Chapter 10

Power Series

In discussing power series it is good to recall a nursery rhyme:

"There was a little girl
Who had a little curl
Right in the middle of her forehead
When she was good
She was very, very good
But when she was bad
She was horrid."

(Robert Strichartz [**14**])

Power series are one of the most useful type of series in analysis. For example, we can use them to define transcendental functions such as the exponential and trigonometric functions (as well as many other less familiar functions).

10.1. Introduction

A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots.$$

where the constants a_n are some coefficients. If all but finitely many of the a_n are zero, then the power series is a polynomial function, but if infinitely many of the a_n are nonzero, then we need to consider the convergence of the power series.

The basic facts are these: Every power series has a radius of convergence $0 \le R \le \infty$, which depends on the coefficients a_n. The power series converges absolutely in $|x| < R$ and diverges in $|x| > R$. Moreover, the convergence is uniform on every interval $|x| < \rho$ where $0 \le \rho < R$. If $R > 0$, then the sum of the power series is infinitely differentiable in $|x| < R$, and its derivatives are given by differentiating the original power series term-by-term.

Power series work just as well for complex numbers as real numbers, and are in fact best viewed from that perspective. We will consider here only real-valued power series, although many of the results extend immediately to complex-valued power series.

Definition 10.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. The power series centered at c with coefficients a_n is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Example 10.2. The following are power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots,$$
$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots,$$
$$\sum_{n=0}^{\infty} (n!) x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \dots,$$
$$\sum_{n=0}^{\infty} (-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + \dots.$$

An example of a power series centered at 1 is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots.$$

The power series in Definition 10.1 is a formal, algebraic expression, since we haven't said anything yet about its convergence. By changing variables $(x-c) \mapsto x$, we can assume without loss of generality that a power series is centered at 0, and we will do so whenever it's convenient.

10.2. Radius of convergence

First, we prove that every power series has a radius of convergence.

Theorem 10.3. Let

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series. There is a non-negative, extended real number $0 \le R \le \infty$ such that the series converges absolutely for $0 \le |x-c| < R$ and diverges for $|x-c| > R$. Furthermore, if $0 \le \rho < R$, then the power series converges uniformly on the interval $|x-c| \le \rho$, and the sum of the series is continuous in $|x-c| < R$.

Proof. We assume without loss of generality that $c = 0$. Suppose the power series

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges for some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Then its terms converge to zero, so they are bounded and there exists $M \geq 0$ such that

$$|a_n x_0^n| \leq M \qquad \text{for } n = 0, 1, 2, \ldots.$$

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M r^n, \qquad r = \left|\frac{x}{x_0}\right| < 1.$$

Comparing the power series with the convergent geometric series $\sum M r^n$, we see that $\sum a_n x^n$ is absolutely convergent. Thus, if the power series converges for some $x_0 \in \mathbb{R}$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$.

Let

$$R = \sup\left\{|x| \geq 0 : \sum a_n x^n \text{ converges}\right\}.$$

If $R = 0$, then the series converges only for $x = 0$. If $R > 0$, then the series converges absolutely for every $x \in \mathbb{R}$ with $|x| < R$, since it converges for some $x_0 \in \mathbb{R}$ with $|x| < |x_0| < R$. Moreover, the definition of R implies that the series diverges for every $x \in \mathbb{R}$ with $|x| > R$. If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.

Finally, let $0 \leq \rho < R$ and suppose $|x| \leq \rho$. Choose $\sigma > 0$ such that $\rho < \sigma < R$. Then $\sum |a_n \sigma^n|$ converges, so $|a_n \sigma^n| \leq M$, and therefore

$$|a_n x^n| = |a_n \sigma^n| \left|\frac{x}{\sigma}\right|^n \leq |a_n \sigma^n| \left|\frac{\rho}{\sigma}\right|^n \leq M r^n,$$

where $r = \rho/\sigma < 1$. Since $\sum M r^n < \infty$, the M-test (Theorem 9.22) implies that the series converges uniformly on $|x| \leq \rho$, and then it follows from Theorem 9.16 that the sum is continuous on $|x| \leq \rho$. Since this holds for every $0 \leq \rho < R$, the sum is continuous in $|x| < R$. □

The following definition therefore makes sense for every power series.

Definition 10.4. If the power series

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

converges for $|x - c| < R$ and diverges for $|x - c| > R$, then $0 \leq R \leq \infty$ is called the radius of convergence of the power series.

Theorem 10.3 does not say what happens at the endpoints $x = c \pm R$, and in general the power series may converge or diverge there. We refer to the set of all points where the power series converges as its interval of convergence, which is one of

$$(c - R, c + R), \quad (c - R, c + R], \quad [c - R, c + R), \quad [c - R, c + R].$$

We won't discuss here any general theorems about the convergence of power series at the endpoints (e.g., the Abel theorem). Also note that a power series need not converge uniformly on $|x - c| < R$.

Theorem 10.3 does not give an explicit expression for the radius of convergence of a power series in terms of its coefficients. The ratio test gives a simple, but useful, way to compute the radius of convergence, although it doesn't apply to every power series.

Theorem 10.5. Suppose that $a_n \neq 0$ for all sufficiently large n and the limit

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists or diverges to infinity. Then the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence R.

Proof. Let

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = |x-c| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

By the ratio test, the power series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result. □

The root test gives an expression for the radius of convergence of a general power series.

Theorem 10.6 (Hadamard)**.** The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

where $R = 0$ if the lim sup diverges to ∞, and $R = \infty$ if the lim sup is 0.

Proof. Let

$$r = \limsup_{n\to\infty} |a_n(x-c)^n|^{1/n} = |x-c| \limsup_{n\to\infty} |a_n|^{1/n}.$$

By the root test, the series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result. □

This theorem provides an alternate proof of Theorem 10.3 from the root test; in fact, our proof of Theorem 10.3 is more-or-less a proof of the root test.

10.3. Examples of power series

We consider a number of examples of power series and their radii of convergence.

Example 10.7. The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

It converges to

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1,$$

and diverges for $|x| > 1$. At $x = 1$, the series becomes

$$1 + 1 + 1 + 1 + \dots$$

and at $x = -1$ it becomes

$$1 - 1 + 1 - 1 + 1 - \dots,$$

so the series diverges at both endpoints $x = \pm 1$. Thus, the interval of convergence of the power series is $(-1, 1)$. The series converges uniformly on $[-\rho, \rho]$ for every $0 \le \rho < 1$ but does not converge uniformly on $(-1, 1)$ (see Example 9.20). Note that although the function $1/(1-x)$ is well-defined for all $x \neq 1$, the power series only converges when $|x| < 1$.

Example 10.8. The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

At $x = 1$, the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which diverges, and at $x = -1$ it is minus the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots,$$

which converges, but not absolutely. Thus the interval of convergence of the power series is $[-1, 1)$. The series converges uniformly on $[-\rho, \rho]$ for every $0 \le \rho < 1$ but does not converge uniformly on $(-1, 1)$.

Example 10.9. The power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x + \frac{1}{3!}x^3 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty,$$

so it converges for all $x \in \mathbb{R}$. The sum is the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

In fact, this power series may be used to define the exponential function. (See Section 10.6.)

Example 10.10. The power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. Its sum $\cos x$ provides an analytic definition of the cosine function.

Example 10.11. The power series

$$\sum_{n=0^\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

has radius of convergence $R = \infty$, and it converges for all $x \in \mathbb{R}$. Its sum $\sin x$ provides an analytic definition of the sine function.

Example 10.12. The power series

$$\sum_{n=0}^{\infty} (n!)x^n = 1 + x + (2!)x + (3!)x^3 + (4!)x^4 + \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so it converges only for $x = 0$. If $x \neq 0$, its terms grow larger once $n > 1/|x|$ and $|(n!)x^n| \to \infty$ as $n \to \infty$.

Example 10.13. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

has radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}/n}{(-1)^{n+2}/(n+1)} \right| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1,$$

so it converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. At the endpoint $x = 2$, the power series becomes the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

which converges. At the endpoint $x = 0$, the power series becomes the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

which diverges. Thus, the interval of convergence is $(0, 2]$.

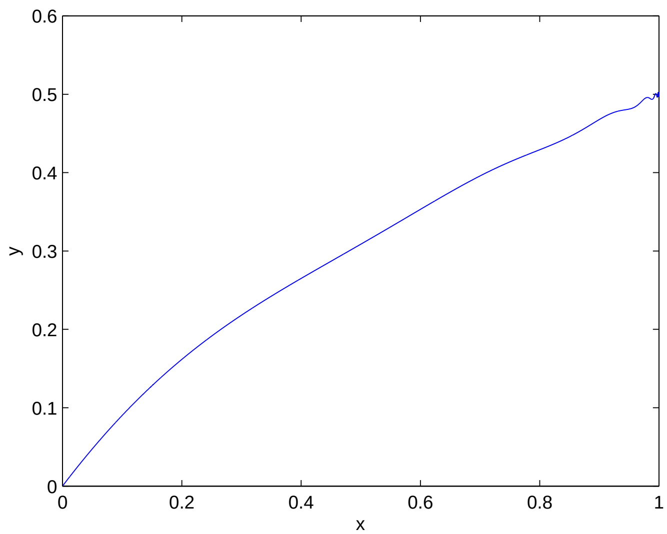

Figure 1. Graph of the lacunary power series $y = \sum_{n=0}^{\infty}(-1)^n x^{2^n}$ on $[0, 1)$. It appears relatively well-behaved; however, the small oscillations visible near $x = 1$ are not a numerical artifact.

Example 10.14. The power series

$$\sum_{n=0}^{\infty}(-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + x^{16} - x^{32} + \dots$$

with

$$a_n = \begin{cases} (-1)^k & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k, \end{cases}$$

has radius of convergence $R = 1$. To prove this, note that the series converges for $|x| < 1$ by comparison with the convergent geometric series $\sum |x|^n$, since

$$|a_n x^n| = \begin{cases} |x|^n & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k \end{cases} \leq |x|^n.$$

If $|x| > 1$, then the terms do not approach 0 as $n \to \infty$, so the series diverges. Alternatively, we have

$$|a_n|^{1/n} = \begin{cases} 1 & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k, \end{cases}$$

so

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1$$

and the Hadamard formula (Theorem 10.6) gives $R = 1$. The series does not converge at either endpoint $x = \pm 1$, so its interval of convergence is $(-1, 1)$.

In this series, there are successively longer gaps (or "lacuna") between the powers with non-zero coefficients. Such series are called lacunary power series, and

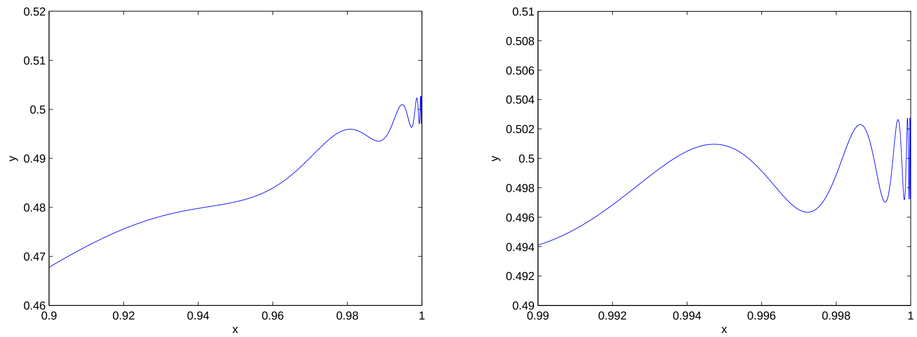

Figure 2. Details of the lacunary power series $\sum_{n=0}^{\infty}(-1)^n x^{2^n}$ near $x = 1$, showing its oscillatory behavior and the nonexistence of a limit as $x \to 1^-$.

they have many interesting properties. For example, although the series does not converge at $x = 1$, one can ask if

$$\lim_{x\to 1^-}\left[\sum_{n=0}^{\infty}(-1)^n x^{2^n}\right]$$

exists. In a plot of this sum on $[0, 1)$, shown in Figure 1, the function appears relatively well-behaved near $x = 1$. However, Hardy (1907) proved that the function has infinitely many, very small oscillations as $x \to 1^-$, as illustrated in Figure 2, and the limit does not exist. Subsequent results by Hardy and Littlewood (1926) showed, under suitable assumptions on the growth of the "gaps" between non-zero coefficients, that if the limit of a lacunary power series as $x \to 1^-$ exists, then the series must converge at $x = 1$. Since the lacunary power series considered here does not converge at 1, its limit as $x \to 1^-$ cannot exist. For further discussion of lacunary power series, see [**4**].

10.4. Algebraic operations on power series

We can add, multiply, and divide power series in a standard way. For simplicity, we consider power series centered at 0.

Proposition 10.15. If $R, S > 0$ and the functions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n \qquad \text{in } |x| < S$$

are sums of convergent power series, then

$$(f+g)(x) = \sum_{n=0}^{\infty}(a_n + b_n)\, x^n \qquad \text{in } |x| < T,$$

$$(fg)(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{in } |x| < T,$$

where $T = \min(R, S)$ and

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k.$$

Proof. The power series expansion of $f + g$ follows immediately from the linearity of limits. The power series expansion of fg follows from the Cauchy product (Theorem 4.38), since power series converge absolutely inside their intervals of convergence, and

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} x^{n-k} \cdot b_k x^k\right) = \sum_{n=0}^{\infty} c_n x^n.$$

□

It may happen that the radius of convergence of the power series for $f+g$ or fg is larger than the radius of convergence of the power series for f, g. For example, if $g = -f$, then the radius of convergence of the power series for $f + g = 0$ is ∞ whatever the radius of convergence of the power series for f.

The reciprocal of a convergent power series that is nonzero at its center also has a power series expansion.

Proposition 10.16. If $R > 0$ and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{in } |x| < R,$$

is the sum of a power series with $a_0 \neq 0$, then there exists $S > 0$ such that

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n x^n \qquad \text{in } |x| < S.$$

The coefficients b_n are determined recursively by

$$b_0 = \frac{1}{a_0}, \qquad b_n = -\frac{1}{a_0}\sum_{k=0}^{n-1} a_{n-k} b_k, \quad \text{for } n \geq 1.$$

Proof. First, we look for a formal power series expansion (i.e., without regard to its convergence)

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

such that the formal Cauchy product fg is equal to 1. This condition is satisfied if

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right) x^n = 1.$$

Matching the coefficients of x^n, we find that

$$a_0 b_0 = 1, \qquad a_0 b_n + \sum_{k=0}^{n-1} a_{n-k} b_k = 0 \qquad \text{for } n \geq 1,$$

which gives the stated recursion relation.

To complete the proof, we need to show that the formal power series for g has a nonzero radius of convergence. In that case, Proposition 10.15 shows that $fg = 1$ inside the common interval of convergence of f and g, so $1/f = g$ has a power series expansion. We assume without loss of generality that $a_0 = 1$; otherwise replace f by f/a_0.

The power series for f converges absolutely and uniformly on compact sets inside its interval of convergence, so the function

$$\sum_{n=1}^{\infty} |a_n|\,|x|^n$$

is continuous in $|x| < R$ and vanishes at $x = 0$. It follows that there exists $\delta > 0$ such that

$$\sum_{n=1}^{\infty} |a_n|\,|x|^n \leq 1 \qquad \text{for } |x| \leq \delta.$$

Then $f(x) \neq 0$ for $|x| < \delta$, since

$$|f(x)| \geq 1 - \sum_{n=1}^{\infty} |a_n|\,|x|^n > 0,$$

so $1/f(x)$ is well defined.

We claim that

$$|b_n| \leq \frac{1}{\delta^n} \qquad \text{for } n = 0, 1, 2, \dots.$$

The proof is by induction. Since $b_0 = 1$, this inequality is true for $n = 0$. If $n \geq 1$ and the inequality holds for b_k with $0 \leq k \leq n-1$, then by taking the absolute value of the recursion relation for b_n, we get

$$|b_n| \leq \sum_{k=1}^{n} |a_k||b_{n-k}| \leq \sum_{k=1}^{n} \frac{|a_k|}{\delta^{n-k}} \leq \frac{1}{\delta^n} \sum_{k=1}^{\infty} |a_k|\delta^k \leq \frac{1}{\delta^n},$$

so the inequality holds for b_k with $0 \leq k \leq n$, and the claim follows.

We then get that

$$\limsup_{n\to\infty} |b_n|^{1/n} \leq \frac{1}{\delta},$$

so the Hadamard formula in Theorem 10.6 implies that the radius of convergence of $\sum b_n x^n$ is greater than or equal to $\delta > 0$, which completes the proof. □

An immediate consequence of these results for products and reciprocals of power series is that quotients of convergent power series are given by convergent power series, provided that the denominator is nonzero.

Proposition 10.17. If $R, S > 0$ and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{in } |x| < R, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{in } |x| < S$$

are the sums of power series with $b_0 \neq 0$, then there exists $T > 0$ and coefficients c_n such that

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} c_n x^n \qquad \text{in } |x| < T.$$

The previous results do not give an explicit expression for the coefficients in the power series expansion of f/g or a sharp estimate for its radius of convergence. Using complex analysis, one can show that radius of convergence of the power series for f/g centered at 0 is equal to the distance from the origin of the nearest singularity of f/g in the complex plane. We will not discuss complex analysis here, but we consider two examples.

Example 10.18. Replacing x by $-x^2$ in the geometric power series from Example 10.7, we get the following power series centered at 0

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n},$$

which has radius of convergence $R = 1$. From the point of view of real functions, it may appear strange that the radius of convergence is 1, since the function $1/(1+x^2)$ is well-defined on $\mathbb{R}$, has continuous derivatives of all orders, and has power series expansions with nonzero radius of convergence centered at every $c \in \mathbb{R}$. However, when $1/(1+z^2)$ is regarded as a function of a complex variable $z \in \mathbb{C}$, one sees that it has singularities at $z = \pm i$, where the denominator vanishes, and $|\pm i| = 1$, which explains why $R = 1$.

Example 10.19. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 1$ and

$$f(x) = \frac{e^x - 1}{x}, \qquad \text{for } x \neq 0$$

has the power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n,$$

with infinite radius of convergence. The reciprocal function $g : \mathbb{R} \to \mathbb{R}$ of f is given by $g(0) = 1$ and

$$g(x) = \frac{x}{e^x - 1}, \qquad \text{for } x \neq 0.$$

Proposition 10.16 implies that

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

has a convergent power series expansion at 0, with $b_0 = 1$ and

$$b_n = -\sum_{k=0}^{n-1} \frac{b_k}{(n-k+1)!} \qquad \text{for } n \geq 1.$$

The numbers $B_n = n! b_n$ are called Bernoulli numbers. They may be defined as the coefficients in the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The function $x/(e^x - 1)$ is called the generating function of the Bernoulli numbers, where we adopt the convention that $x/(e^x - 1) = 1$ at $x = 0$.

A number of properties of the Bernoulli numbers follow from their generating function. First, we observe that

$$\frac{x}{e^x - 1} + \frac{1}{2}x = \frac{1}{2}x\left(\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}\right)$$

is an even function of x. It follows that

$$B_0 = 1, \qquad B_1 = -\frac{1}{2},$$

and $B_n = 0$ for all odd $n \geq 3$. Thus, the power series expansion of $x/(e^x - 1)$ has the form

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}x^{2n}.$$

The recursion formula for b_n can be written in terms of B_n as

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0,$$

which implies that the Bernoulli numbers are rational. For example, one finds that

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

As the sudden appearance of the large irregular prime number 691 in the numerator of B_{12} suggests, there is no simple pattern for the values of B_{2n}, although they continue to alternate in sign.[1] The Bernoulli numbers have many surprising connections with number theory and other areas of mathematics; for example, as noted in Section 4.5, they give the values of the Riemann zeta function at even natural numbers.

Using complex analysis, one can show that the radius of convergence of the power series for $z/(e^z - 1)$ at $z = 0$ is equal to 2π, since the closest zeros of the denominator $e^z - 1$ to the origin in the complex plane occur at $z = \pm 2\pi i$, where $|z| = 2\pi$. Given this fact, the Hadamard formula (Theorem 10.6) implies that

$$\limsup_{n\to\infty} \left|\frac{B_n}{n!}\right|^{1/n} = \frac{1}{2\pi},$$

which shows that at least some of the Bernoulli numbers B_n grow very rapidly (factorially) as $n \to \infty$.

Finally, we remark that we have proved that algebraic operations on convergent power series lead to convergent power series. If one is interested only in the formal algebraic properties of power series, and not their convergence, one can introduce a purely algebraic structure called the ring of formal power series (over the field $\mathbb{R}$) in a variable x,

$$\mathbb{R}[[x]] = \left\{\sum_{n=0}^{\infty} a_n x^n : a_n \in \mathbb{R}\right\},$$

[1] A prime number p is said to be irregular if it divides the numerator of B_{2n}, expressed in lowest terms, for some $2 \leq 2n \leq p - 3$; otherwise it is regular. The smallest irregular prime number is 37, which divides the numerator of $B_{32} = -7709321041217/5100$, since $7709321041217 = 37 \cdot 683 \cdot 305065927$. There are infinitely many irregular primes, and it is conjectured that there are infinitely many regular primes. A proof of this conjecture is, however, an open problem.

with sums and products on $\mathbb{R}[[x]]$ defined in the obvious way:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n,$$
$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right) x^n.$$

10.5. Differentiation of power series

We saw in Section 9.4.3 that, in general, one cannot differentiate a uniformly convergent sequence or series. We can, however, differentiate power series, and they behaves as nicely as one can imagine in this respect. The sum of a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

is infinitely differentiable inside its interval of convergence, and its derivative

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

is given by term-by-term differentiation. To prove this result, we first show that the term-by-term derivative of a power series has the same radius of convergence as the original power series. The idea is that the geometrical decay of the terms of the power series inside its radius of convergence dominates the algebraic growth of the factor n that comes from taking the derivative.

Theorem 10.20. Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence R. Then the power series

$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

also has radius of convergence R.

Proof. Assume without loss of generality that $c = 0$, and suppose $|x| < R$. Choose ρ such that $|x| < \rho < R$, and let

$$r = \frac{|x|}{\rho}, \qquad 0 < r < 1.$$

To estimate the terms in the differentiated power series by the terms in the original series, we rewrite their absolute values as follows:

$$\left|n a_n x^{n-1}\right| = \frac{n}{\rho}\left(\frac{|x|}{\rho}\right)^{n-1} |a_n \rho^n| = \frac{n r^{n-1}}{\rho} |a_n \rho^n|.$$

The ratio test shows that the series $\sum n r^{n-1}$ converges, since

$$\lim_{n\to\infty}\left[\frac{(n+1)r^n}{n r^{n-1}}\right] = \lim_{n\to\infty}\left[\left(1 + \frac{1}{n}\right) r\right] = r < 1,$$

so the sequence (nr^{n-1}) is bounded, by M say. It follows that

$$\left|na_n x^{n-1}\right| \leq \frac{M}{\rho}|a_n \rho^n| \qquad \text{for all } n \in \mathbb{N}.$$

The series $\sum |a_n \rho^n|$ converges, since $\rho < R$, so the comparison test implies that $\sum na_n x^{n-1}$ converges absolutely.

Conversely, suppose $|x| > R$. Then $\sum |a_n x^n|$ diverges (since $\sum a_n x^n$ diverges) and

$$\left|na_n x^{n-1}\right| \geq \frac{1}{|x|}|a_n x^n|$$

for $n \geq 1$, so the comparison test implies that $\sum na_n x^{n-1}$ diverges. Thus the series have the same radius of convergence. □

Theorem 10.21. Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \qquad \text{for } |x-c| < R$$

has radius of convergence $R > 0$ and sum f. Then f is differentiable in $|x-c| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \qquad \text{for } |x-c| < R.$$

Proof. The term-by-term differentiated power series converges in $|x-c| < R$ by Theorem 10.20. We denote its sum by

$$g(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}.$$

Let $0 < \rho < R$. Then, by Theorem 10.3, the power series for f and g both converge uniformly in $|x-c| < \rho$. Applying Theorem 9.18 to their partial sums, we conclude that f is differentiable in $|x-c| < \rho$ and $f' = g$. Since this holds for every $0 \leq \rho < R$, it follows that f is differentiable in $|x-c| < R$ and $f' = g$, which proves the result. □

Repeated application of Theorem 10.21 implies that the sum of a power series is infinitely differentiable inside its interval of convergence and its derivatives are given by term-by-term differentiation of the power series. Furthermore, we can get an expression for the coefficients a_n in terms of the function f; they are simply the Taylor coefficients of f at c.

Theorem 10.22. If the power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

has radius of convergence $R > 0$, then f is infinitely differentiable in $|x-c| < R$ and

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Proof. We assume $c = 0$ without loss of generality. Applying Theorem 10.22 to the power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots$$

k times, we find that f has derivatives of every order in $|x| < R$, and

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \ldots, \\
f''(x) &= 2a_2 + (3\cdot 2)a_3 x + \cdots + n(n-1)a_n x^{n-2} + \ldots, \\
f'''(x) &= (3\cdot 2\cdot 1)a_3 + \cdots + n(n-1)(n-2)a_n x^{n-3} + \ldots, \\
&\vdots \\
f^{(k)}(x) &= (k!)a_k + \cdots + \frac{n!}{(n-k)!}x^{n-k} + \ldots,
\end{aligned}$$

where all of these power series have radius of convergence R. Setting $x = 0$ in these series, we get

$$a_0 = f(0), \quad a_1 = f'(0), \quad \ldots \quad a_k = \frac{f^{(k)}(0)}{k!}, \quad \ldots$$

which proves the result (after replacing 0 by c). □

One consequence of this result is that power series with different coefficients cannot converge to the same sum.

Corollary 10.23. If two power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n, \qquad \sum_{n=0}^{\infty} b_n (x-c)^n$$

have nonzero-radius of convergence and are equal in some neighborhood of 0, then $a_n = b_n$ for every $n = 0, 1, 2, \ldots$.

Proof. If the common sum in $|x - c| < \delta$ is $f(x)$, we have

$$a_n = \frac{f^{(n)}(c)}{n!}, \qquad b_n = \frac{f^{(n)}(c)}{n!},$$

since the derivatives of f at c are determined by the values of f in an arbitrarily small open interval about c, so the coefficients are equal. □

10.6. The exponential function

We showed in Example 10.9 that the power series

$$E(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \ldots.$$

has radius of convergence ∞. It therefore defines an infinitely differentiable function $E : \mathbb{R} \to \mathbb{R}$.

Term-by-term differentiation of the power series, which is justified by Theorem 10.21, implies that

$$E'(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{(n-1)!}x^{(n-1)} + \ldots,$$

so $E' = E$. Moreover $E(0) = 1$. As we show below, there is a unique function with these properties, which are shared by the exponential function e^x. Thus, this power series provides an analytical definition of $e^x = E(x)$. All of the other familiar properties of the exponential follow from its power-series definition, and we will prove a few of them here

First, we show that $e^x e^y = e^{x+y}$. For the moment, we continue to write the function e^x as $E(x)$ to emphasise that we use nothing beyond its power series definition.

Proposition 10.24. For every $x, y \in \mathbb{R}$,

$$E(x)E(y) = E(x + y).$$

Proof. We have

$$E(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \qquad E(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

Multiplying these series term-by-term and rearranging the sum as a Cauchy product, which is justified by Theorem 4.38, we get

$$\begin{aligned} E(x)E(y) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{j!\,k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!\,k!}. \end{aligned}$$

From the binomial theorem,

$$\sum_{k=0}^{n} \frac{x^{n-k} y^k}{(n-k)!\,k!} = \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} x^{n-k} y^k = \frac{1}{n!}\left(x+y\right)^n.$$

Hence,

$$E(x)E(y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y),$$

which proves the result. □

In particular, since $E(0) = 1$, it follows that

$$E(-x) = \frac{1}{E(x)}.$$

We have $E(x) > 0$ for all $x \geq 0$, since all of the terms in its power series are positive, so $E(x) > 0$ for all $x \in \mathbb{R}$.

Next, we prove that the exponential is characterized by the properties $E' = E$ and $E(0) = 1$. This is a simple uniqueness result for an initial value problem for a linear ordinary differential equation.

Proposition 10.25. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$f' = f, \qquad f(0) = 1.$$

Then $f = E$.

Proof. Suppose that $f' = f$. Using the equation $E' = E$, the fact that E is nonzero on $\mathbb{R}$, and the quotient rule, we get

$$\left(\frac{f}{E}\right)' = \frac{fE' - Ef'}{E^2} = \frac{fE - Ef}{E^2} = 0.$$

It follows from Theorem 8.34 that f/E is constant on $\mathbb{R}$. Since $f(0) = E(0) = 1$, we have $f/E = 1$, which implies that $f = E$. □

In view of this result, we now write $E(x) = e^x$. The following proposition, which we use below in Section 10.7.2, shows that e^x grows faster than any power of x as $x \to \infty$.

Proposition 10.26. Suppose that n is a non-negative integer. Then

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof. The terms in the power series of e^x are positive for $x > 0$, so for every $k \in \mathbb{N}$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} > \frac{x^k}{k!} \qquad \text{for all } x > 0.$$

Taking $k = n + 1$, we get for $x > 0$ that

$$0 < \frac{x^n}{e^x} < \frac{x^n}{x^{(n+1)}/(n+1)!} = \frac{(n+1)!}{x}.$$

Since $1/x \to 0$ as $x \to \infty$, the result follows. □

The logarithm $\log : (0, \infty) \to \mathbb{R}$ can be defined as the inverse of the exponential function $\exp : \mathbb{R} \to (0, \infty)$, which is strictly increasing on $\mathbb{R}$ since its derivative is strictly positive. Having the logarithm and the exponential, we can define the power function for all exponents $p \in \mathbb{R}$ by

$$x^p = e^{p \log x}, \qquad x > 0.$$

Other transcendental functions, such as the trigonometric functions, can be defined in terms of their power series, and these can be used to prove their usual properties. We will not carry all this out in detail; we just want to emphasize that, once we have developed the theory of power series, we can define all of the functions arising in elementary calculus from the first principles of analysis.

10.7. * Smooth versus analytic functions

The power series theorem, Theorem 10.22, looks similar to Taylor's theorem, Theorem 8.46, but there is a fundamental difference. Taylor's theorem gives an expression for the error between a function and its Taylor polynomials. No question of convergence is involved. On the other hand, Theorem 10.22 asserts the convergence of an infinite power series to a function f. The coefficients of the Taylor polynomials and the power series are the same in both cases, but Taylor's theorem approximates f by its Taylor polynomials $P_n(x)$ of degree n at c in the limit $x \to c$ with n fixed, while the power series theorem approximates f by $P_n(x)$ in the limit $n \to \infty$ with x fixed.

10.7.1. Taylor's theorem and power series. To explain the difference between Taylor's theorem and power series in more detail, we introduce an important distinction between smooth and analytic functions: smooth functions have continuous derivatives of all orders, while analytic functions are sums of power series.

Definition 10.27. Let $k \in \mathbb{N}$. A function $f : (a, b) \to \mathbb{R}$ is C^k on (a, b), written $f \in C^k(a, b)$, if it has continuous derivatives $f^{(j)} : (a, b) \to \mathbb{R}$ of orders $1 \leq j \leq k$. A function f is smooth (or C^∞, or infinitely differentiable) on (a, b), written $f \in C^\infty(a, b)$, if it has continuous derivatives of all orders on (a, b).

In fact, if f has derivatives of all orders, then they are automatically continuous, since the differentiability of $f^{(k)}$ implies its continuity; on the other hand, the existence of k derivatives of f does not imply the continuity of $f^{(k)}$. The statement "f is smooth" is sometimes used rather loosely to mean "f has as many continuous derivatives as we want," but we will use it to mean that f is C^∞.

Definition 10.28. A function $f : (a, b) \to \mathbb{R}$ is analytic on (a, b) if for every $c \in (a, b)$ the function f is the sum in a neighborhood of c of a power series centered at c with nonzero radius of convergence.

Strictly speaking, this is the definition of a real analytic function, and analytic functions are complex functions that are sums of power series. Since we consider only real functions here, we abbreviate "real analytic" to "analytic."

Theorem 10.22 implies that an analytic function is smooth: If f is analytic on (a, b) and $c \in (a, b)$, then there is an $R > 0$ and coefficients (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \qquad \text{for } |x - c| < R.$$

Then Theorem 10.22 implies that f has derivatives of all orders in $|x - c| < R$, and since $c \in (a, b)$ is arbitrary, f has derivatives of all orders in (a, b). Moreover, it follows that the coefficients a_n in the power series expansion of f at c are given by Taylor's formula.

What is less obvious is that a smooth function need not be analytic. If f is smooth, then we can define its Taylor coefficients $a_n = f^{(n)}(c)/n!$ at c for every $n \geq 0$, and write down the corresponding Taylor series $\sum a_n (x - c)^n$. The problem is that the Taylor series may have zero radius of convergence if the derivatives of f grow too rapidly as $n \to \infty$, in which case it diverges for every $x \neq c$, or the Taylor series may converge, but not to f.

10.7.2. A smooth, non-analytic function. In this section, we give an example of a smooth function that is not the sum of its Taylor series.

It follows from Proposition 10.26 that if

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

is any polynomial function, then

$$\lim_{x \to \infty} \frac{p(x)}{e^x} = \sum_{k=0}^{n} a_k \lim_{x \to \infty} \frac{x^k}{e^x} = 0.$$

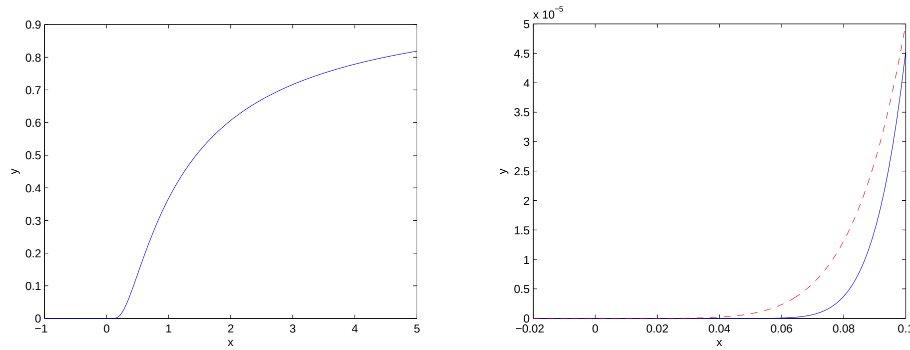

Figure 3. Left: Plot $y = \phi(x)$ of the smooth, non-analytic function defined in Proposition 10.29. Right: A detail of the function near $x = 0$. The dotted line is the power-function $y = x^6/50$. The graph of ϕ near 0 is "flatter' than the graph of the power-function, illustrating that $\phi(x)$ goes to zero faster than any power of x as $x \to 0$.

We will use this limit to exhibit a non-zero function that approaches zero faster than every power of x as $x \to 0$. As a result, all of its derivatives at 0 vanish, even though the function itself does not vanish in any neighborhood of 0. (See Figure 3.)

Proposition 10.29. Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Then ϕ has derivatives of all orders on $\mathbb{R}$ and

$$\phi^{(n)}(0) = 0 \qquad \text{for all } n \ge 0.$$

Proof. The infinite differentiability of $\phi(x)$ at $x \neq 0$ follows from the chain rule. Moreover, its nth derivative has the form

$$\phi^{(n)}(x) = \begin{cases} p_n(1/x)\exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where $p_n(1/x)$ is a polynomial of degree $2n$ in $1/x$. This follows, for example, by induction, since differentiation of $\phi^{(n)}$ shows that p_n satisfies the recursion relation

$$p_{n+1}(z) = z^2 \left[p_n(z) - p_n'(z)\right], \qquad p_0(z) = 1.$$

Thus, we just have to show that ϕ has derivatives of all orders at 0, and that these derivatives are equal to zero.

First, consider $\phi'(0)$. The left derivative $\phi'(0^-)$ of ϕ at 0 is 0 since $\phi(0) = 0$ and $\phi(h) = 0$ for all $h < 0$. To find the right derivative, we write $1/h = x$ and use

Proposition 10.26, which gives

$$\begin{aligned}\phi'(0^+) &= \lim_{h\to 0^+}\left[\frac{\phi(h)-\phi(0)}{h}\right]\\ &= \lim_{h\to 0^+}\frac{\exp(-1/h)}{h}\\ &= \lim_{x\to\infty}\frac{x}{e^x}\\ &= 0.\end{aligned}$$

Since both the left and right derivatives equal zero, we have $\phi'(0) = 0$.

To show that all the derivatives of ϕ at 0 exist and are zero, we use a proof by induction. Suppose that $\phi^{(n)}(0) = 0$, which we have verified for $n = 1$. The left derivative $\phi^{(n+1)}(0^-)$ is clearly zero, so we just need to prove that the right derivative is zero. Using the form of $\phi^{(n)}(h)$ for $h > 0$ and Proposition 10.26, we get that

$$\begin{aligned}\phi^{(n+1)}(0^+) &= \lim_{h\to 0^+}\left[\frac{\phi^{(n)}(h)-\phi^{(n)}(0)}{h}\right]\\ &= \lim_{h\to 0^+}\frac{p_n(1/h)\exp(-1/h)}{h}\\ &= \lim_{x\to\infty}\frac{xp_n(x)}{e^x}\\ &= 0,\end{aligned}$$

which proves the result. □

Corollary 10.30. The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases}\exp(-1/x) & \text{if } x > 0,\\ 0 & \text{if } x \le 0,\end{cases}$$

is smooth but not analytic on $\mathbb{R}$.

Proof. From Proposition 10.29, the function ϕ is smooth, and the nth Taylor coefficient of ϕ at 0 is $a_n = 0$. The Taylor series of ϕ at 0 therefore converges to 0, so its sum is not equal to ϕ in any neighborhood of 0, meaning that ϕ is not analytic at 0. □

The fact that the Taylor polynomial of ϕ at 0 is zero for every degree $n \in \mathbb{N}$ does not contradict Taylor's theorem, which says that for for every $n \in \mathbb{N}$ and $x > 0$ there exists $0 < \xi < x$ such that

$$\phi(x) = \frac{\phi^{(n)}(\xi)}{n!}x^n.$$

Since the derivatives of ϕ are bounded, it follows that there is a constant C_n, depending on n, such that

$$|\phi(x)| \le C_n x^n \qquad \text{for all } 0 < x < \infty.$$

Thus, $\phi(x) \to 0$ as $x \to 0$ faster than any power of x. But this inequality does not imply that $\phi(x) = 0$ for $x > 0$ since C_n grows rapidly as n increases, and $C_n x^n \not\to 0$ as $n \to \infty$ for any $x > 0$, however small.

We can construct other smooth, non-analytic functions from ϕ.

Example 10.31. The function

$$\psi(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is infinitely differentiable on $\mathbb{R}$, since $\psi(x) = \phi(x^2)$ is a composition of smooth functions.

The function in the next example is useful in many parts of analysis. Before giving the example, we introduce some terminology.

Definition 10.32. A function $f : \mathbb{R} \to \mathbb{R}$ has compact support if there exists $R \geq 0$ such that $f(x) = 0$ for all $x \in \mathbb{R}$ with $|x| \geq R$.

It isn't hard to construct continuous functions with compact support; one example that vanishes for $|x| \geq 1$ is the piecewise-linear, triangular (or 'tent') function

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

By matching left and right derivatives of piecewise-polynomial functions, we can similarly construct C^1 or C^k functions with compact support. Using ϕ, however, we can construct a smooth (C^∞) function with compact support, which might seem unexpected at first sight.

Example 10.33. The function

$$\eta(x) = \begin{cases} \exp[-1/(1 - x^2)] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

is infinitely differentiable on $\mathbb{R}$, since $\eta(x) = \phi(1 - x^2)$ is a composition of smooth functions. Moreover, it vanishes for $|x| \geq 1$, so it is a smooth function with compact support. Figure 4 shows its graph. This function is sometimes called a 'bump' function.

The function ϕ defined in Proposition 10.29 illustrates that knowing the values of a smooth function and all of its derivatives at one point does not tell us anything about the values of the function at nearby points. This behavior contrasts with, and highlights, the remarkable property of analytic functions that the values of an analytic function and all of its derivatives at a single point of an interval determine the function on the whole interval.

We make this principle of analytic continuation precise in the following proposition. The proof uses a common trick of going from a local result (equality of functions in a neighborhood of a point) to a global result (equality of functions on the whole of their connected domain) by proving that an appropriate subset is open, closed, and non-empty.

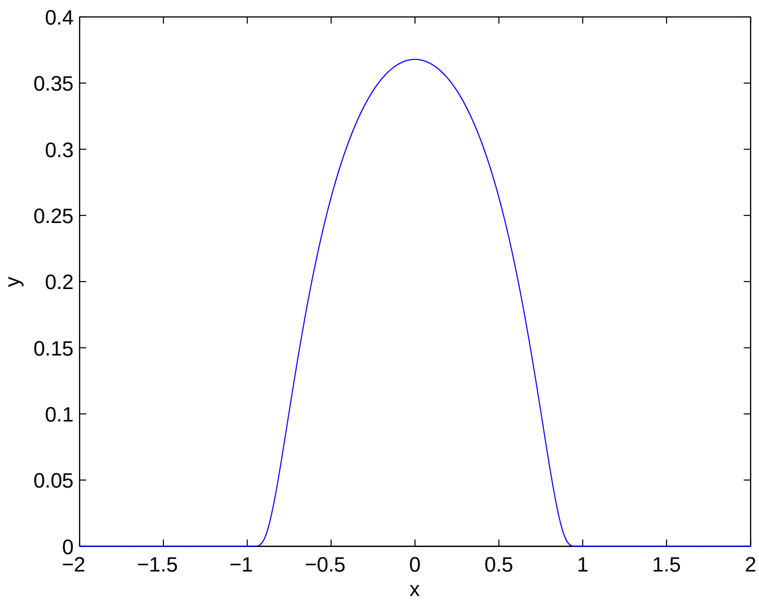

Figure 4. Plot of the smooth, compactly supported "bump" function defined in Example 10.33.

Proposition 10.34. Suppose that $f, g : (a, b) \to \mathbb{R}$ are analytic functions on an open interval (a, b). If $f^{(n)}(c) = g^{(n)}(c)$ for all $n \geq 0$ at some point $c \in (a, b)$, then $f = g$ on (a, b).

Proof. Let

$$E = \left\{x \in (a, b) : f^{(n)}(x) = g^{(n)}(x) \text{ all } n \geq 0\right\}.$$

The continuity of the derivatives $f^{(n)}$, $g^{(n)}$ implies that E is closed in (a, b): If $x_k \in E$ and $x_k \to x \in (a, b)$, then

$$f^{(n)}(x) = \lim_{k\to\infty} f^{(n)}(x_k) = \lim_{k\to\infty} g^{(n)}(x_k) = g^{(n)}(x),$$

so $x \in E$, and E is closed.

The analyticity of f, g implies that E is open in (a, b): If $x \in E$, then $f = g$ in some open interval $(x - r, x + r)$ with $r > 0$, since both functions have the same Taylor coefficients and convergent power series centered at x, so $f^{(n)} = g^{(n)}$ in $(x - r, x + r)$, meaning that $(x - r, x + r) \subset E$, and E is open.

From Theorem 5.63, the interval (a, b) is connected, meaning that the only subsets that are open and closed in (a, b) are the empty set and the entire interval. But $E \neq \varnothing$ since $c \in E$, so $E = (a, b)$, which proves the result. □

It is worth noting the choice of the set E in the preceding proof. For example, the proof would not work if we try to use the set

$$\tilde{E} = \{x \in (a, b) : f(x) = g(x)\}$$

instead of E. The continuity of f, g implies that $\tilde{E}$ is closed, but $\tilde{E}$ is not, in general, open.

One particular consequence of Proposition 10.34 is that a non-zero analytic function on $\mathbb{R}$ cannot have compact support, since an analytic function on $\mathbb{R}$ that is equal to zero on any interval $(a, b) \subset \mathbb{R}$ must equal zero on $\mathbb{R}$. Thus, the non-analyticity of the 'bump'-function η in Example 10.33 is essential.