Unknot diagrams requiring a quadratic number of Reidemeister moves to untangle

Joel Hass* and Tahl Nowik

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Abstract

We present a sequence of diagrams of the unknot, for which the minimum number of Reidemeister moves required to pass to the trivial diagram, both in $\mathbb{S}^2$ and in $\mathbb{R}^2$, is quadratic with respect to the number of crossings.

1 Introduction

Knots in $\mathbb{R}^3$ or $\mathbb{S}^3$ are commonly represented by a knot diagram, a generic projection of the knot to a plane or 2-sphere. A diagram is an immersed oriented planar or spherical curve with finitely many double points, called crossings. Each crossing is marked to indicate a strand, called the overcrossing, that lies above the second strand, called the undercrossing. The original knot can be recovered, up to isotopy, by constructing a curve with the overcrossing arcs pushed slightly above the plane of the diagram and the remainder of the diagram lying in this plane.

Alexander and Briggs [2] and independently Reidemeister [11] showed that two diagrams of the same knot can be connected through a sequence of moves of three types, commonly called Reidemeister moves. The number of such moves required to connect two equivalent diagrams is difficult to estimate. An exponential upper bound is obtained in [6], where it is shown that there is a positive constant c_1 such that given an unknot diagram $\mathcal{D}$ with n crossings, no more than $2^{c_1 n}$ Reidemeister moves are required to transform $\mathcal{D}$ to the trivial knot diagram. See Figure 1.

We can get some lower bounds by looking at classical invariants of diagrams such as crossing numbers, writhes and winding numbers, since a

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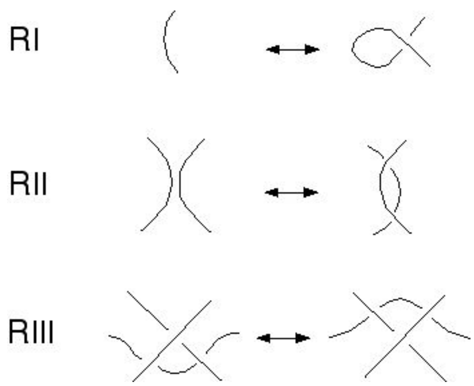

Figure 1: The three types of Reidemeister moves

single Reidemeister move changes these numbers by 0, 1 or 2. However non-trivial lower bounds are difficult to obtain. This is a common situation in complexity theory. While upper bounds can be established by careful analysis of one procedure, lower bounds require somehow bounding from below the running time of all possible procedures. It is quite difficult to get lower bounds for any particular pair of equivalent diagrams, as seen in [4], [5], [9] and [10]. We note that in [8] examples are constructed that show that PL spanning disks for unknots can require exponentially many faces compared to the number of edges or vertices of the diagram. However these diagrams can be transformed to the trivial diagram using linearly many Reidemeister moves.

Given two knot diagrams $\mathcal{D}, \mathcal{E}$ of the same knot, we define the *Reidemeister distance* $d(\mathcal{D}, \mathcal{E})$ between $\mathcal{D}$ and $\mathcal{E}$ to be the minimal number of Reidemeister moves required to pass from $\mathcal{D}$ to $\mathcal{E}$. One may consider this notion in either $\mathbb{S}^2$ or $\mathbb{R}^2$, and our result will hold in both settings. If we wish to distinguish them we will refer to $d_{\mathbb{R}^2}(\mathcal{D}, \mathcal{E})$ and $d_{\mathbb{S}^2}(\mathcal{D}, \mathcal{E})$.

Our main tool is an invariant of knot diagrams developed in [7], and used there to obtain new linear lower bounds on $d_{\mathbb{R}^2}$ and $d_{\mathbb{S}^2}$. In this paper we give a family of examples of unknot diagrams D_n with D_n having $7n$ crossings and with at least $2n^2 + 3n - 2$ Reidemeister moves required to transform D_n to the trivial diagram. These are the first examples for which a non-linear lower bound has been demonstrated.

2 The example

Let U denote the trivial knot diagram. We will present a sequence D_n of diagrams of the unknot, for which $d(D_n, U)$ grows quadratically with respect to the number of crossings of D_n. More precisely, we prove:

Theorem 2.1. *In both $\mathbb{S}^2$ and $\mathbb{R}^2$, the diagram D_n of Figure 2, which has $7n$ crossings, satisfies:*

$$2n^2 + 3n - 2 \quad \leq \quad d(D_n, U) \quad \leq \quad 2n^2 + 4n + 1.$$

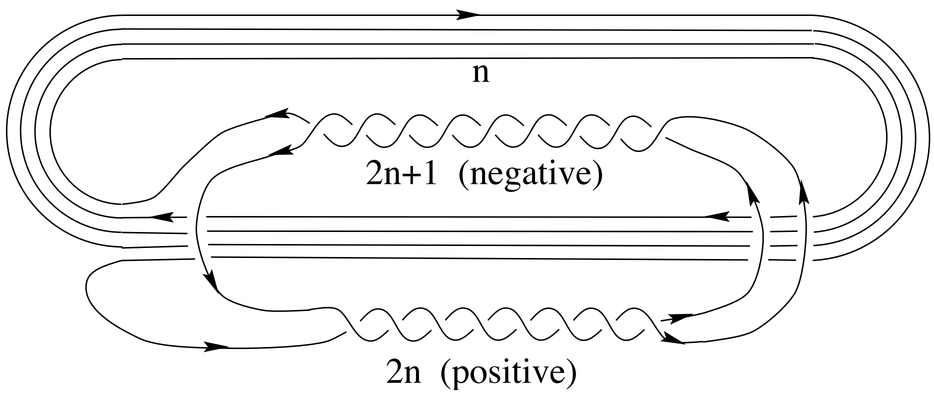

Figure 2: The diagram D_n for $n = 4$.

We will prove the lower bound in $\mathbb{S}^2$, and the upper bound in $\mathbb{R}^2$, and it follows that both bounds hold in both settings. We recall the definition of the invariant of knot diagrams in $\mathbb{S}^2$ introduced in [7]. We denote the set of all knot diagrams in $\mathbb{S}^2$ by $\mathcal{G}$. We denote the set of all two component links in $\mathbb{R}^3$ by $\mathcal{L}$. Given a knot diagram $D \in \mathcal{G}$ and a crossing a in D, define the smoothing $D^a \in \mathcal{L}$, to be the two component link obtained by smoothing the crossing a. Given a knot diagram D, denote by D_+ the set of all positive crossings in D and by D_- the set of all negative crossings. Given an invariant of two component links $\phi : \mathcal{L} \to S$ where S is any set, let $\mathbb{G}_S$ denote the free abelian group with basis $\{X_s, Y_s\}_{s \in S}$. We then define the invariant $I_\phi : \mathcal{G} \to \mathbb{G}_S$ as follows:

$$I_\phi(D) = \sum_{a \in D_+} X_{\phi(D^a)} + \sum_{a \in D_-} Y_{\phi(D^a)}.$$

In this work ϕ is taken to be the linking number, $lk : \mathcal{L} \to \mathbb{Z}$, that is, we use the invariant $I_{lk} : \mathcal{G} \to \mathbb{G}_{\mathbb{Z}}$. It is shown in [7] that the change in the value of I_{lk} following a Reidemeister move, is of the following form. For an RI move: X_0 or Y_0. For an RII move: $X_k + Y_k$ or $X_k + Y_{k+1}$. For an RIII move: $X_k - X_{k+1}$ or $Y_k - Y_{k+1}$.

Let R be the set of elements in $\mathbb{G}_{\mathbb{Z}}$ of the form X_0, Y_0, X_k+Y_k, X_k+Y_{k+1}, $X_k - X_{k+1}$, $Y_k - Y_{k+1}$, and their negatives. That is, R is the set of all elements of $\mathbb{G}_{\mathbb{Z}}$ that may appear as the change in the value of $I_{lk}(D)$ when performing a Reidemeister move on D. The set R generates $\mathbb{G}_{\mathbb{Z}}$. The length of an element of $\mathbb{G}_{\mathbb{Z}}$ with respect to the generating set R is called its R-*length*. Given two diagrams D, E of the same knot, the R-length of $I_{lk}(D) - I_{lk}(E)$ is a lower bound for $d(D, E)$ in $\mathbb{S}^2$, and so a fortiori, also in $\mathbb{R}^2$. In particular, if D is a diagram of the unknot, then since $I_{lk}(U) = 0$, the R-length of $I_{lk}(D)$ gives a lower bound for $d(D, U)$. We use this procedure to obtain our lower bound for $d(D_n, U)$.

By direct inspection we see that

$$I_{lk}(D_n) = nX_n + nX_{-n} + 2nX_{-1} + (3n + 1)Y_0.$$

Indeed, each crossing in the bottom horizontal string of $2n$ crossings contributes X_{-1}, each crossing in the top horizontal string of $2n + 1$ crossings contributes Y_0, each crossing in the left vertical line of n crossings contributes Y_0 (the bottom one requires separate checking), each crossing in the middle vertical line of n crossing contributes X_{-n}, and each crossing in the right vertical line of n crossings contributes X_n. Together this gives $nX_n + nX_{-n} + 2nX_{-1} + (3n + 1)Y_0$, and we denote this element of $\mathbb{G}_{\mathbb{Z}}$ by v_n.

Theorem 2.1 will be proved by first showing that the R-length of v_n is at least $2n^2 + 3n - 1$, and then demonstrating an explicit sequence of $2n^2 + 4n + 1$ Reidemister moves in $\mathbb{R}^2$ from D_n to U.

Let $g : \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ be the homomorphism defined by $g(X_k) = 1 + |k|$ and $g(Y_k) = -1 - |k|$ for all k. Then $|g(r)| \leq 1$ for all $r \in R$, and $g(v_n) = 2n^2 + 3n - 1$. It follows that the R-length of v_n is at least $2n^2 + 3n - 1$.

It remains to demonstrate an explicit sequence of $2n^2 + 4n + 1$ Reidemeister moves in $\mathbb{R}^2$, from D_n to the trivial diagram. Start by sliding the bottom horizontal string of $2n$ crossings, in the counterclockwise direction, across the n horizontal strands. This requires $2n^2$ RIII moves. Then cancel these $2n$ positive crossings with $2n$ of the negative crossings, now lying to the left of them, via $2n$ RII moves. With n RII moves, push the loop now lying above the n horizontal strands, into the center. Finally, perform $n+1$ RI moves to get to the trivial diagram. These are illustrated in Figure 3.

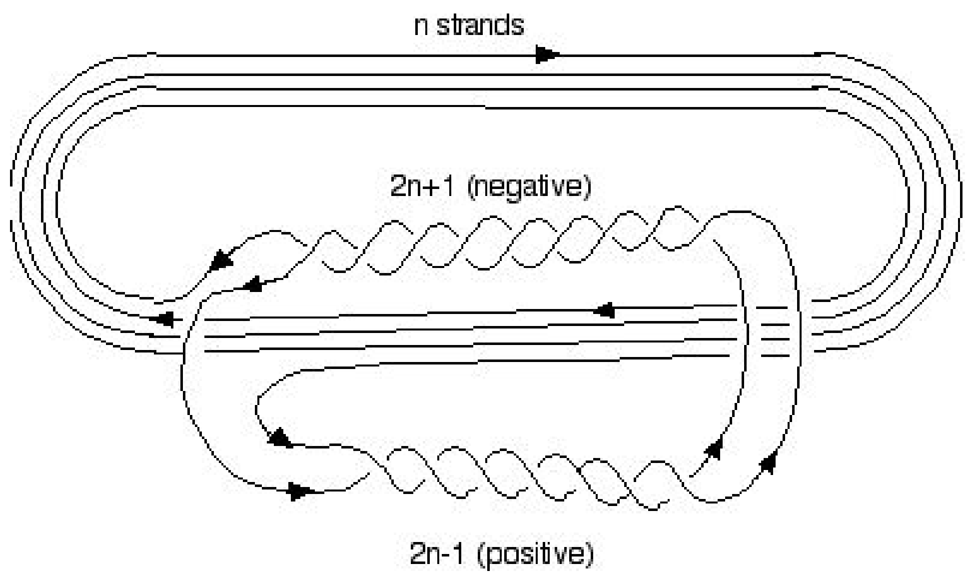

Figure 3: Initial diagram

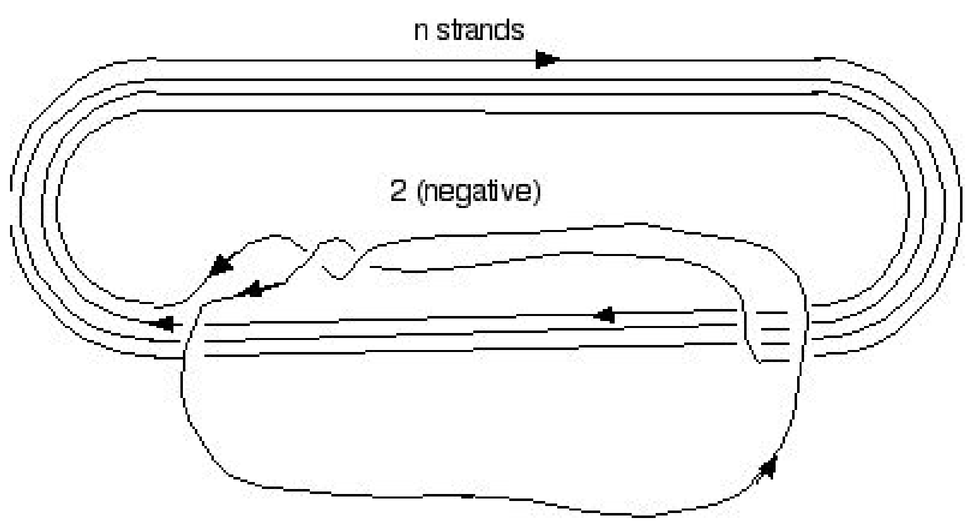

Figure 4: Stage 2: We can do R2 moves on the outside, in $\mathbb{S}^2$.

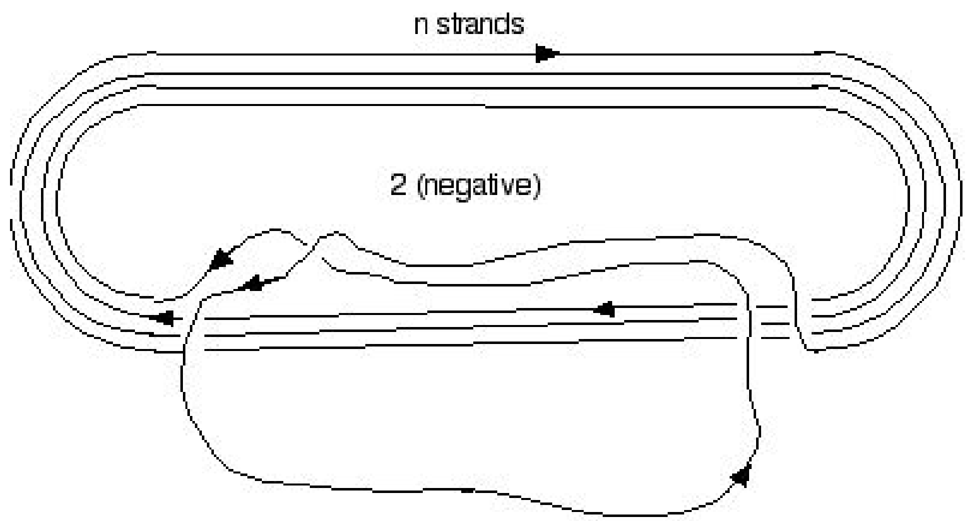

Figure 5: Stage 3 if we do Reidemeister moves in R^2 rather than $\mathbb{S}^2$

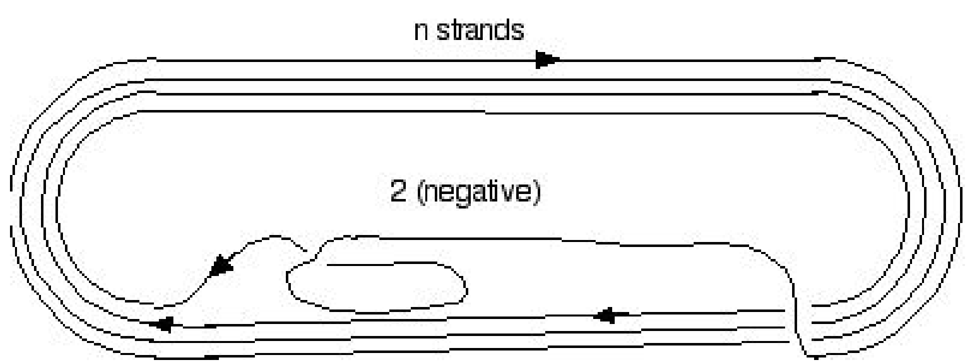

Figure 6: Stage 4

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email: hass@math.ucdavis.edu
tahl@math.biu.ac.il

address: J. Hass
Department of Mathematics
University of California, Davis
Davis, CA 95616

T. Nowik
Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900, Israel