by a choice of a corresponding plane \( P' \) in \( \tilde{M'} \). We need to specify how the choice of \( P' \) is to be made. Note that if the quotient group \( N(\pi_1(\Sigma))/\pi_1(\Sigma) \) is \( Z \) then there are infinitely many planes parallel to \( P \) in \( \tilde{M} \), i.e. having the same stabilizer as \( P \). There is correspondingly an infinite number of choices for \( P' \). However, in this case no element of \( N(\pi_1(\Sigma)) \) interchanges the ends of \( M_1 \) so any choice will do as in the preceding argument. Standard results in 3-manifold theory show that the quotient group \( N(\pi_1(\Sigma))/\pi_1(\Sigma) \) is one of \( 1, Z, Z_2, \) or \( Z_2 \ast Z_2 \) [13] [28]. In the last case the quotient \( W \) of \( \tilde{M} \) by \( N(\pi_1(\Sigma)) \) is the union of two twisted 1-bundles over closed surfaces, glued along \( \Sigma \). This again presents the problem that in \( \tilde{M} \) there are infinitely many parallel planes above \( \Sigma \). We handle this by noting that the corresponding quotient \( W' \) of \( \tilde{M'} \) is homeomorphic to \( W \) by Waldhausen [30] as they are Haken manifolds. This
homeomorphism lifts to a \( N(\pi_1(\Sigma)) \)-equivariant homeomorphism of \( \tilde{M} \) with \( \tilde{M}' \), giving a bijection between the translates of \( P \) by \( N(\pi_1(\Sigma)) \) and some of the planes in \( M' \) above \( f'(\Sigma) \). Let \( P' \) correspond to \( P \) under this bijection, and choose corresponding orientations. If \( g \) lies in \( N(\pi_1(\Sigma)) \) and \( gP \) is on the positive side of \( P \), then \( gP' \) lies on the positive side of \( P' \). Now the bijection between all translates of \( P \) and \( P' \) yields an equivariant homeomorphism as in the first part of the proof.

If \( N(\pi_1(\Sigma))/\pi_1(\Sigma) \) is \( \mathbb{Z}_2 \), we need a slight modification of this argument as the quotients of \( \tilde{M} \) and \( \tilde{M}' \) by \( N(\pi_1(\Sigma)) \) are not known to be homeomorphic. Let \( P \) and \( gP \) denote the two planes in \( M \) stabilized by \( \pi_1(\Sigma) \). Note that \( g^2P = P \). There are two corresponding planes in \( \tilde{M}' \) and we need to choose which one corresponds to \( P \). To do this consider the quotient \( M_1 \) of \( \tilde{M} \) by \( \pi_1(\Sigma) \) and the corresponding quotient \( M'_1 \) of \( \tilde{M}' \). Our homotopy equivalence of \( M \) with \( M' \) determines a correspondence between the two ends of \( M_1 \) and the two ends of \( M'_1 \). \( P \) and \( gP \) project to disjoint embedded surfaces \( F \) and \( gF \) in \( M_1 \). Let \( e \) denote the end of \( M_1 - gF \) which contains \( F \), i.e. the end closer to \( F \). We choose \( P' \) so that the corresponding end of \( M'_1 - gF' \) contains \( F' \). We also choose the orientation on \( P' \), which is not determined by the homotopy equivalence in this case, to ensure that \( gP \) is on the positive side of \( P \) if and only if \( gP' \) is on the positive side of \( P' \). This allows us to complete the proof of Lemma 4.1.

\textbf{Theorem 4.2.} Let \( \{c_i\} \) be a collection of essential closed curves in general position on an orientable closed surface \( \Sigma \), satisfying the 1-line-intersection and 3-line properties. Let \( \{c'_i\} \) be another collection of the same number of essential closed curves on \( \Sigma \), also in general position, satisfying the 1-line-intersection and 3-line properties, and such that \( c'_i \) is homotopic to \( c_i \) for each \( i \). Then there is a permutation \( \sigma \) of the indices and an isotopy of \( \Sigma \) carrying \( \{c_i\} \) to \( \{c_{\sigma(i)}\} \).

\textbf{Remark.} The same result also holds in the non-orientable case. The proof needs much the same idea as the second part of the proof of Lemma 4.1.

\textbf{Proof.} Let \( C \) denote the union of \( c_1, \ldots, c_n \) and \( C' \) the union of \( c'_1, \ldots, c'_n \) and let \( N(C), N(C') \) denote regular neighborhoods of \( C \) and \( C' \). First consider the case when \( n = 1 \). We argue as in the proof of Lemma 4.1. Let \( f \) and \( f' \) be homotopic immersions of \( S^1 \) into \( \Sigma \). If \( f \) were homotopic to a map which factors through a covering of circles and \( f_2 : S^1 \to \Sigma \), then the covering would be of degree 2 and \( f_2 \) would be one-sided. As \( \Sigma \) is orientable, \( f \) cannot be homotopic to such a map. Thus the first part of the proof of Lemma 4.1 shows that there is a homeomorphism \( h \) of \( N(C) \) to \( N(C') \) which induces the identity on the images of their fundamental groups in \( \pi_1(\Sigma) \). This homeomorphism extends to a homeomorphism of \( \Sigma \) which induces the identity on \( \pi_1(\Sigma) \). For if a boundary curve \( S \) of \( N(C) \) is inessential in \( \Sigma \) the corresponding curve \( S' \) in \( \partial N(C') \) will also be inessential, so that \( h \) extends to the union of \( N(C) \) with the 2-disk bounded by \( S \). After repeating, we can assume that all components of \( \partial N(C) \) are essential. This implies that \( \partial N(C) \) is homotopic to \( \partial N(C') \) and hence isotopic to \( \partial N(C') \), so the rest of the extension problem is clear. The resulting homeomorphism of \( \Sigma \) is homotopic to the identity and hence isotopic to the identity. This provides the required ambient isotopy of \( \Sigma \).

For \( n > 1 \), the proof that disjoint lines in the universal cover which correspond to covers of the curves in \( C \) and \( C' \) lie on determined sides of one another may fail. But it fails only when two of these lines project to distinct but homotopic curves. In this case we may have to do some relabeling. If \( \Sigma \) is not the torus, a given line in the universal cover can only be parallel to finitely many other lines. Thus the required re-ordering of the labels is clear. If \( \Sigma \) is the torus, we will have infinitely many lines all parallel in the universal cover even in the
case $n = 1$. In this case we can argue as follows. As each $c_i$ is not homotopic to a proper power in $\pi_1(\Sigma)$, we see that each $c_i$ is a simple closed curve. The 3-line property implies that there are at most two isotopy classes of the $c_i$'s. Thus either the $c_i$'s are all disjoint or they divide $\Sigma$ into quadrilaterals. The same comments apply to the $c_i$ family, so that the required result is clear.

**Theorem 4.3.** Suppose that $f$ and $f'$ are least area maps in general position and that $f: \Sigma \to M$ satisfies the 1-line-intersection and 4-plane properties. Then $f$ and $f'$ can be homotoped so that the 2-complex formed by the union of the planes $\{ P_i \}$ in $\widetilde{M}$ is equivariantly isomorphic to the 2-complex formed by the union of the planes $\{ P'_i \}$ in $\widetilde{M}'$.

**Proof.** As in the proof of Lemma 4.1, we consider the case when $f$ is not homotopic to a degree two cover of a 1-sided surface. We can extend to the general case by using the same arguments as in the second part of the proof of Lemma 4.1.

Theorem 3.1 tells us that we can homotop $f$ and $f'$ preserving the 1-line-intersection and 4-plane properties so that the double curves have the 1-point-intersection property. We claim that for a surface in general position in $M$ satisfying all these conditions, the existence of a 3-gon prism region in $\widetilde{M}$ is equivalent to the existence of three planes which meet pairwise and whose stabilizers have a common infinite cyclic subgroup. This is immediate from the definitions if $\widetilde{M}$ contains a 3-gon prism region. Conversely, suppose that $\widetilde{M}$ contains planes $P_1, P_2$ and $P_3$ such that $P_i$ meets $P_j$ in a line $l_{ij}$ with a common stabilizer generated by $\alpha$. Then each pair of these lines must be disjoint or meet in infinitely many points. As we are assuming that the double curves of $f$ have the 1-point-intersection property, these lines must be disjoint or coincide. As $f$ is in general position they must be disjoint, so that there is a 3-gon prism region in $\widetilde{M}$ bounded by $P_1, P_2$ and $P_3$. Now it follows that the absence of 3-gon prism regions in $\widetilde{M}$ implies the absence of 3-gon prism regions in $\widetilde{M}'$.

For clarity of presentation, we will assume temporarily that there are no 3-gon prism regions in $\widetilde{M}$. This assumption can often be shown to hold in specific situations, such as those of Examples 1.5 and 1.6, and makes the completion of the proof much simpler. For we will see that this implies that the unions of the families of planes in $\widetilde{M}$ and $\widetilde{M}'$ are equivariantly homeomorphic without further homotopies of $f$ and $f'$. After handling this case we will return to the case where there are 3-gon prism regions and show how to deal with the additional complications.

We begin with a plane $P_1$ and consider a plane $P_2$ which crosses it in a line $\lambda$. The corresponding planes $P'_1$ and $P'_2$ also cross in a line $\lambda'$. If any other planes cross $\lambda$ transversely, then the order in which they do so is the same as that with which the corresponding planes of $\{ P_i \}$ cross $\lambda'$, as any two planes crossing $\lambda$ are disjoint by the 4-plane property, and we saw in Lemma 4.1 that the order of disjoint planes is the same in $\widetilde{M}$ and $\widetilde{M}'$. Thus we can construct a homeomorphism of a neighborhood of $\lambda$ taking planes $P_1$ in $\widetilde{M}$ to the corresponding planes $P'_1$ in $\widetilde{M}'$. This can then be extended, line by line, to a neighborhood of the component of $P_1 \cap \{ \cup P_i \}$ containing $\lambda$.

Next suppose that $P_3$ is a plane intersecting $P_1$ in a line $\mu$ disjoint from $\lambda$. If $P_3$ is disjoint from $P_2$ then it lies either on the positive or negative side of $P_2$ and the corresponding statement holds for $P'_3$ and $P'_2$ in $\widetilde{M}'$. If $P_3$ does meet $P_2$ then it does so in a line $\nu$ disjoint from $P_1$. The planes $P'_1, P'_2$ and $P'_3$ meet similarly in disjoint lines $\lambda', \mu'$, $\nu'$. Consider the quotient $M_2$ of $M$ by the stabilizer of $P_2$. Each of $P_1$ and $P_3$ project to cylinders running from one end of $M_2$ to the other. The intersection line $\mu$ of $P_1$ and $P_3$ projects to either
a circle or a line in $M_2$, according to whether the stabilizers of the three planes have cyclic or trivial intersection. If $\mu$ projects to a circle, then there is a 3-gon prism region in $M$, because $P_1, P_2$ and $P_3$ intersect pairwise and have stabilizers which intersect in an infinite cyclic group. This contradicts our assumption that there are no 3-gon prism regions. If the intersection line $\mu$ projects to a line in $M_2$, then this line is contained in one end of $M_2$. It does not lie within a finite distance of the projection of $P_2$ to $M_2$, and thus $\mu$ does not lie within a finite distance of $\lambda$. By Lemma 2.5, $\mu'$ lies on the positive side of $\lambda'$, since $\lambda'$ lies within a finite distance of $\varphi(\lambda)$. Thus we can extend the homeomorphism to the component of $P_1 \cap \{ \cup P_i \}$ containing $\mu$. Continuing, we get a homeomorphism of $P_1$ to $P_1'$ taking the intersections of planes $P_i$ and $P_1$ in $M$ to the corresponding intersections of planes $P_1'$ and $P_1''$ in $M'$. We can now extend to planes $P_2, P_3, \ldots$ obtaining an equivariant homeomorphism of the two 2-complexes as desired.

Finally we consider the situation where 3-gon prism regions may exist in $\tilde{M}$. If a 3-gon prism region exists, we aim to collapse this region to a line. This move is rather different from our previous moves in that repetition of it may destroy the 4-plane property and 1-line-intersection properties by causing two planes in $\tilde{M}$ to intersect which were previously disjoint. This could occur if one of the edge lines of the prism region in $\tilde{M}$ has a curve of triple points on its boundary from a previous collapse, leading to a curve of quadruple points when the prism region is collapsed. Two previously disjoint planes now meet non-transversely along this line. As this collapsing process is repeated, circles of $k$-tuple points may be created in $M$.

Our collapsing process has to be defined carefully, as it is not clear that even one collapse can be made. For technical reasons, it seems simplest to collapse $n$-gon prism regions for all values of $n$. We restrict our attention to those prism regions whose edges contain no triple points. We call such a prism region a good prism region. Any plane which meets the interior of a good prism region divides it into two good prism regions. Note that under the equivariant bijection between the planes in $\tilde{M}$ and in $\tilde{M}'$, $n$-gon prism regions correspond and good $n$-gon prism regions correspond. This is by the same reasoning by which we showed that 3-gon prism regions in $\tilde{M}$ correspond to 3-gon prism regions in $\tilde{M}'$.

Recall that a prism region is innermost if it contains no other prism region. Let $X$ be an innermost good prism region in $\tilde{M}$. Then $X$ is the closure of a component of $\tilde{M}$ minus the planes in $\tilde{M}$. Thus the image in $M$ of $X$ is the closure of a component of $M - f(\Sigma)$. It will be called a compact prism region in $M$. As translates of $X$ can only meet $X$ in some union of strips and edges in $\partial X$, it follows that a regular neighborhood of a compact prism region in $M$ is a Seifert fiber space in which the images of the double lines are fibers. If we let $N$ denote a regular neighborhood of the union of all the compact prism regions in $M$, it follows that $N$ is a Seifert fiber space, not necessarily connected. Note that the pre-image in $\tilde{M}$ of $N$ is a regular neighborhood of the union of all the good prism regions, not just the good innermost regions.

As $M$ is not a Seifert fiber space, $N$ cannot equal $M$. As $M$ is not Haken, each component of $\partial N$ is compressible in $M$. Each component of $\partial N$ carries a loop essential in $M$, namely a fiber of the Seifert fibration of $N$. Thus no component of $\partial N$ can lie in a ball. Hence each component of $\partial N$ bounds a solid torus in $M$. Let $N_0$ denote a component of $N$ and let $T$ be a component of $\partial N_0$. Let $V$ be the solid torus in $M$ bounded by $T$. We will show that $V$ must equal $N_0$. Note that it is conceivable that $N_0$ is contained in and not equal to $V$, or that $V \cap N_0 = T$. Let $V_1$ denote the union of all components of $M - f(\Sigma)$ which meet $V$ and let $U_1$ denote a component of the pre-image in $\tilde{M}$ of $V_1$. As the stabilizer of $U_1$ is infinite cyclic and the quotient by this stabilizer is compact, it follows that $U_1$ is a prism region. Now
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$U_1$ must be a good prism region as it meets the good prism regions in the pre-image of $N_0$. This implies that $V$ lies in $N$ and hence in $N_0$. We conclude that each component of $N$ is a solid torus.

We will collapse each component of $N$ to a circle by a homotopy of $f$, chosen so as to induce an isotopy of all of the $P_i$'s in $\tilde{M}$. This can be done as the Seifert fibration on $N$ is such that $f(\Sigma)$ meets $N$ in a union of fibers. This isotopy will have the effect of simultaneously collapsing every good prism region in $\tilde{M}$ to a line.

In order to keep track of the collapsing that occurs, we will define an equivalence relation on the double lines $l_{ij}$ formed by transversely intersecting planes $P_i$ and $P_j$. If the planes $P_1, P_2, \ldots, P_n$ together bound a good $n$-gon prism region, then we define the intersection lines of $P_i$ and $P_{i+1}$ to be equivalent. We consider the family of equivalence classes generated by this relation. We consider a similarly defined equivalence class of lines $l'_{ij}$ in $M'$. The lines in an equivalence class are precisely the lines which collapse to a common line in $\tilde{M}$ when all good prism regions are collapsed. It follows that an equivalence class of lines in $\tilde{M}$ corresponds to one in $M'$.

Before we can define the required equivariant homeomorphism, we need to prove some technical facts about the results of our collapse. These are contained in the following results.

**Lemma 4.4.** Let $P_1$ and $P_2$ be two planes in $\tilde{M}$.

(a) If $P_1$ and $P_2$ were disjoint before the collapse, then after the collapse their intersection consists of disjoint lines of non-transverse intersection.

(b) If $P_1$ and $P_2$ crossed before the collapse, then after the collapse they still meet in only one line and the intersection is still transverse.

**Proof.** (a) This is clear as the collapse of each component of $N$ produces at most one line of intersection between $P_1$ and $P_2$ and this intersection is non-transverse.

(b) Suppose that after the collapse $P_1$ and $P_2$ meet transversely in a line $l$ and non-transversely in a line $\lambda$. These lines must be disjoint. Then before the collapse, $P_1$ and $P_2$ met transversely in $l$ and both also met some component $U$ of the union of all good prism regions in $\tilde{M}$. Now $P_1 \cap U$ and $P_2 \cap U$ must each consist of a finite union of strips in $U$. In particular, some element $\alpha$ which stabilizes $U$ must also stabilize both $P_1$ and $P_2$. As the intersection of their stabilizers is infinite cyclic by Lemma 2.3, it follows that $\alpha$ stabilizes $l$. But this implies that there is a prism region $W$ in $\tilde{M}$ with $l$ as an edge which meets $U$. As $W$ meets $U$, it is a good prism region. We conclude that $l$ lies in $U$ and that $l$ equals $\lambda$, a contradiction. This completes the proof of Lemma 4.4.

Next we consider a line $\lambda$ in $\tilde{M}$ which is the intersection of planes $P_1, \ldots, P_n$. We know that the corresponding planes $P_i', \ldots, P_n'$ in $\tilde{M}'$ intersect in a line $\lambda'$ corresponding to $\lambda$. We need to know that these families of planes are arranged in the same way around this common line. Let $N$ be a regular neighborhood of $\lambda$ in $\tilde{M}$ which is invariant under the stabilizer of $\lambda$. Choose an orientation of $\lambda$ and let $P_{i+}$ and $P_{i-}$ denote the components of $P_i - \lambda$, with notation chosen according to some orientation convention. A choice of orientation for $\lambda$ is equivalent to a choice of a generator for $\text{stab}(\lambda)$. Thus we can make the corresponding choice of orientation of $\lambda'$ and choose $P_{i+'}$ and $P_{i-'}$ according to the same orientation convention as before. This means that when we homotop $\hat{\phi}(P_i)$ to $P_i'$ by our homotopy which moves points a bounded distance, we can arrange to homotop $\hat{\phi}(\lambda)$ to $\lambda'$, $\hat{\phi}(P_{i+})$ to $P_{i+'}$ and $\hat{\phi}(P_{i-})$ to $P_{i-'}$. The precise result we need can be stated as follows.
LEMMA 4.5. There is a homeomorphism of $N$ with $N'$ sending $P_i \cap N$ to $P_i \cap N'$, $1 \leq i \leq n$, which is equivariant with respect to the actions of the stabilizers of $\lambda$ and $\lambda'$, sends $P_i+$ to $P_i+$ and sends $P_i-$ to $P_i-$, $1 \leq i \leq n$.

Proof. We consider first any two planes $P_1$ and $P_2$. Lemma 4.4, shows that $P_1$ crosses $P_2$ at $\lambda$ if and only if $P_1'$ crosses $P_2'$ at $\lambda'$, and orientations are preserved, so we can construct a homeomorphism from $N$ to $N'$ carrying $P_i$ to $P_i'$, $1 \leq i \leq n$, which is equivariant with respect to the actions of the stabilizers of $\lambda$ and $\lambda'$, sends $P_i+$ to $P_i+$ and sends $P_i-$ to $P_i-$, $1 \leq i \leq n$.

We proceed by induction. Consider the case of adding an additional plane $P_k$ to the planes $P_1, \ldots, P_{k-1}$. We assume that there is a homeomorphism of $N$ to $N'$ sending $P_i \cap N$ to $P_i \cap N'$ for $i = 1, \ldots, k - 1$. We will show that there is a homeomorphism of $N$ to $N'$ which in addition sends $P_k$ to $P_k'$. The planes $P_1, \ldots, P_{k-1}$ split $N$ into $2k - 2$ sectors which we label $S_1, \ldots, S_{2k-2}$, and by induction there are sectors $S_1', \ldots, S_{2k-2}'$ in $N'$ corresponding by the homeomorphism of $N$ to $N'$. To carry out the induction step it suffices to show that if a half-plane $P_i$ is contained in a sector $S_j$, the half-plane $P_i'$ must lie in $S_j$.

Note by the $k = 2$ case that $P_k$ lies on the positive side of a plane $P_i$ if and only if $P_k'$ lies on the positive side of $P_i'$. Suppose that $P_k'$ lies in a distinct sector $S_j$. Then we can form a plane $P$ by joining the two possible placments of the image of $P_k$ in $N'$. $P$ lies in $S_j' \cup S_i$, and does not cross any of $P_i, \ldots, P_{i-1}$, since each of its two halves lie on the same side of each of $P_i, \ldots, P_{k-1}$. It follows that there is a plane $P_j$, $1 < j < k$ which does not cross $P_i, \ldots, P_{i-1}$, a contradiction. This establishes Lemma 4.5.

We now continue the proof of Theorem 4.3. We proceed to define an equivariant isomorphism of the 2-complex formed by the union of the planes $\{P_i\}$ in $M$ with the 2-complex formed by the union of the planes $\{P_i\}$ in $M'$. We begin again with a plane $P_1$ and consider a plane $P_2$ which meets it in a line $\lambda = l_{12}$. Lemma 4.5 shows that there is a homeomorphism of a neighborhood of $\lambda$ to a neighborhood of $\lambda'$ which carries the planes $P_1$ meeting $\lambda$ to the planes $P_1'$ meeting $\lambda'$. If $\lambda$ was not obtained by collapsing, then only $P_2$ crosses $P_1$ along $\lambda$ and there may be other components of $P_1 \cap \{\cup P_i\}$ crossing $\lambda$. As before, the order of these lines corresponds in $M$ and $M'$. If $\lambda$ was obtained by collapsing then there are three planes mutually crossing along $\lambda$ so that there are no other components of $P_1 \cap \{\cup P_i\}$ crossing $\lambda$ by the four plane property. In either case this homeomorphism can then be extended, line by line, to a neighborhood of the component of $P_1 \cap \{\cup P_i\}$ containing $\lambda$.

Suppose that $\mu$ is a line of $P_1 \cap \{\cup P_i\}$ disjoint from the component of $P_1 \cap \{\cup P_i\}$ containing $\lambda$. There is at least one plane $P_2$ crossing $P_1$ transversely at $\mu$. We will show that $\mu$ lies on the same side of $\lambda$ as $\mu'$ does of $\lambda'$. Note that $P_3$ meets $P_1$ along exactly one line. We consider two possibilities as to how the three planes meet.

Case 1. If $P_3$ does not cross $P_2$ transversely then it lies either on the positive or negative side of $P_2$, and the corresponding statement holds for $P_3'$ and $P_2'$ in $M'$, so that $\mu$ lies on the same side of $\lambda$ as $\mu'$.

Case 2. If $P_3$ does cross $P_2$ transversely, then it crosses $P_2$ in a line $\nu$ distinct from $\lambda$ and $\mu$. The planes $P_1', P_2'$ and $P_3'$ meet similarly in distinct lines $\lambda', \mu', \nu'$. Consider the quotient
$M_2$ of $M$ by the stabilizer of $P_2$. By Lemma 2.3 each of $P_1$ and $P_3$ project to cylinders running from one end of $M_2$ to the other. The intersection line $\mu$ of $P_1$ and $P_3$ projects to either a circle or a line in $M_2$, according to whether the stabilizers of the three planes have cyclic or trivial intersection. If $\mu$ projects to a circle, then $P_1$, $P_2$ and $P_3$ intersect transversely pairwise and have stabilizers which intersect in an infinite cyclic group. Since we have collapsed all 3-gon prism regions, the three lines of intersection of these planes in $\tilde{M}$ are already coincident, contradicting the assumption that $\lambda$ and $\mu$ are disjoint. So the intersection line $\mu$ projects to a line in $M_2$. This line is contained in one end of $M_2$, so it does not lie within a finite distance of the projection of $P_2$ to $M_2$. Thus $\mu$ does not lie within a finite distance of $\lambda$. It follows from Lemma 2.5 that $\lambda'$ lies on the positive side of $\lambda$ if and only if $\mu$ lies on the positive side of $\lambda$.

Thus the side of $\lambda$ on which $\mu$ lies is determined, and we can extend the homeomorphism to the component of $P_1 \cap \{ \cup P_i \}$ containing $\mu$. Continuing, we get a homeomorphism of $P_1$ to $P_1'$ taking the intersections of planes $P_1$ and $P_1$ in $M$ to the corresponding intersections of planes $P_1'$ and $P_1'$ in $M'$. We can now extend to planes $P_2, P_3, \ldots$ obtaining an isomorphism of the two 2-complexes. Moreover, the isomorphism thus obtained extends to a regular neighborhood of these 2-complexes in $\tilde{M}$ and $M'$. This completes the proof of Theorem 4.3.

Remark. Instead of collapsing the solid torus components of $N$ to circles, we could add $N$ to a regular neighborhood of $f(\Sigma)$. Essentially the same arguments will then produce a homeomorphism from this union to the corresponding union in $M'$.

§5. HOMEOMORPHISMS OF NON-HAKEN 3-MANIFOLDS

In §3 and §4, we assumed that all our maps of surfaces were in general position. A least area immersion $f$ need not be in general position, see [4]. It can be perturbed to a general position immersion $f'$, but $f'$ may not be least area. However, we can choose $f'$ so as to have the same key combinatorial properties as $f$. These properties can be summarized as follows. Let $P_1$ and $P_2$ be planes in $M$ in the pre-image of $f(\Sigma)$ which cross and let $G$ denote the intersection of the stabilizers of $P_1$ and $P_2$. If $S_1$ and $S_2$ denote the images of $P_1$ and $P_2$ in $M_G$, the quotient of $M$ by $G$, then there are no compact product regions in $M_0$ between $S_1$ and $S_2$. As $f$ has the 1-line-intersection property, all the intersections of $f$ are transverse. Thus a small perturbation $f'$ of $f$ must have the same combinatorial properties as $f$. Note that $f'$ need not have the same number of triple points as $f$. Now these combinatorial properties of a least area surface are all that is needed to prove the results of Sections 3 and 4. It follows that the results in those sections can be applied to a map $f'$ which is a small perturbation of a least area map with the 1-line-intersection property.

The following is proved in Lemma 1.4 of [11].

**Lemma 5.1.** Let $M$ be a non-Haken irreducible orientable 3-manifold and let $f: \Sigma \to M$ be a least area surface immersed in $M$. Let $N(F)$ be a regular neighborhood of $f(F)$. Then the closures of the components of $M - N(F)$ are handlebodies whose fundamental groups inject into $\pi_1(M)$.

The proof of Lemma 1.4 in [11] does not require that $f$ be in general position, nor that $f$ be least area. It goes through as long as the planes in the universal cover of $M$ have no closed curves of intersection. In particular, it applies to the 2-complex resulting from
collapsing of prism regions which may have lines of k-tuple points and lines of non-transverse intersection of planes.

We now restate and prove the main theorem.

**Theorem 5.2.** Let $M$ be a closed orientable irreducible 3-manifold containing an immersed surface $F$ satisfying the 4-plane and 1-line-intersection properties. Let $M'$ be a closed irreducible 3-manifold homotopy equivalent to $M$. Then $M'$ is homeomorphic to $M$.

**Proof.** By passing to a double cover of $F$ if necessary, we can assume that $F$ is orientable.

We fix a metric on $M$ and take $f: F \rightarrow M$ to be a least area surface in its homotopy class. $f$ still satisfies the 1-line-intersection and 4-plane properties by Lemma 2.4. We then perturb $f$ so that it lies in general position. By the discussion at the start of this section, we can continue to assume that $f$ has the key combinatorial properties of a least area surface which are used in Sections 3 and 4.

We first modify $f$ to put it in a canonical position. We eliminate all 2-gons between double curves as in Theorem 3.1 and then collapse all good prism regions as in Theorem 4.3.

Now Theorem 4.3 yields a homeomorphism from a regular neighborhood $N$ of $f(\Sigma)$ to a regular neighborhood $N'$ of $f'(\Sigma)$. We need to extend this homeomorphism to a homeomorphism from $M$ to $M'$. Lemma 5.1 shows that $M$ and $M'$ are obtained from the regular neighborhoods $N$ and $N'$ by gluing in handlebodies. We need to specify how the handlebodies are glued in. This is determined by specifying which curves of a component $S$ of $\partial N$ are homotopically trivial in the handlebody attached to it. We determine if a curve $\alpha$ is trivial by attempting to lift it to the cover $\tilde{N}$ of $N$ which is the lift of $N$ to $\tilde{M}$. If $\alpha$ lifts to $\tilde{M}$ then the curve is trivial in the handlebody, and if it does not lift, it is non-trivial. Theorem 4.3 shows that $N$ and $\tilde{N}$ are homeomorphic to $N'$ and $\tilde{N}'$ respectively, so that the same set of curves lifts in each case. Thus we see that $M$ and $M'$ are homeomorphic.

§6. HYPERBOLIC MANIFOLDS

In this section we examine the relation between the 4-plane property and totally geodesic surfaces in hyperbolic manifolds.

**Lemma 6.1.** If four hyperbolic planes in hyperbolic 3-space or four Euclidean planes in Euclidean 3-space all intersect pairwise, then one of their dihedral angles is smaller than or equal to $\tau$, the dihedral angle of a regular Euclidean tetrahedron.

**Remark.** $\tau \approx 0.39183\pi \approx 70.5288^\circ$.

**Proof.** First note that if three planes in Euclidean or hyperbolic space intersect pairwise but have no common point of intersection, then there exists a mutually perpendicular plane which intersects them in a triangle. This triangle has angles equal to the corresponding dihedral angles, thus one is no greater than $\pi/3$, which is less than $\tau$, and we are done. Thus we can assume that any three of the four planes meet in a common point. It follows that they bound a tetrahedral region. In hyperbolic space we can scale down the tetrahedron, increasing all dihedral angles. In the limit the 4 planes intersect in a single point, reducing us to the Euclidean case.

We claim that if a Euclidean tetrahedron $T$ has all dihedral angles $\geq \tau$, then $T$ is regular. In particular, any Euclidean tetrahedron has at least one dihedral angle which is $\leq \tau$. 
Although this result is probably known, we have not found an explicit statement in the
literature, so a proof is presented.

We represent the situation at a vertex $v$ of $T$ by a geodesic triangle on the unit sphere
with angles equal to the dihedral angles of $T$ at $v$. The edge lengths of this triangle are the
face angles at $v$. Hence an equilateral spherical triangle with all angles equal to $\tau$ has edge
lengths equal to $\pi/3$. The following fact from spherical geometry will be needed.

**Lemma 6.2.** Let $\Delta$ be a triangle on the sphere of radius one with all angles $\geq \tau$. Then
either all the edge lengths of $\Delta$ are $\geq \pi/3$, or two of the edge lengths are $\geq \pi/2$.

*Remark.* The case of two edge lengths $> \pi/2$ and one $< \pi/3$ can occur.

*Proof.* Let $ABC$ be an equilateral triangle on the sphere of radius one with all angles
equal to $\tau$, and with $A$ at the North pole. We let $A'$, $B'$ and $C'$ denote the vertices of the given
triangle $\Delta$ and we assume that $A'$ equals $A$, that $C'$ lies on $AC$ or its geodesic extension and
that the angle $B'A'C'$ contains $BAC$. We will assume that at most one edge of $\Delta$ has length
$\geq \pi/2$, or the lemma is already proved.

**Case 1.** At least two edges of $\Delta$ have length $< \pi/3$. By relabelling vertices, we can
assume that $A'C'$ and $A'B'$ are $< \pi/3$. See Fig. 8. We claim that one of the angles $A'B'C'$,
$A'C'B'$ is $< \tau$. For if $AB = AC' < \pi/3$ and $B'A'C' \geq \tau$, then $A'B'C'$ and $A'C'B'$ are equal and
$< \tau$. Now moving $B'$ towards $A$ reduces the angle $A'C'B'$ and moving $C'$ towards $A$
reduces the angle $A'B'C'$. Thus at least one of the angles $A'B'C'$, $A'C'B'$ is $< \tau$, as claimed.

**Case 2.** Only one edge of $\Delta$ has length $< \pi/3$. Relabel $\Delta$ so that $AC < \pi/3$ and
$AB' < \pi/2$. Choose $B''$ on $AB'$ so that $AB'' = \pi/3$. See Fig. 9. Then angle $A'B'C' < angle
AB'C'$ as we are assuming that $AB < \pi/2$. Now angle $AB'C' < angle ACB' < angle ACB = \tau$. Thus we have a contradiction to our hypothesis that all angles of
$\Delta$ are $\geq \tau$.

This completes the proof of Lemma 6.2.

Let $T$ be a Euclidean tetrahedron with all dihedral angles $\geq \tau$. Lemma 6.2 shows that
for the three face angles at a vertex of $T$ either all three angles are $\geq \pi/3$, or two of them are
$\geq \pi/2$. Consideration of the possible cases shows that all the face angles must equal $\pi/3$, so
that $T$ is regular, as claimed.
Corollary 6.5. Let $M$ be a hyperbolic 3-manifold containing an immersed totally geodesic surface $F$ whose self-intersections at each point form an angle larger than $\tau$. If $M'$ is a closed irreducible 3-manifold homotopy equivalent to $M$, then $M'$ is homeomorphic to $M$.

Proof. It suffices to check that the 4-plane and 1-line-intersection conditions hold for $F$ in order to apply Theorem 5.2. The 1-line-intersection condition is clear, and the 4-plane condition follows from Lemma 6.1.

Question 6.6. Does every non-Haken irreducible 3-manifold with infinite fundamental group contain a surface satisfying the 1-line-intersection and 4-plane properties? In particular, do hyperbolic 3-manifolds contain such surfaces?

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