Chapter 2

Continuous Functions

In Chapter 1, we introduced the notion of a normed linear space, with finite-dimensional Euclidean space $\mathbb{R}^n$ as the main example. In this chapter, we study linear spaces of continuous functions on a compact set equipped with the uniform norm. These function spaces are our first examples of infinite-dimensional normed linear spaces, and we explore the concepts of convergence, completeness, density, and compactness in this context. As an application of compactness, we prove an existence result for initial value problems for ordinary differential equations.

2.1 Convergence of functions

Suppose that (f_n) is a sequence of real-valued functions $f_n : X \to \mathbb{R}$ defined on a metric space X. What would we mean by $f_n \to f$? Two natural ways to answer this question are the following.

(a) The functions f_n are defined by their values, so the functions converge if the values converge. That is, we say $f_n \to f$ if $f_n(x) \to f(x)$ for all $x \in X$. This definition reduces the convergence of real-valued functions to the convergence of real numbers, with which we are already familiar. This type of convergence is called *pointwise convergence*.

(b) We define a suitable notion of the distance between functions, and say that $f_n \to f$ if the distance between f_n and f tends to zero. In this approach, we regard the functions as points in a metric space, and use *metric convergence*.

Both of these ideas are useful. It turns out, however, that they are not compatible. For most domains X — for example, any uncountable domain — pointwise convergence cannot be expressed as convergence with respect to a metric. The next example shows that pointwise convergence is not a good notion of convergence to use for continuous functions because it does not preserve continuity.

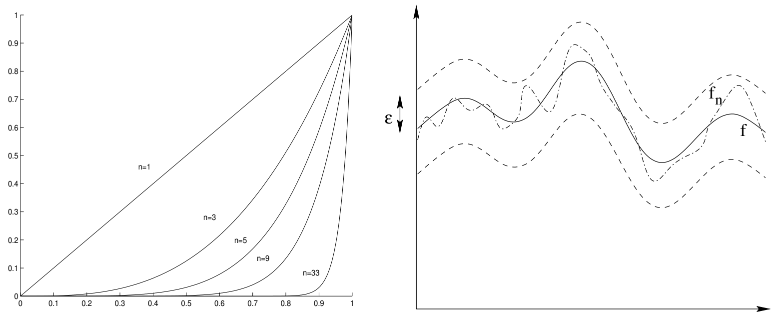

Fig. 2.1 Left: the sequence of functions $f_n(x) = x^n$ converges pointwise but not uniformly on $[0,1]$. Right: graphically, uniform convergence means that for an arbitrarily narrow tubular neighborhood of the limiting function, the functions f_n will be contained in it for all sufficiently large n.

Example 2.1 We define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = x^n.$$

As illustrated in Figure 2.1, the sequence (f_n) converges pointwise to the function f given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

The pointwise limit f is discontinuous at $x = 1$.

In view of these somewhat pathological features of pointwise convergence, we consider metric convergence. As we will see, there are many different ways to define a distance between functions, and different metrics or norms usually lead to different types of convergence. A natural norm on spaces of continuous functions is the *uniform* or *sup* norm, which is defined by

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{2.1}$$

The norm $\|f\|$ is finite if and only if f is bounded. The uniform norm is often denoted by $\|\cdot\|_{\text{sup}}$ or $\|\cdot\|_\infty$. The reason for the latter notation will become clear when we study L^p spaces in Chapter 12. In this chapter, we only use the uniform norm, so we denote it by $\|\cdot\|$ without ambiguity.

As illustrated in Figure 2.1, two functions are close in the metric associated with the uniform norm if their pointwise values are uniformly close. Metric convergence with respect to the uniform norm is called *uniform convergence.*

Definition 2.2 A sequence of bounded, real-valued functions (f_n) on a metric space X *converges uniformly* to a function f if

$$\lim_{n\to\infty} \|f_n - f\| = 0,$$

where $\|\cdot\|$ is defined in (2.1).

Uniform convergence implies pointwise convergence. The sequence defined in Example 2.1 shows that the opposite implication does not hold, since $f_n \to f$ pointwise but $\|f_n - f\| = 1$ for every n. Unlike pointwise convergence, uniform convergence preserves continuity.

Theorem 2.3 Let (f_n) be a sequence of bounded, continuous, real-valued functions on a metric space (X, d). If $f_n \to f$ uniformly, then f is continuous.

Proof. In order to show that f is continuous at $x \in X$, we need to prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. By the triangle inequality, we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

Since $f_n \to f$ uniformly, there is an n such that

$$|f(x) - f_n(x)| < \frac{\epsilon}{3}, \quad |f_n(y) - f(y)| < \frac{\epsilon}{3} \qquad \text{for all } x, y \in X.$$

Since f_n is continuous at x, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that

$$|f_n(y) - f_n(x)| < \frac{\epsilon}{3}.$$

It follows that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$, so f is continuous at x. □

The "$\epsilon/3$-trick" used in this proof has many other applications. The proof fails if $f_n \to f$ pointwise but not uniformly.

2.2 Spaces of continuous functions

Let X be a metric space. We denote the set of continuous, real-valued functions $f : X \to \mathbb{R}$ by $C(X)$. The set $C(X)$ is a real linear space under the pointwise addition of functions and the scalar multiplication of functions by real numbers. That is, for $f, g \in C(X)$ and $\lambda \in \mathbb{R}$, we define

$$(f + g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda (f(x)).$$

From Theorem 1.68, a continuous function f on a compact metric space K is bounded, so the uniform norm $\|f\|$ is finite for $f \in C(K)$. It is straightforward to check that $C(K)$ equipped with the uniform norm is a normed linear space. For example, the triangle inequality holds because

$$\|f + g\| = \sup_{x \in K} |f(x) + g(x)| \leq \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)| = \|f\| + \|g\|.$$

We will always use the uniform norm on $C(K)$, unless we state explicitly otherwise. A basic property of $C(K)$ is that it is complete, and therefore a Banach space.

Theorem 2.4 Let K be a compact metric space. The space $C(K)$ is complete.

Proof. Let (f_n) be a Cauchy sequence in $C(K)$ with respect to the uniform norm. We have to show that (f_n) converges uniformly. We do this in two steps. First, we construct a candidate function f for the limit of the sequence, as the pointwise limit of the sequence. Second, we show that the sequence converges uniformly to f.

First, the fact that (f_n) is Cauchy in $C(K)$ implies that the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$ for each $x \in K$. Since $\mathbb{R}$ is complete, the sequence of pointwise values converges, and we can define a function $f : K \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

For the second step, we use the fact that (f_n) is Cauchy in $C(K)$ to prove that it converges uniformly to f. Since $f_m(x) \to f(x)$ as $m \to \infty$, we have

$$\begin{aligned} \|f_n - f\| &= \sup_{x\in K} |f_n(x) - f(x)| \\ &= \sup_{x\in K} \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\leq \liminf_{m\to\infty} \sup_{x\in K} |f_n(x) - f_m(x)|. \end{aligned} \tag{2.2}$$

The fact that (f_n) is Cauchy in the uniform norm means that for all $\epsilon > 0$ there is an N such that

$$\sup_{x\in K} |f_n(x) - f_m(x)| < \epsilon \quad \text{for all } m, n \geq N.$$

It follows from (2.2) that $\|f_n - f\| \leq \epsilon$ for $n \geq N$, which proves that $\|f_n - f\| \to 0$ as $n \to \infty$. By Theorem 2.3, the limit function f is continuous, and therefore belongs to $C(K)$. Hence, $C(K)$ is complete. □

Example 2.5 Suppose $K = \{x_1, \ldots, x_n\}$ is a finite space, with metric d defined by $d(x_i, x_j) = 1$ for $i \neq j$. A function $f : K \to \mathbb{R}$ can be identified with a point $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, where $f(x_i) = y_i$, and

$$\|f\| = \max_{1\leq i\leq n} |y_i|.$$

Thus, the space $C(K)$ is linearly isomorphic to the finite-dimensional space $\mathbb{R}^n$ with the maximum norm, which we have already observed is a Banach space. If K contains infinitely many points, for example if $K = [0, 1]$, then $C(K)$ is an infinite-dimensional Banach space.

The same proof applies to complex-valued functions $f : K \to \mathbb{C}$, and the space of complex-valued continuous functions on a compact metric space is also a Banach space with the uniform norm (2.1).

The pointwise product of two continuous functions is continuous, so $C(K)$ has an *algebra* structure. The product is compatible with the norm, in the sense that

$$\|fg\| \leq \|f\| \, \|g\|. \tag{2.3}$$

We say that $C(K)$ is a *Banach algebra*. Strict inequality may occur in (2.3); for example, the product of two functions that are nonzero on disjoint sets is zero.

Equation (2.1) does not define a norm on $C(X)$ when X is not compact, since continuous funtions may be unbounded. The space $C_b(X)$ of bounded continuous functions on X is a Banach space with respect to the uniform norm.

Definition 2.6 The *support*, $\operatorname{supp} f$, of a function $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$) on a metric space X is the closure of the set on which f is nonzero,

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

We say that f has *compact support* if $\operatorname{supp} f$ is a compact subset of X, and denote the space of continuous functions on X with compact support by $C_c(X)$.

The space $C_c(X)$ is a linear subspace of $C_b(X)$, but it need not be closed, in which case it is not a Banach space. We denote the closure of $C_c(X)$ in $C_b(X)$ by $C_0(X)$. Since $C_0(X)$ is a closed linear subspace of a Banach space, it is also a Banach space. (We warn the reader that the notation $C_0(X)$ is often used to denote the space $C_c(X)$ of functions with compact support.) We have the following inclusions between these spaces of continuous functions:

$$C(X) \supset C_b(X) \supset C_0(X) \supset C_c(X).$$

If X is compact, then these spaces are equal.

Example 2.7 A function $f : \mathbb{R}^n \to \mathbb{R}$ has compact support if there is an $R > 0$ such that $f(x) = 0$ for all x with $\|x\| > R$. The space $C_0(\mathbb{R}^n)$ consists of continuous functions that vanish at infinity, meaning that for every $\epsilon > 0$ there is an $R > 0$ such that $\|x\| > R$ implies that $|f(x)| < \epsilon$. We write this condition as $\lim_{\|x\| \to \infty} f(x) = 0$.

Example 2.8 Consider real functions $f : \mathbb{R} \to \mathbb{R}$. Then $f(x) = x^2$ is in $C(\mathbb{R})$ but not $C_b(\mathbb{R})$. The constant function $f(x) = 1$ is in $C_b(\mathbb{R})$ but not $C_0(\mathbb{R})$. The function $f(x) = e^{-x^2}$ is in $C_0(\mathbb{R})$ but not $C_c(\mathbb{R})$. The function

$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

is in $C_c(\mathbb{R})$.

2.3 Approximation by polynomials

A *polynomial* $p : [a, b] \to \mathbb{R}$ on a closed, bounded interval $[a, b]$ is a function of the form

$$p(x) = \sum_{k=0}^{n} c_k x^k,$$

where the coefficients c_k are real numbers. If $c_n \neq 0$, the integer $n \geq 0$ is called the *degree* of p. The Weierstrass Approximation Theorem states that every continuous function $f : [a, b] \to \mathbb{R}$ can be approximated by a polynomial with arbitrary accuracy in the uniform norm.

Theorem 2.9 (Weierstrass approximation) The set of polynomials is dense in $C([a, b])$.

Proof. We need to show that for any $f \in C([a, b])$ there is a sequence of polynomials (p_n) such that $p_n \to f$ uniformly.

We first show that, by shifting and rescaling x, it is sufficient to prove the theorem in the case $[a, b] = [0, 1]$. We define $T : C([a, b]) \to C([0, 1])$ by

$$(Tf)(x) = f\left(a + (b - a)x\right).$$

Then T is linear and invertible, with inverse

$$\left(T^{-1} f\right)(x) = f\left(\frac{x - a}{b - a}\right).$$

Moreover, T is an isometry, since $\|Tf\| = \|f\|$, and for any polynomial p both Tp and $T^{-1}p$ are polynomials. If polynomials are dense in $C([0, 1])$, then for any $f \in C([a, b])$ we have polynomials p_n such that $p_n \to Tf$ in $C([0, 1])$. It follows that the polynomials $T^{-1}p_n$ converge to f in $C([a, b])$.

To show that polynomials are dense in $C([0, 1])$, we use a proof by Bernstein, which gives an explicit formula for a sequence of polynomials converging to a function f in $C([0, 1])$. These polynomials are called the Bernstein polynomials of f, and are defined by

$$B_n(x; f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}. \tag{2.4}$$

Notice that each term $x^k(1 - x)^{n-k}$, attains its maximum at $x = k/n$. This is illustrated in Figure 2.2 for $n = 20$ and some values of k. The value of $B_n(x; f)$ for x near k/n, is therefore predominantly determined by the values of $f(x)$ near $x = k/n$. In (2.4), we use the standard notation for the binomial coefficients,

$$\binom{n}{k} = \frac{n!}{(n - k)!k!}.$$

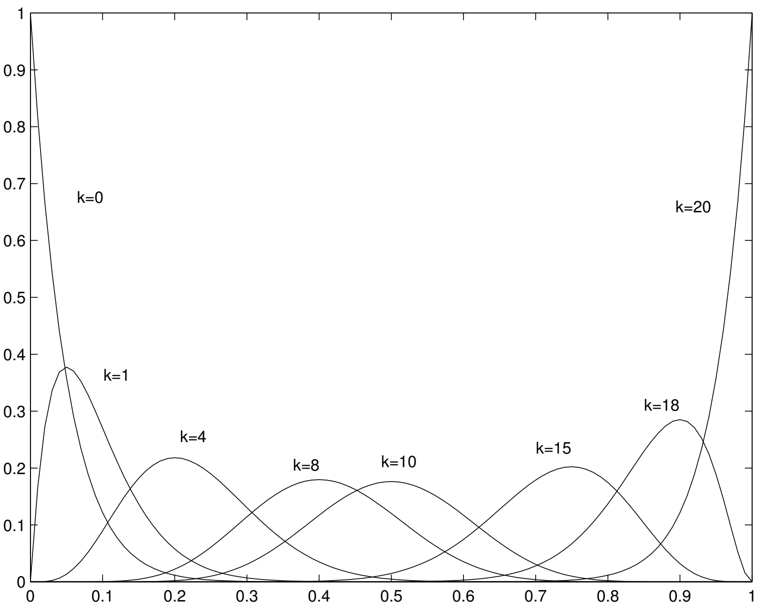

Fig. 2.2 The polynominals $x^k(1-x)^{n-k}$, for the case $n = 20$, appearing in the definition of the Bernstein polynomials (2.4). Note that they attain their maximum at $x = k/n$.

The binomial theorem implies that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

Therefore, the difference between f and its nth Bernstein polynomial can be written as

$$B_n(x; f) - f(x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] \binom{n}{k} x^k (1-x)^{n-k}. \tag{2.5}$$

Taking the supremum with respect to x of the absolute value of this equation, we get

$$\|B_n(\cdot\,; f) - f\| \leq \sup_{0 \leq x \leq 1} \left[\sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \right]. \tag{2.6}$$

Here, we use $B_n(x; f)$ to denote the value of the Bernstein polynomial at x, and $B_n(\cdot\,; f)$ to denote the corresponding polynomial function.

Let $\epsilon > 0$ be an arbitrary positive number. From Theorem 1.67, the function f is uniformly continuous, so there is a $\delta > 0$ such that

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \epsilon, \tag{2.7}$$

for all $x, y \in [0,1]$. To estimate the right hand side of (2.6), we divide the terms in the series into two groups. We let

$$\begin{aligned} I(x) &= \{k \mid 0 \leq k \leq n \text{ and } |x - (k/n)| < \delta\}, \\ J(x) &= \{k \mid 0 \leq k \leq n \text{ and } |x - (k/n)| \geq \delta\}. \end{aligned} \tag{2.8}$$

From (2.6), (2.7), and (2.8), we get the following estimate,

$$\begin{aligned} \|B_n(\cdot\,; f) - f\| &\leq \epsilon \sup_{0 \leq x \leq 1} \left[\sum_{k \in I(x)} \binom{n}{k} x^k (1-x)^{n-k} \right] \\ &\quad + \sup_{0 \leq x \leq 1} \left[\sum_{k \in J(x)} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \right] \\ &\leq \epsilon + 2\|f\| \sup_{0 \leq x \leq 1} \left[\sum_{k \in J(x)} \binom{n}{k} x^k (1-x)^{n-k} \right]. \end{aligned} \tag{2.9}$$

Since $[x - (k/n)]^2 \geq \delta^2$ for $k \in J(x)$, the sum on the right hand side of (2.9) can be estimated as follows:

$$\begin{aligned} &\sup_{0 \leq x \leq 1} \left[\sum_{k \in J(x)} \binom{n}{k} x^k (1-x)^{n-k} \right] \\ &\quad \leq \frac{1}{\delta^2} \sup_{0 \leq x \leq 1} \left[\sum_{k \in J(x)} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \right] \\ &\quad \leq \frac{1}{\delta^2} \sup_{0 \leq x \leq 1} \left[\sum_{k=0}^{n} \left(x^2 - \frac{2k}{n} x + \frac{k^2}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k} \right] \\ &\quad \leq \frac{1}{\delta^2} \sup_{0 \leq x \leq 1} \left[x^2 B_n(x;1) - 2x B_n(x;x) + B_n(x;x^2) \right]. \end{aligned} \tag{2.10}$$

To find an expression for the Bernstein polynomials $B_n(x;1)$, $B_n(x;x)$, and $B_n(x;x^2)$ of the polynomials 1, x, and x^2, we write out the binomial expansion of $(x+y)^n$, compute the first and second derivatives of the expansion with respect to x, and rearrange the results. This gives

$$\begin{aligned} (x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \\ x(x+y)^{n-1} &= \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} x^k y^{n-k}, \\ \left(\frac{n-1}{n}\right) x^2 (x+y)^{n-2} + \left(\frac{1}{n}\right) x(x+y)^{n-1} &= \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k y^{n-k}. \end{aligned}$$

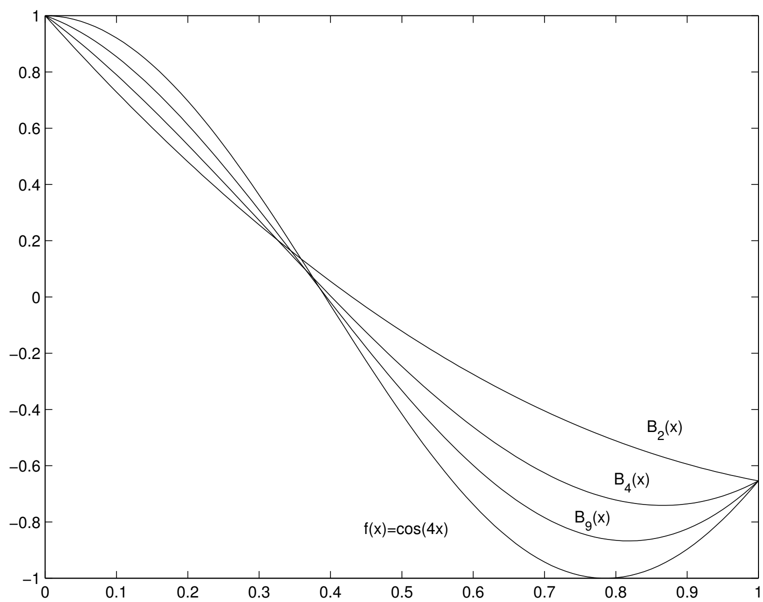

Fig. 2.3 Some approximations of the function $x \mapsto \cos(4x)$ by its Bernstein polynomials.

Evaluation of these equations at $y = 1 - x$, and the use of (2.4) gives

$$\begin{aligned} B_n(x;1) &= 1, \\ B_n(x;x) &= x, \\ B_n(x;x^2) &= \left(\frac{n-1}{n}\right)x^2 + \left(\frac{1}{n}\right)x, \end{aligned} \tag{2.11}$$

for all $n \geq 1$. Using (2.10) and (2.11) in (2.9), we obtain the estimate

$$\|B_n(\cdot\,;f) - f\| \leq \epsilon + \frac{\|f\|}{2n\delta^2}.$$

Taking the lim sup of this equation as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|B_n(\cdot\,;f) - f\| \leq \epsilon.$$

Since ϵ is arbitrary, it follows that $\limsup_{n\to\infty} \|B_n(\cdot\,;f) - f\| = 0$, so the polynomials $B_n(\cdot\,;f)$ converge uniformly to f. □

The first few approximations by Bernstein polynomials of the function $f(x) = \cos(4x)$ are graphed in Figure 2.3. Note that we could have formulated the theorem for complex-valued functions with the same proof.

The Weierstrass approximation theorem differs from Taylor's theorem, which states that a function with sufficiently many derivatives can be approximated locally by its Taylor polynomial. The Weierstrass approximation theorem applies to a continuous function, which need not be differentiable, and states that there is a global polynomial approximation of the function on the whole interval $[a, b]$.

An analogous result is the density of trigonometric polynomials in the space of periodic continuous functions on the circle, which we prove below in Theorem 7.3. Both of these theorems are special cases of the Stone-Weierstrass theorem (see Rudin [48]).

2.4 Compact subsets of $C(K)$

The proof of the Heine-Borel theorem, that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded, uses the finite-dimensionality of $\mathbb{R}^n$ in an essential way. Compact subsets of infinite-dimensional normed spaces are also closed and bounded, but these properties are no longer sufficient. In this section, we prove the Arzelà-Ascoli theorem, which characterizes the compact subsets of $C(K)$. To state the theorem, we introduce the notion of *equicontinuity*.

Definition 2.10 Let $\mathcal{F}$ be a family of functions from a metric space (X, d) to a metric space (Y, d). The family $\mathcal{F}$ is *equicontinuous* if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d\left(f(x), f(y)\right) < \epsilon$ for all $f \in \mathcal{F}$.

The crucial point in this definition is that δ does not depend on f, although it may depend on x. If δ can be chosen independent of x as well, then the family is said to be *uniformly equicontinuous*. The following theorem is a generalization of Theorem 1.67.

Theorem 2.11 An equicontinuous family of functions from a compact metric space to a metric space is uniformly equicontinuous.

Proof. Suppose that K is a compact metric space, and $\mathcal{F}$ is a family of functions $f : K \to Y$ that is not uniformly equicontinuous. We will prove that $\mathcal{F}$ is not equicontinuous.

Since $\mathcal{F}$ is not uniformly equicontinuous, there is an $\epsilon > 0$, such that for every $n \in \mathbb{N}$ there are points $x_n, y_n \in K$ and a function $f_n \in \mathcal{F}$ with

$$d\left(x_n, y_n\right) < \frac{1}{n} \quad \text{and} \quad d\left(f_n(y_n), f_n(x_n)\right) \geq 2\epsilon. \tag{2.12}$$

Since K is compact, the sequence (x_n) has a convergent subsequence, which we also denote by (x_n). Suppose that $x_n \to x$ as $n \to \infty$. Then (2.12) implies that $y_n \to x$ as well. Hence, for all $\delta > 0$, there are points x_n, y_n such that $d(x_n, x) < \delta$ and $d(y_n, x) < \delta$. But, from (2.12), we must have either $d\left(f_n(x_n), f_n(x)\right) \geq \epsilon$ or $d\left(f_n(y_n), f_n(x)\right) \geq \epsilon$, so $\mathcal{F}$ is not equicontinuous at x. □

Next, we give necessary and sufficient conditions for compactness in $C(K)$.

Theorem 2.12 (Arzelà-Ascoli) Let K be a compact metric space. A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.

Proof. Recall that a set is precompact if its closure is compact, and that a set is compact if and only if it is closed and precompact. We will prove that a subset of $C(K)$ is precompact if and only if it is bounded and equicontinuous.

We divide the proof into three parts. First, we show that an unbounded subset is not precompact. Second, we show that a precompact subset is equicontinuous. Third, we show that a bounded, equicontinuous subset is precompact.

For the first part, suppose that $\mathcal{F}$ is an unbounded subset of $C(K)$. Then there is a sequence of functions $f_n \in \mathcal{F}$, with $\|f_{n+1}\| \geq \|f_n\| + 1$, so that $\|f_n - f_m\| \geq 1$ for all $n \neq m$. It follows that (f_n) has no Cauchy subsequence, and therefore no convergent subsequence, so $\mathcal{F}$ is not precompact.

For the second part, suppose that $\mathcal{F}$ is a precompact subset of $C(K)$. Fix $\epsilon > 0$. Since $\mathcal{F}$ is dense in $\overline{\mathcal{F}}$, we have

$$\overline{\mathcal{F}} \subset \bigcup_{f \in \mathcal{F}} B_{\epsilon/3}(f).$$

Since $\overline{\mathcal{F}}$ is compact, there is a finite subset $\{f_1, \ldots, f_k\}$ of $\mathcal{F}$ such that

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{k} B_{\epsilon/3}(f_i).$$

Each f_i is uniformly continuous by Theorem 1.67, so there is a $\delta_i > 0$ such that $d(x, y) < \delta_i$ implies that $|f_i(x) - f_i(y)| < \epsilon/3$ for all $x, y \in K$. We define δ by

$$\delta = \min_{1 \leq i \leq k} \delta_i.$$

Since δ is the minimum of a finite set of $\delta_i > 0$, we have $\delta > 0$. For every $f \in \mathcal{F}$, there is an $1 \leq i \leq k$ such that $\|f - f_i\| < \epsilon/3$. We conclude that for $d(x, y) < \delta$

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \epsilon.$$

Since ϵ is arbitrary and δ is independent of f, the set $\mathcal{F}$ is equicontinuous.

For the third part, suppose that $\mathcal{F}$ is a bounded, equicontinuous subset of $C(K)$. We will show that every sequence (f_n) in $\mathcal{F}$ has a convergent subsequence. By Lemma 1.63 there is a countable dense set $\{x_1, x_2, x_3, \ldots\}$ in the compact domain K. We choose a subsequence $(f_{1,n})$ of (f_n) such that the sequence of values $(f_{1,n}(x_1))$ converges in $\mathbb{R}$. Such a subsequence exists because $(f_n(x_1))$ is bounded in $\mathbb{R}$, since $\mathcal{F}$ is bounded in $C(K)$. We choose a subsequence $(f_{2,n})$ of $(f_{1,n})$ such that $(f_{2,n}(x_2))$ converges, which exists for the same reason. Repeating this procedure, we obtain sequences $(f_{k,n})_{n=1}^{\infty}$ for $k = 1, 2, \ldots$ such that $(f_{k+1,n})$ is a subsequence of $(f_{k,n})$, and $(f_{k,n}(x_k))$ converges as $n \to \infty$. Finally, we define a "diagonal" subsequence (g_k) by $g_k = f_{k,k}$. By construction, the sequence (g_k) is a subsequence of (f_n) with the property that $g_k(x_i)$ converges in $\mathbb{R}$ as $k \to \infty$ for all x_i in a dense subset of K.

So far, we have only used the boundedness of $\mathcal{F}$. The equicontinuity of $\mathcal{F}$ is needed to ensure the uniform convergence of (g_k). Let $\epsilon > 0$. Since $\mathcal{F}$ is equicontinuous and K is compact, Theorem 2.11 implies that $\mathcal{F}$ is uniformly equicontinuous. Consequently, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}.$$

Since $\{x_i\}$ is dense in K, we have

$$K \subset \bigcup_{i=1}^{\infty} B_\delta(x_i).$$

Since K is compact, there is a finite subset of $\{x_i\}$, which we denote by $\{x_1, \ldots, x_n\}$, such that

$$K \subset \bigcup_{i=1}^{n} B_\delta(x_i).$$

The sequence $(g_k(x_i))$ is convergent for each $i = 1, \ldots, n$, and hence Cauchy, so there is an N such that

$$|g_j(x_i) - g_k(x_i)| < \frac{\epsilon}{3}$$

for all $j, k \geq N$ and $i = 1, \ldots, n$. For any $x \in K$, there is an i such that $x \in B_\delta(x_i)$. Then, for $j, k \geq N$, we have

$$|g_j(x) - g_k(x)| \leq |g_j(x) - g_j(x_i)| + |g_j(x_i) - g_k(x_i)| + |g_k(x_i) - g_k(x)| < \epsilon.$$

It follows that (g_k) is a Cauchy sequence for the uniform norm and, since $C(K)$ is complete, it converges. □

In the proof of this theorem, we again used Cantor's diagonal argument, and the "$\epsilon/3$-trick."

Example 2.13 For each $n \in \mathbb{N}$, we define a function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 2^{-n}, \\ 2^{n+1}(x - 2^{-n}) & \text{if } 2^{-n} \leq x \leq 3 \cdot 2^{-(n+1)}, \\ 2^{n+1}(2^{-(n-1)} - x) & \text{if } 3 \cdot 2^{-(n+1)} \leq x \leq 2^{-(n-1)}, \\ 0 & \text{if } 2^{-(n-1)} \leq x \leq 1. \end{cases} \tag{2.13}$$

These functions consist of 'tent' functions of height one that move from right to left across the interval $[0, 1]$, becoming narrower and steeper as they do so.

The first two functions are shown in Figure 2.4. Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$. Then $\|f_n\| = 1$ for all $n \geq 1$, so $\mathcal{F}$ is bounded, but $\|f_m - f_n\| = 1$ for all $m \neq n$, so the sequence (f_n) does not have any convergent subsequences. Hence, the set $\mathcal{F}$ is a closed, bounded subset of $C([0, 1])$ which is not compact. Note that $\mathcal{F}$ is not

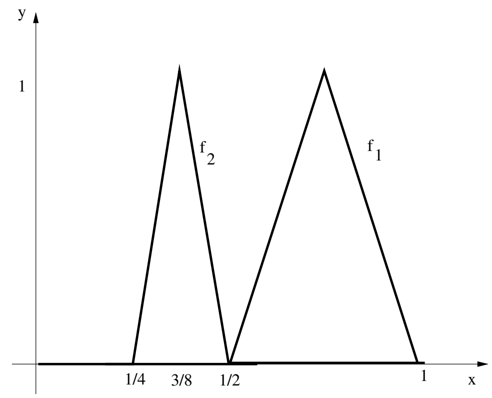

Fig. 2.4 The sequence of bounded continuous functions defined in Example 2.13 is not equicontinuous.

equicontinuous either, because the graphs of the f_n become steeper as n gets larger. The same phenomenon occurs for the set $\mathcal{F} = \{\sin(n\pi x) \mid n \in \mathbb{N}\}$.

Heuristically, a subset of an infinite-dimensional linear space is precompact if it is "almost" contained in a bounded subset of a finite-dimensional subspace. Without making this statement more precise at the moment (but see Theorem 9.17), we illustrate it with the following example.

Example 2.14 Let $\mathcal{F}$ be the subset of $C([0,1])$ that consists of functions f of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{with} \quad \sum_{n=1}^{\infty} n|a_n| \leq 1.$$

The series defining f converges uniformly, so f is an element of $C([0,1])$. The set $\mathcal{F}$ is bounded in $C(K)$, since for any $f \in \mathcal{F}$ we have

$$\|f\| \leq \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} n|a_n| \leq 1.$$

By the mean value theorem, for any $x < y \in \mathbb{R}$ there is a $x < \xi < y$ with

$$\sin x - \sin y = (\cos \xi)\,(x - y).$$

Hence, for all $x, y \in \mathbb{R}$ we have

$$|\sin x - \sin y| \leq |x - y|.$$

Thus, every $f \in \mathcal{F}$ satisfies

$$|f(x) - f(y)| \leq \sum_{n=1}^{\infty} |a_n|\,|\sin(n\pi x) - \sin(n\pi y)| \leq \sum_{n=1}^{\infty} \pi n|a_n|\,|x - y| \leq \pi\,|x - y|\,.$$

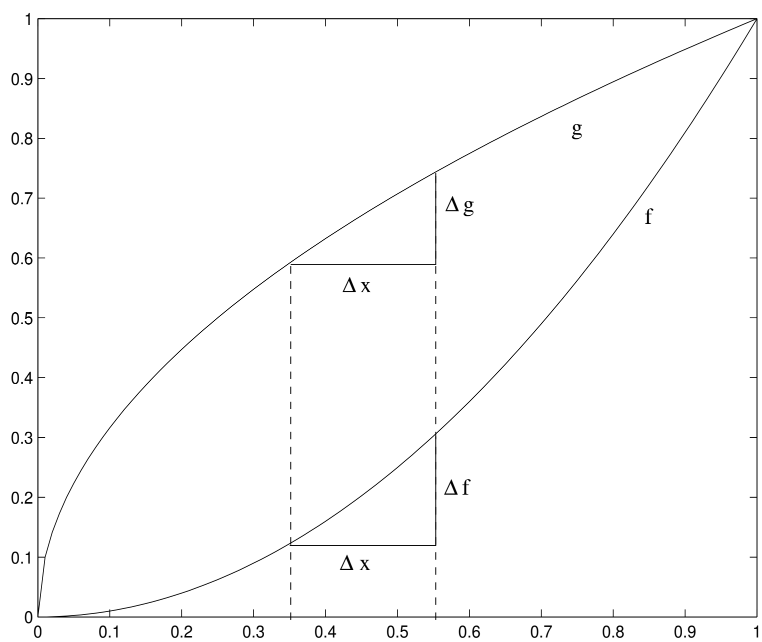

Fig. 2.5 The graph of two continuous functions on $[0,1]$: $f(x) = x^2$, and $g(x) = \sqrt{x}$. f is Lipschitz on $[0,1]$, but g is not Lipschitz at the point 0. The ratio $\Delta f/\Delta x$ is bounded for arbitrarily small Δx everywhere in $[0,1]$, but $\Delta g/\Delta x$ is unbounded for small Δx near $x = 0$.

Therefore, given $\epsilon > 0$, we can pick $\delta = \epsilon/\pi$, and then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$. From the Arzelà-Ascoli theorem, $\mathcal{F}$ is a precompact subset of $C([0,1])$. For large N, the subset $\mathcal{F}$ is "almost" contained in the unit ball in the finite-dimensional subspace spanned by $\{\sin \pi x, \sin 2\pi x, \ldots, \sin N\pi x\}$.

The previous example illustrates a useful sufficient condition for equicontinuity, which we now describe. We begin by defining *Lipschitz continuous* functions.

Definition 2.15 A function $f : X \to \mathbb{R}$ on a metric space X is *Lipschitz continuous* on X if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C d(x,y) \qquad \text{for all } x, y \in X. \tag{2.14}$$

We will often abbreviate the term "Lipschitz continuous" to "Lipschitz." Every Lipschitz continuous function is uniformly continuous, but there are uniformly continuous functions that are not Lipschitz.

Example 2.16 As illustrated in Figure 2.5, the square function $f(x) = x^2$ is Lipschitz continuous on $[0,1]$, but the uniformly continuous square-root function

$g(x) = \sqrt{x}$ is not, because

$$\lim_{x \to 0^+} \frac{|g(x) - g(0)|}{|x - 0|} = \infty.$$

If $f : X \to \mathbb{R}$ is a Lipschitz function, then we define the *Lipschitz constant* $\mathrm{Lip}(f)$ of f by

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Equivalently, $\mathrm{Lip}(f)$ is the smallest constant C that works in the Lipschitz condition (2.14),

$$\mathrm{Lip}(f) = \inf \{C \mid |f(x) - f(y)| \leq C d(x, y) \text{ for all } x, y \in X\}.$$

Suppose that K is a compact metric space and $M > 0$. We define a subset $\mathcal{F}_M$ of $C(K)$ by

$$\mathcal{F}_M = \{f \mid f \text{ is Lipschitz on } K \text{ and } \mathrm{Lip}(f) \leq M\}. \tag{2.15}$$

The set $\mathcal{F}_M$ is equicontinuous, since if $\epsilon > 0$ and $\delta = \epsilon / M$, then

$$d(x, y) < \delta \quad \text{implies} \quad |f(x) - f(y)| < \epsilon \quad \text{for all } f \in \mathcal{F}_M.$$

The set $\mathcal{F}_M$ is closed, since if (f_n) is a sequence in $\mathcal{F}_M$ that converges uniformly to f in $C(K)$, then

$$\begin{aligned}
\mathrm{Lip}(f) &= \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \\
&= \sup_{x \neq y} \left[\lim_{n \to \infty} \frac{|f_n(x) - f_n(y)|}{d(x, y)}\right] \\
&\leq \liminf_{n \to \infty} \left[\sup_{x \neq y \in K} \frac{|f_n(x) - f_n(y)|}{d(x, y)}\right] \\
&\leq M.
\end{aligned}$$

Thus, the limit f belongs to $\mathcal{F}_M$. The set $\mathcal{F}_M$ is not bounded, since the constant functions belong to $\mathcal{F}_M$ and their sup-norms are arbitrarily large. Consequently, although $\mathcal{F}_M$ itself is not compact, the Arzelà-Ascoli theorem implies that every closed, bounded subset of $\mathcal{F}_M$ is compact, and every bounded subset of $\mathcal{F}_M$ is precompact.

Example 2.17 Suppose that x_0 is a point in a compact metric space K. Let

$$\mathcal{B}_M = \{f \in \mathcal{F}_M \mid f(x_0) = 0\}.$$

Then $\mathcal{B}_M$ is bounded because for every $f \in \mathcal{B}_M$ we have

$$\|f\| = \sup_{x \in K} |f(x) - f(x_0)| \leq M \sup_{x \in K} |x - x_0| \leq M \operatorname{diam} K,$$

where $\operatorname{diam} K$ is finite since K is compact, and hence bounded. The set $\mathcal{B}_M$ is closed, since if $f_n(x_0) = 0$ and $f_n \to f$ in $C(K)$, then

$$f(x_0) = \lim_{n\to\infty} f_n(x_0) = 0.$$

Therefore, the set $\mathcal{B}_M$ is a compact subset of $C(K)$.

Lemma 2.18 Suppose that $f : C \to \mathbb{R}$ is a continuously differentiable function on an open, convex subset C of $\mathbb{R}^n$, and that the partial derivatives of f are bounded on C. Then, for all $x, y \in C$, we have

$$|f(x) - f(y)| \le M\|x - y\|, \tag{2.16}$$

where $\|\cdot\|$ denotes the Euclidean norm and

$$M = \sup_{z\in C} \|\nabla f(z)\|. \tag{2.17}$$

Proof. Since C is convex, the point $tx + (1-t)y$ lies in C for all $x, y \in C$ and $0 \le t \le 1$. The fundamental theorem of calculus and the chain rule imply that

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \frac{d}{dt} f\left(tx + (1-t)y\right)\, dt \\ &= \int_0^1 \nabla f\left(tx + (1-t)y\right) \cdot (x - y)\, dt, \end{aligned} \tag{2.18}$$

where

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

is the gradient of f with respect to $x = (x_1, \dots, x_n)$. We take the absolute value of equation (2.18), and estimate the resulting integral, to obtain

$$|f(x) - f(y)| \le \sup_{0\le t\le 1} \left\{\|\nabla f\left(tx + (1-t)y\right)\|\right\} \|x - y\|.$$

The use of (2.17) in this equation gives (2.16). □

From Lemma 2.18, a continuously differentiable function with bounded partial derivatives is Lipschitz. A Lipschitz continuous function need not be differentiable everywhere, however, since its graph may have "corners."

Example 2.19 The absolute value function $f(x) = |x|$ is Lipschitz continuous, with Lipschitz constant one, because

$$|f(x) - f(y)| = \left|\,|x| - |y|\,\right| \le |x - y|.$$

The absolute value function is not differentiable at $x = 0$.

Lemma 2.18 implies that a family of continuously differentiable functions with uniformly bounded derivatives is equicontinuous. If the family is also bounded, then it is precompact. The idea that a uniform bound on suitable norms of the derivatives of a family of functions implies that the family is precompact will reappear when we study Sobolev spaces in Chapter 12.9.

Example 2.20 Let $C^1([0,1])$ denote the space of all continuous functions f on $[0,1]$ with continuous derivative f'. For constants $M > 0$ and $N > 0$, we define the subset $\mathcal{F}$ of $C([0,1])$ by

$$\mathcal{F} = \left\{ f \in C^1([0,1]) \mid \|f\| \leq M, \|f'\| \leq N \right\},$$

where $\|\cdot\|$ denotes the sup-norm. Then $\mathcal{F}$ is precompact in $C(K)$. It is not closed, however, because the uniform limit of continuously differentiable functions need not be differentiable. Thus, $\mathcal{F}$ is not compact. Its closure in $C([0,1])$ is the compact set

$$\overline{\mathcal{F}} = \{ f \in C([0,1]) \mid \|f\| \leq M, \mathrm{Lip}(f) \leq N \}.$$

2.5 Ordinary differential equations

A differential equation is an equation that relates the values of a function and its derivatives at each point. We distinguish between ordinary differential equations (ODEs) for functions of a single variable, and partial differential equations (PDEs) for functions of several variables. In this section, we discuss the existence and uniqueness of solutions of ODEs.

To focus on the central ideas in the simplest setting, we consider a scalar, first order ODE for a real-valued function $u(t)$ of the form

$$\dot{u} = f(t, u). \tag{2.19}$$

In (2.19), we use $\dot{u}(t)$ to denote the derivative of $u(t)$ with respect to t, and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function. We say that (2.19) is a *linear ODE* if $f(t,u)$ is a linear (strictly speaking, we should say "affine") function of u of the form $f(t,u) = a(t)u + b(t)$. Otherwise, we say that (2.19) is a *nonlinear ODE.*

A solution of (2.19), defined in an open interval $I \subset \mathbb{R}$, is a continuously differentiable function $u : I \to \mathbb{R}$ such that

$$\dot{u}(t) = f(t, u(t)) \qquad \text{for all } t \in I.$$

If the solution is defined on the whole of $\mathbb{R}$, then we call it a *global solution.* If the solution is defined only on a subinterval of $\mathbb{R}$, then we call it a *local solution.*

We will refer to the independent variable t in (2.19) as "time." Equation (2.19) determines the rate of change of the function u at each time in terms of the value of u. We expect that if we know the value of u at some time, then the ODE determines the values of u at nearby times, and by repetition of this process, we expect that there is a unique solution of the *initial value problem* (IVP)

$$\begin{aligned} \dot{u} &= f(t, u), \\ u(t_0) &= u_0. \end{aligned} \tag{2.20}$$

Here, t_0 is a given initial time, and u_0 is a given initial value. As the following examples show, however, the question of the existence and uniqueness of solutions (2.20) is not always as straightforward as this naive discussion might suggest.

Example 2.21 Consider the linear initial value problem,

$$\begin{aligned} \dot{u} &= au, \\ u(0) &= u_0, \end{aligned} \tag{2.21}$$

where $a \in \mathbb{R}$ is a constant. This initial value problem has a unique, global solution $u(t) = u_0 e^{at}$. Equation (2.21) has a simple interpretation in terms of population growth. It states that the growth rate $\dot{u}$ of a population is proportional to the population u. If the per capita growth rate $\dot{u}/u = a$ is positive, then the population grows exponentially in time, as Malthus observed in 1798.

Example 2.22 Consider the nonlinear initial value problem,

$$\begin{aligned} \dot{u} &= u^2, \\ u(0) &= u_0. \end{aligned} \tag{2.22}$$

The unique solution is

$$u(t) = \frac{u_0}{1 - u_0 t}.$$

This solution becomes arbitrarily large as $t \to 1/u_0$. For $u_0 > 0$, the initial value problem in (2.22) has a local solution defined in the interval $-\infty < t < 1/u_0$, but it does not have a global solution. This phenomenon is called "blow-up," and is a fundamental difficulty in the study of nonlinear differential equations. When interpreted as a population model, equation (2.22) describes the growth of a population in which the per capita growth rate is equal to the population. Thus, as the population increases the growth rate increases both because the population is larger and because the per capita growth rate is larger. As a result, the solution tends to infinity in finite time.

Example 2.23 Consider the initial value problem

$$\begin{aligned} \dot{u} &= \sqrt{|u|}, \\ u(0) &= 0. \end{aligned} \tag{2.23}$$

The zero function $u(t) = 0$ is a global solution, but it is not the only one. The following function satisfies (2.23) for any $a \geq 0$,

$$u(t) = \begin{cases} 0 & \text{if } t \leq a, \\ (t-a)^2/4 & \text{if } t > a. \end{cases}$$

In this example, the function $f(u) = \sqrt{|u|}$ is a continuous function of u, but it is not Lipschitz continuous at the initial value $u = 0$.

These examples show that the most we can hope for, if f is an arbitrary continuous function, is the existence of a local solution of the initial value problem (2.20). If f is smooth, the solution is unique, as we will see, but it may not exist globally.

For general f, we cannot prove the existence of a solution by giving an explicit analytical formula for it, as we did in the simple examples above. Instead we use a compactness argument, analogous to the one used in the proof of Theorem 1.68. We construct a family $\{u_\epsilon\}$ of functions that satisfy (2.20) in a suitable approximate sense. Since the functions are approximate solutions of the ordinary differential equation, their derivatives are uniformly bounded, and the Arzelà-Ascoli theorem implies that they form a precompact set. Consequently, there is a subsequence of approximate solutions that converges uniformly as $\epsilon \to 0$ to a function u. We then show that u is a solution of (2.20).

Theorem 2.24 Suppose that $f(t, u)$ is a continuous function on $\mathbb{R}^2$. Then, for every (t_0, u_0), there is an open interval $I \subset \mathbb{R}$ that contains t_0, and a continuously differentiable function $u : I \to \mathbb{R}$ that satisfies the initial value problem (2.20).

Proof. We say that $u_\epsilon(t)$ is an *ϵ-approximate solution* of (2.20) in an interval I containing t_0 if:

(a) $u_\epsilon(t_0) = u_0$;
(b) $u_\epsilon(t)$ is a continuous function of t that is differentiable at all but finitely many points of I;
(c) at every point $t \in I$ where $\dot{u}_\epsilon(t)$ exists, we have

$$|\dot{u}_\epsilon(t) - f(t, u_\epsilon(t))| \leq \epsilon.$$

To construct an ϵ-approximate solution u_ϵ, we first pick $T_1 > 0$, and let

$$I_1 = \{t \mid |t - t_0| \leq T_1\}.$$

We partition I_1 into $2N$ subintervals of length h, where $T_1 = Nh$, and let

$$t_k = t_0 + kh \qquad \text{for } -N \leq k \leq N.$$

We denote the values of the approximate solution at the times t_k by $u_\epsilon(t_k) = a_k$. We define these values by the following finite difference approximation of the ODE,

$$\frac{a_{k+1} - a_k}{h} = f(t_k, a_k),$$

$$a_0 = u_0.$$

This discretization of (2.20) is called the forward Euler method. It is not an accurate numerical method for the solution of (2.20), but its simplicity makes it convenient for an existence proof.

Inside the subinterval $t_k \leq t \leq t_{k+1}$, we define $u_\epsilon(t)$ to be the linear function of t that takes the appropriate values at the endpoints. That is,

$$u_\epsilon(t) = a_k + b_k\,(t - t_k) \qquad \text{for } t_k \leq t \leq t_{k+1},$$

where the parameters a_k and b_k are defined recursively by

$$\begin{aligned} a_0 &= u_0, & a_k &= a_{k-1} + b_{k-1}h, \\ b_0 &= f(t_0, u_0), & b_k &= f(t_k, a_k). \end{aligned}$$

Thus, $u_\epsilon(t)$ is a continuous, piecewise linear function of t that is differentiable except possibly at the points $t = t_k$, and $\dot{u}_\epsilon(t) = b_k$ for $t_k < t < t_{k+1}$. For $t_k < t < t_{k+1}$, we have

$$|\dot{u}_\epsilon(t) - f(t, u_\epsilon(t))| = |f\,(t_k, a_k) - f\,(t, a_k + b_k(t - t_k))\,|, \tag{2.24}$$

$$|t - t_k| \leq h, \quad |a_k + b_k(t - t_k) - a_k| \leq |b_k|\,h. \tag{2.25}$$

We choose an $L > 0$, and a $T \leq T_1$ such that the graph of every u_ϵ with $|t - t_0| \leq T$ is contained in the rectangle $R \subset \mathbb{R}^2$ given by

$$R = \{(t, u) \mid |t - t_0| \leq T,\, |u - u_0| \leq L\}\,.$$

To do this, we consider the closed rectangle $R_1 \subset \mathbb{R}^2$, centered at (t_0, u_0), defined by

$$R_1 = \{(t, u) \mid |t - t_0| \leq T_1,\, |u - u_0| \leq L\}\,.$$

We let

$$M = \sup\{|f(t, u)| \mid (t, u) \in R_1\}\,, \qquad T = \min\left(T_1, L/M\right).$$

It follows that, for $|t - t_0| \leq T$, the slopes b_k of the linear segments of u_ϵ are less than or equal to M, and the graph of u_ϵ lies in the cone bounded by the lines $u - u_0 = M(t - t_0)$ and $u - u_0 = -M(t - t_0)$. Figure 2.6 shows why this is true.

Since R is compact, the function f is uniformly continuous on R. Therefore, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(s, u) - f(t, v)| \leq \epsilon$$

for all $(s, u), (t, v) \in R$ such that $|s - t| \leq \delta$ and $|u - v| \leq \delta$. Using (2.24)–(2.25), we see that u_ϵ is an ϵ-approximate solution when $h \leq \delta$ and $Mh \leq \delta$.

Each u_ϵ is Lipschitz continuous, and its Lipschitz constant is bounded uniformly by M, independently of ϵ. We also have $u_\epsilon(t_0) = u_0$ for all ϵ. From Example 2.17, the set $\{u_\epsilon\}$ is precompact in $C([t_0 - T, t_0 + T])$. Hence there is a continuous

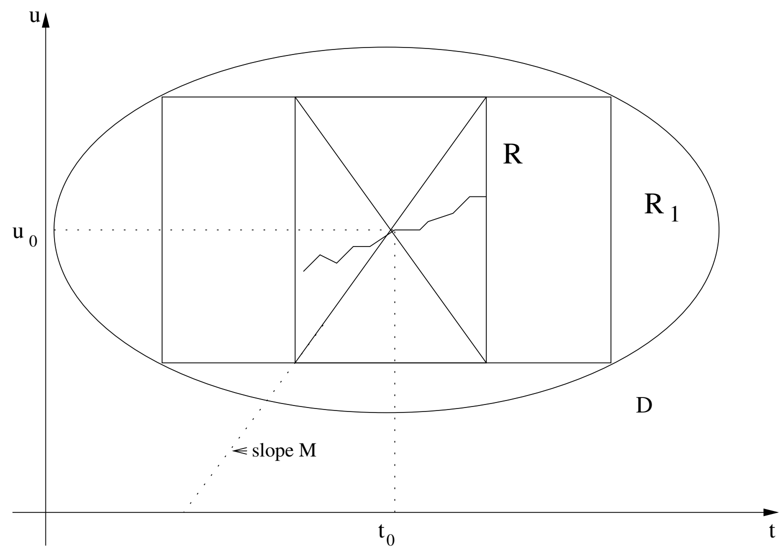

Fig. 2.6 The construction of the rectangle R used in the proof of Theorem 2.24.

function u and a sequence (ϵ_n) with $\epsilon_n \to 0$ as $n \to \infty$ such that $u_{\epsilon_n} \to u$ as $n \to \infty$ uniformly on $[t_0 - T, t_0 + T]$.

It remains to show that the limiting function u solves (2.20). Since u_ϵ is piecewise linear, we have

$$\begin{aligned} u_\epsilon(t) &= u_\epsilon(t_0) + \int_{t_0}^{t} \dot{u}_\epsilon(s)\, ds \\ &= u_0 + \int_{t_0}^{t} f(s, u_\epsilon(s))\, ds + \int_{t_0}^{t} \left[\dot{u}_\epsilon - f(s, u_\epsilon(s))\right] ds. \end{aligned} \tag{2.26}$$

Here, $\dot{u}_\epsilon$ is not necessarily defined at the points t_k, but this does not affect the value of the integral. We set $\epsilon = \epsilon_n$ in (2.26), and let $n \to \infty$. Using Exercise 2.2 to take the limit, we find that

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds. \tag{2.27}$$

The fundamental theorem of calculus implies that the right hand side of (2.27) is continuously differentiable. Therefore, the function u is also continuously differentiable in $|t - t_0| < T$, and $\dot{u}(t) = f(t, u(t))$. □

More generally, the same proof applies if f is continuous only in some open set $D \subset \mathbb{R}^2$ which contains the initial point (t_0, u_0), provided we choose the rectangles R_1 and R so that they are contained in D (see Figure 2.6).

This proof shows the existence of a local solution in some interval about the initial time, but the solution need not be global. A solution has, however, a maximal open interval of existence that contains t_0.

As shown by Example 2.23, the continuity of f does not guarantee uniqueness, but if $f(t,u)$ is Lipschitz continuous in u, then the solution is unique. The condition that f is Lipschitz continuous is a mild one, and is met in nearly all applications, where f is typically a smooth function. To prove this fact, we use the following result, called *Gronwall's inequality.*

Theorem 2.25 (Gronwall's inequality) Suppose that $u(t) \geq 0$ and $\varphi(t) \geq 0$ are continuous, real-valued functions defined on the interval $0 \leq t \leq T$ and $u_0 \geq 0$ is a constant. If u satisfies the inequality

$$u(t) \leq u_0 + \int_0^t \varphi(s)u(s)\,ds \qquad \text{for } t \in [0,T], \tag{2.28}$$

then

$$u(t) \leq u_0 \exp\left(\int_0^t \varphi(s)\,ds\right) \qquad \text{for } t \in [0,T].$$

In particular, if $u_0 = 0$ then $u(t) = 0$.

Proof. Suppose first that $u_0 > 0$. Let

$$U(t) = u_0 + \int_0^t \varphi(s)u(s)\,ds.$$

Then, since $u(t) \leq U(t)$, we have that

$$\dot{U} = \varphi u \leq \varphi U, \qquad U(0) = u_0.$$

Since $U(t) > 0$, it follows that

$$\frac{d}{dt}\log U = \frac{\dot{U}}{U} \leq \varphi.$$

Hence

$$\log U(t) \leq \log u_0 + \int_0^t \varphi(s)\,ds,$$

so

$$u(t) \leq U(t) \leq u_0 \exp\left(\int_0^t \varphi(s)\,ds\right). \tag{2.29}$$

If the inequality (2.28) holds for $u_0 = 0$, then it also holds for all $u_0 > 0$, so (2.29) holds for all $u_0 > 0$. Taking the limit of (2.29) as $u_0 \to 0^+$, we conclude that $u(t) = 0$, which proves the result when $u_0 = 0$. □

Theorem 2.26 Suppose that $f(t,u)$ is continuous in the rectangle

$$R = \{(t,u) \mid |t - t_0| \leq T,\, |u - u_0| \leq L\},$$

and that

$$|f(t,u)| \leq M \qquad \text{if } (t,u) \in R.$$

Let $\delta = \min(T, L/M)$. If $u(t)$ is any solution of (2.20), then

$$|u(t) - u_0| \leq L \qquad \text{when } |t - t_0| \leq \delta. \tag{2.30}$$

Suppose, in addition, that f is a Lipschitz continuous function of u, uniformly in t, meaning that there is a constant C such that

$$|f(t,u) - f(t,v)| \leq C|u - v| \qquad \text{for all } (t,u) \in R.$$

Then the solution of (2.20) is unique in the interval $|t - t_0| \leq \delta$.

Proof. The result in (2.30) is intuitively obvious: if a solution $u(t)$ stays inside the interval $|u(t) - u_0| \leq L$, then its derivative is bounded by M, so the solution cannot escape the interval in less time than L/M. To avoid circularity in the proof, we use a "continuous induction" argument. We consider the set D defined by

$$D = \{0 \leq \eta \leq \delta \mid |u(t) - u_0| \leq L \text{ for all } |t - t_0| \leq \eta\}.$$

Then $0 \in D$, and if $\eta \in D$, then $\eta' \in D$ for all $0 \leq \eta' \leq \eta$. Thus, D is a nonempty interval. Moreover, D is closed in $[0, \delta]$ because $u(t)$ is a continuous function of t. If $\eta \in D$ and $\eta < \delta$, then $f(t, u(t)) \leq M$ for $|t - t_0| \leq \eta$, so

$$|u(t) - u_0| \leq \left| \int_{t_0}^{t} f(s, u(s))\, ds \right| \leq M\eta < M\delta = L.$$

Since we have strict inequality, and u is continuous, it follows that there is an $\epsilon > 0$ such that $|u(t) - u_0| \leq L$ when $|t - t_0| \leq \eta + \epsilon$. Thus, D is open in $[0, \delta]$, from which we conclude that $D = [0, \delta]$. This proves the first part of the theorem.

To prove the uniqueness part, we use a common strategy: we derive an equation for the difference of two solutions which shows that it is zero. Suppose that u and v are solutions of (2.20) on a interval I that contains t_0. Then subtraction and integration of the ODEs satisfied by u and v implies that

$$u(t) - v(t) = \int_{t_0}^{t} \left[f(s, u(s)) - f(s, v(s)) \right] ds.$$

Taking the absolute value of this equation, and estimating the result, we find that $w = |u - v|$ satisfies the inequality

$$w(t) \leq \int_{t_0}^{t} \left| f(s, u(s)) - f(s, v(s)) \right| ds. \tag{2.31}$$

By the first part of the theorem, the graph of any solution remains in R for $|t-t_0| \leq \delta$. The Lipschitz continuity of f in R therefore implies that

$$|f(t, u(t)) - f(t, v(t))| \leq C|u(t) - v(t)| = Cw(t).$$

The use of this inequality in (2.31) implies that $w \geq 0$ satisfies

$$w(t) \leq C \int_{t_0}^{t} w(s)\, ds.$$

Therefore, from Gronwall's inequality, we have $w = 0$, and $u = v$. □

We will give another proof of the existence and uniqueness of solutions of the initial value problem for ODEs in the next chapter, as a consequence of the contraction mapping theorem. The existence theorem above, based on compactness, is called the *Peano existence theorem*, while the theorem in the next chapter, based on the contraction mapping theorem, is called the *Picard existence theorem.*

2.6 References

Most of the material in this chapter is also covered in Rudin [47] and Marsden and Hoffman [37]. For an introduction to the theory of ordinary differential equations, see Hirsch and Smale [21].

2.7 Exercises

Exercise 2.1 Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational,} \\ p \sin(1/q) & \text{if } x = p/q, \text{ where } p,q \text{ are relatively prime integers.} \end{cases}$$

Determine the set of points where f is continuous.

Exercise 2.2 Let $f_n \in C([a, b])$ be a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Give a counterexample to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply the convergence of the corresponding integrals.

Exercise 2.3 Suppose that $f : G \to \mathbb{R}$ is a uniformly continuous function defined on an open subset G of a metric space X. Prove that f has a unique extension to a continuous function $\overline{f} : \overline{G} \to \mathbb{R}$ defined on the closure $\overline{G}$ of G. Show that such an extension need not exist if f is continuous but not uniformly continuous on G.

Exercise 2.4 Give a counterexample to show that $f_n \to f$ in $C([0,1])$ and f_n continuously differentiable does not imply that f is continuously differentiable.

Exercise 2.5 Consider the space of continuously differentiable functions,

$$C^1([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f,\, f' \text{ are continuous}\},$$

with the C^1-norm,

$$\|f\| = \sup_{a \le x \le b} |f(x)| + \sup_{a \le x \le b} |f'(x)|.$$

Prove that $C^1([a,b])$ is a Banach space.

Exercise 2.6 Show that the space $C([a,b])$ equipped with the L^1-norm $\|\cdot\|_1$ defined by

$$\|f\|_1 = \int_a^b |f(x)|\, dx,$$

is incomplete. Show that if $f_n \to f$ with respect to the sup-norm $\|\cdot\|_\infty$, then $f_n \to f$ with respect to $\|\cdot\|_1$. Give a counterexample to show that the converse statement is false.

Exercise 2.7 Prove that the set of Lipschitz continuous functions on $[0,1]$ with Lipschitz constant less than or equal one and zero integral is compact in $C([0,1])$.

Exercise 2.8 Prove that $C([a,b])$ is separable.

Exercise 2.9 Let $w : [0,1] \to \mathbb{R}$ be a nonnegative, continuous function. For $f \in C([0,1])$, we define the *weighted supremum norm* by

$$\|f\|_w = \sup_{0 \le x \le 1} \{w(x)|f(x)|\}.$$

If $w(x) > 0$ for $0 < x < 1$, show that $\|\cdot\|_w$ is a norm on $C([0,1])$. If $w(x) > 0$ for $0 \le x \le 1$, show that $\|\cdot\|_w$ is equivalent to the usual sup-norm, corresponding to $w = 1$. (See Definition 5.21 for the definition of equivalent norms.) Show that the norm $\|\cdot\|_x$ corresponding to $w(x) = x$ is not equivalent to the usual sup-norm. Is the space $C([0,1])$ equipped with the weighted norm $\|\cdot\|_x$ a Banach space?

Exercise 2.10 Let $C_0(\mathbb{R}^n)$ be the closure of the space $C_c(\mathbb{R}^n)$ of continuous, compactly supported functions with respect to the uniform norm. Prove that $C_0(\mathbb{R}^n)$ is the space of continuous functions that vanish at infinity.

Exercise 2.11 Suppose $f_n \in C([0,1])$ is a monotone decreasing sequence that converges pointwise to $f \in C([0,1])$. Prove that f_n converges uniformly to f. This result is called *Dini's monotone convergence theorem.*

Exercise 2.12 Let $\{f_n \in C([0,1]) \mid n \in \mathbb{N}\}$ be equicontinuous. If $f_n \to f$ pointwise, prove that f is continuous.

Exercise 2.13 Consider the scalar initial value problem,

$$\begin{aligned} \dot{u}(t) &= |u(t)|^\alpha, \\ u(0) &= 0. \end{aligned}$$

Show that the solution is unique if $\alpha \geq 1$, but not if $0 \leq \alpha < 1$.

Exercise 2.14 Suppose that $f(t,u)$ is a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|f(t,u) - f(t,v)| \leq K|u - v| \qquad \text{for all } t, u, v \in \mathbb{R}.$$

Also suppose that

$$M = \sup\left\{|f(t,u_0)| \mid |t - t_0| \leq T\right\}.$$

Prove that the solution $u(t)$ of the initial value problem

$$\dot{u} = f(t,u), \qquad u(t_0) = u_0$$

satisfies the estimate

$$|u(t) - u_0| \leq MTe^{KT} \qquad \text{for } |t - t_0| \leq T.$$

Explicitly check this estimate for the linear initial value problem

$$\dot{u} = Ku, \qquad u(t_0) = u_0.$$