Chapter 5

Topology of the Real Numbers

In this chapter, we define some topological properties of the real numbers $\mathbb{R}$ and its subsets.

5.1. Open sets

Open sets are among the most important subsets of $\mathbb{R}$. A collection of open sets is called a topology, and any property (such as convergence, compactness, or continuity) that can be defined entirely in terms of open sets is called a topological property.

Definition 5.1. A set $G \subset \mathbb{R}$ is open if for every $x \in G$ there exists a $\delta > 0$ such that $G \supset (x - \delta, x + \delta)$.

The entire set of real numbers $\mathbb{R}$ is obviously open, and the empty set $\varnothing$ is open since it satisfies the definition vacuously (there is no $x \in \varnothing$).

Example 5.2. The open interval $I = (0, 1)$ is open. If $x \in I$, then

$$I \supset (x - \delta, x + \delta), \qquad \delta = \min\left(\frac{x}{2}, \frac{1-x}{2}\right) > 0.$$

Similarly, every finite or infinite open interval (a, b), $(-\infty, b)$, or (a, ∞) is open.

Example 5.3. The half-open interval $J = (0, 1]$ isn't open, since $1 \in J$ and $(1 - \delta, 1 + \delta)$ isn't a subset of J for any $\delta > 0$, however small.

The next proposition states a characteristic property of open sets.

Proposition 5.4. An arbitrary union of open sets is open, and a finite intersection of open sets is open.

Proof. Suppose that $\{A_i \subset \mathbb{R} : i \in I\}$ is an arbitrary collection of open sets. If $x \in \bigcup_{i\in I} A_i$, then $x \in A_i$ for some $i \in I$. Since A_i is open, there is $\delta > 0$ such that $A_i \supset (x-\delta, x+\delta)$, and therefore

$$\bigcup_{i\in I} A_i \supset (x-\delta, x+\delta),$$

which proves that $\bigcup_{i\in I} A_i$ is open.

Suppose that $\{A_i \subset \mathbb{R} : i = 1, 2, \dots, n\}$ is a finite collection of open sets. If $x \in \bigcap_{i=1}^n A_i$, then $x \in A_i$ for every $1 \le i \le n$. Since A_i is open, there is $\delta_i > 0$ such that $A_i \supset (x-\delta_i, x+\delta_i)$. Let

$$\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\} > 0.$$

Then we see that

$$\bigcap_{i=1}^n A_i \supset (x-\delta, x+\delta),$$

which proves that $\bigcap_{i=1}^n A_i$ is open. □

The previous proof fails for an infinite intersection of open sets, since we may have $\delta_i > 0$ for every $i \in \mathbb{N}$ but $\inf\{\delta_i : i \in \mathbb{N}\} = 0$.

Example 5.5. The interval

$$I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is open for every $n \in \mathbb{N}$, but

$$\bigcap_{n=1}^\infty I_n = \{0\}$$

is not open.

In fact, every open set in $\mathbb{R}$ is a countable union of disjoint open intervals, but we won't prove it here.

5.1.1. Neighborhoods. Next, we introduce the notion of the neighborhood of a point, which often gives clearer, but equivalent, descriptions of topological concepts than ones that use open intervals.

Definition 5.6. A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if

$$U \supset (x-\delta, x+\delta)$$

for some $\delta > 0$. The open interval $(x-\delta, x+\delta)$ is called a δ-neighborhood of x.

A neighborhood of x needn't be an open interval about x, it just has to contain one. Some people require than a neighborhood is also an open set, but we don't; we'll specify that a neighborhood is open if it's needed.

Example 5.7. If $a < x < b$, then the closed interval $[a, b]$ is a neighborhood of x, since it contains the interval $(x-\delta, x+\delta)$ for sufficiently small $\delta > 0$. On the other hand, $[a, b]$ is not a neighborhood of the endpoints a, b since no open interval about a or b is contained in $[a, b]$.

We can restate the definition of open sets in terms of neighborhoods as follows.

Definition 5.8. A set $G \subset \mathbb{R}$ is open if every $x \in G$ has a neighborhood U such that $G \supset U$.

In particular, an open set is itself a neighborhood of each of its points.

We can restate Definition 3.10 for the limit of a sequence in terms of neighborhoods as follows.

Proposition 5.9. A sequence (x_n) of real numbers converges to a limit $x \in \mathbb{R}$ if and only if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Proof. First suppose the condition in the proposition holds. Given $\epsilon > 0$, let $U = (x - \epsilon, x + \epsilon)$ be an ϵ-neighborhood of x. Then there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$, which means that $|x_n - x| < \epsilon$. Thus, $x_n \to x$ as $n \to \infty$.

Conversely, suppose that $x_n \to x$ as $n \to \infty$, and let U be a neighborhood of x. Then there exists $\epsilon > 0$ such that $U \supset (x - \epsilon, x + \epsilon)$. Choose $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n > N$. Then $x_n \in U$ for all $n > N$, which proves the condition. □

5.1.2. Relatively open sets. We define relatively open sets by restricting open sets in $\mathbb{R}$ to a subset.

Definition 5.10. If $A \subset \mathbb{R}$ then $B \subset A$ is relatively open in A, or open in A, if $B = A \cap G$ where G is open in $\mathbb{R}$.

Example 5.11. Let $A = [0, 1]$. Then the half-open intervals $(a, 1]$ and $[0, b)$ are open in A for every $0 \le a < 1$ and $0 < b \le 1$, since

$$(a, 1] = [0, 1] \cap (a, 2), \qquad [0, b) = [0, 1] \cap (-1, b)$$

and $(a, 2)$, $(-1, b)$ are open in $\mathbb{R}$. By contrast, neither $(a, 1]$ nor $[0, b)$ is open in $\mathbb{R}$.

The neighborhood definition of open sets generalizes to relatively open sets. First, we define relative neighborhoods in the obvious way.

Definition 5.12. If $A \subset \mathbb{R}$ then a relative neighborhood in A of a point $x \in A$ is a set $V = A \cap U$ where U is a neighborhood of x in $\mathbb{R}$.

As we show next, a set is relatively open if and only if it contains a relative neighborhood of every point.

Proposition 5.13. A set $B \subset A$ is relatively open in A if and only if every $x \in B$ has a relative neighborhood V in A such that $B \supset V$.

Proof. Assume that B is open in A. Then $B = A \cap G$ where G is open in $\mathbb{R}$. If $x \in B$, then $x \in G$, and since G is open, there is a neighborhood U of x in $\mathbb{R}$ such that $G \supset U$. Then $V = A \cap U$ is a relative neighborhood of x with $B \supset V$.

Conversely, assume that every point $x \in B$ has a relative neighborhood $V_x = A \cap U_x$ in A such that $V_x \subset B$, where U_x is a neighborhood of x in $\mathbb{R}$. Since U_x is a neighborhood of x, it contains an open neighborhood $G_x \subset U_x$. We claim that that $B = A \cap G$ where

$$G = \bigcup_{x \in B} G_x.$$

It then follows that G is open, since it's a union of open sets, and therefore $B = A \cap G$ is relatively open in A.

To prove the claim, we show that $B \subset A \cap G$ and $B \supset A \cap G$. First, $B \subset A \cap G$ since $x \in A \cap G_x \subset A \cap G$ for every $x \in B$. Second, $A \cap G_x \subset A \cap U_x \subset B$ for every $x \in B$. Taking the union over $x \in B$, we get that $A \cap G \subset B$. □

5.2. Closed sets

Sets are not doors. (Attributed to James Munkres.)

Closed sets are defined topologically as complements of open sets.

Definition 5.14. A set $F \subset \mathbb{R}$ is closed if $F^c = \{x \in \mathbb{R} : x \notin F\}$ is open.

Example 5.15. The closed interval $I = [0, 1]$ is closed since

$$I^c = (-\infty, 0) \cup (1, \infty)$$

is a union of open intervals, and therefore it's open. Similarly, every finite or infinite closed interval $[a, b]$, $(-\infty, b]$, or $[a, \infty)$ is closed.

The empty set $\varnothing$ and $\mathbb{R}$ are both open and closed; they're the only such sets. Most subsets of $\mathbb{R}$ are neither open nor closed (so, unlike doors, "not open" doesn't mean "closed" and "not closed" doesn't mean "open").

Example 5.16. The half-open interval $I = (0, 1]$ isn't open because it doesn't contain any neighborhood of the right endpoint $1 \in I$. Its complement

$$I^c = (\infty, 0] \cup (1, \infty)$$

isn't open either, since it doesn't contain any neighborhood of $0 \in I^c$. Thus, I isn't closed either.

Example 5.17. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed. It isn't open because every neighborhood of a rational number contains irrational numbers, and its complement isn't open because every neighborhood of an irrational number contains rational numbers.

Closed sets can also be characterized in terms of sequences.

Proposition 5.18. A set $F \subset \mathbb{R}$ is closed if and only if the limit of every convergent sequence in F belongs to F.

Proof. First suppose that F is closed and (x_n) is a convergent sequence of points $x_n \in F$ such that $x_n \to x$. Then every neighborhood of x contains points $x_n \in F$. It follows that $x \notin F^c$, since F^c is open and every $y \in F^c$ has a neighborhood $U \subset F^c$ that contains no points in F. Therefore, $x \in F$.

Conversely, suppose that the limit of every convergent sequence of points in F belongs to F. Let $x \in F^c$. Then x must have a neighborhood $U \subset F^c$; otherwise for every $n \in \mathbb{N}$ there exists $x_n \in F$ such that $x_n \in (x - 1/n, x + 1/n)$, so $x = \lim x_n$, and x is the limit of a sequence in F. Thus, F^c is open and F is closed. □

Example 5.19. To verify that the closed interval $[0, 1]$ is closed from Proposition 5.18, suppose that (x_n) is a convergent sequence in $[0, 1]$. Then $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$, and since limits preserve (non-strict) inequalities, we have

$$0 \leq \lim_{n\to\infty} x_n \leq 1,$$

meaning that the limit belongs to $[0, 1]$. On the other hand, the half-open interval $I = (0, 1]$ isn't closed since, for example, $(1/n)$ is a convergent sequence in I whose limit 0 doesn't belong to I.

Closed sets have complementary properties to those of open sets stated in Proposition 5.4.

Proposition 5.20. An arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.

Proof. If $\{F_i : i \in I\}$ is an arbitrary collection of closed sets, then every F_i^c is open. By De Morgan's laws in Proposition 1.23, we have

$$\left(\bigcap_{i\in I} F_i\right)^c = \bigcup_{i\in I} F_i^c,$$

which is open by Proposition 5.4. Thus $\bigcap_{i\in I} F_i$ is closed. Similarly, the complement of a finite union of closed sets is open, since

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c,$$

so a finite union of closed sets is closed. □

The union of infinitely many closed sets needn't be closed.

Example 5.21. If I_n is the closed interval

$$I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right],$$

then the union of the I_n is an open interval

$$\bigcup_{n=1}^{\infty} I_n = (0, 1).$$

If A is a subset of $\mathbb{R}$, it is useful to consider different ways in which a point $x \in \mathbb{R}$ can belong to A or be "close" to A.

Definition 5.22. Let $A \subset \mathbb{R}$ be a subset of $\mathbb{R}$. Then $x \in \mathbb{R}$ is:

(1) an interior point of A if there exists $\delta > 0$ such that $A \supset (x - \delta, x + \delta)$;

(2) an isolated point of A if $x \in A$ and there exists $\delta > 0$ such that x is the only point in A that belongs to the interval $(x - \delta, x + \delta)$;

(3) a boundary point of A if for every $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains points in A and points not in A;

(4) an accumulation point of A if for every $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point in A that is distinct from x.

When the set A is understood from the context, we refer, for example, to an "interior point."

Interior and isolated points of a set belong to the set, whereas boundary and accumulation points may or may not belong to the set. In the definition of a boundary point x, we allow the possibility that x itself is a point in A belonging to $(x-\delta, x+\delta)$, but in the definition of an accumulation point, we consider only points in A belonging to $(x-\delta, x+\delta)$ that are distinct from x. Thus an isolated point is a boundary point, but it isn't an accumulation point. Accumulation points are also called cluster points or limit points.

We illustrate these definitions with a number of examples.

Example 5.23. Let $I = (a, b)$ be an open interval and $J = [a, b]$ a closed interval. Then the set of interior points of I or J is (a, b), and the set of boundary points consists of the two endpoints $\{a, b\}$. The set of accumulation points of I or J is the closed interval $[a, b]$ and I, J have no isolated points. Thus, I, J have the same interior, isolated, boundary and accumulation points, but J contains its boundary points and all of its accumulation points, while I does not.

Example 5.24. Let $a < c < b$ and suppose that

$$A = (a, c) \cup (c, b)$$

is an open interval punctured at c. Then the set of interior points is A, the set of boundary points is $\{a, b, c\}$, the set of accumulation points is the closed interval $[a, b]$, and there are no isolated points.

Example 5.25. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then every point of A is an isolated point, since a sufficiently small interval about $1/n$ doesn't contain $1/m$ for any integer $m \neq n$, and A has no interior points. The set of boundary points of A is $A \cup \{0\}$. The point $0 \notin A$ is the only accumulation point of A, since every open interval about 0 contains $1/n$ for sufficiently large n.

Example 5.26. The set $\mathbb{N}$ of natural numbers has no interior or accumulation points. Every point of $\mathbb{N}$ is both a boundary point and an isolated point.

Example 5.27. The set $\mathbb{Q}$ of rational numbers has no interior or isolated points, and every real number is both a boundary and accumulation point of $\mathbb{Q}$.

Example 5.28. The Cantor set C defined in Section 5.5 below has no interior points and no isolated points. The set of accumulation points and the set of boundary points of C is equal to C.

The following proposition gives a sequential definition of an accumulation point.

Proposition 5.29. A point $x \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$ if and only if there is a sequence (x_n) in A with $x_n \neq x$ for every $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$.

Proof. Suppose $x \in \mathbb{R}$ is an accumulation point of A. Definition 5.22 implies that for every $n \in \mathbb{N}$ there exists $x_n \in A \setminus \{x\}$ such that $x_n \in (x - 1/n, x + 1/n)$. It follows that $x_n \to x$ as $n \to \infty$.

Conversely, if x is the limit of a sequence (x_n) in A with $x_n \neq x$, and U is a neighborhood of x, then $x_n \in U \setminus \{x\}$ for sufficiently large $n \in \mathbb{N}$, which proves that x is an accumulation point of A. □

Example 5.30. If

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

then 0 is an accumulation point of A, since $(1/n)$ is a sequence in A such that $1/n \to 0$ as $n \to \infty$. On the other hand, 1 is not an accumulation point of A since the only sequences in A that converges to 1 are the ones whose terms eventually equal 1, and the terms are required to be distinct from 1.

We can also characterize open and closed sets in terms of their interior and accumulation points.

Proposition 5.31. A set $A \subset \mathbb{R}$ is:

(1) open if and only if every point of A is an interior point;

(2) closed if and only if every accumulation point belongs to A.

Proof. If A is open, then it is an immediate consequence of the definitions that every point in A is an interior point. Conversely, if every point $x \in A$ is an interior point, then there is an open neighborhood $U_x \subset A$ of x, so

$$A = \bigcup_{x \in A} U_x$$

is a union of open sets, and therefore A is open.

If A is closed and x is an accumulation point, then Proposition 5.29 and Proposition 5.18 imply that $x \in A$. Conversely, if every accumulation point of A belongs to A, then every $x \in A^c$ has a neighborhood with no points in A, so A^c is open and A is closed. □

5.3. Compact sets

The significance of compact sets is not as immediately apparent as the significance of open sets, but the notion of compactness plays a central role in analysis. One indication of its importance already appears in the Bolzano-Weierstrass theorem (Theorem 3.57).

Compact sets may be characterized in many different ways, and we will give the two most important definitions. One is based on sequences (every sequence has a convergent subsequence), and the other is based on open sets (every open cover has a finite subcover).

We will prove that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. For example, every closed, bounded interval $[a, b]$ is compact. There are, however, many other compact subsets of $\mathbb{R}$. In Section 5.5 we describe a particularly interesting example called the Cantor set.

We emphasize that although the compact sets in $\mathbb{R}$ are exactly the closed and bounded sets, this isn't their fundamental definition; rather it's an explicit description of what compact sets look like in $\mathbb{R}$. In more general spaces than $\mathbb{R}$, closed and bounded sets need not be compact, and it's the properties defining compactness that are the crucial ones. Chapter 13 has further explanation.

5.3.1. Sequential definition. Intuitively, a compact set confines every sequence of points in the set so much that the sequence must accumulate at some point of the set. This implies that a subsequence converges to an accumulation point and leads to the following definition.

Definition 5.32. A set $K \subset \mathbb{R}$ is sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.

Note that we require that the subsequence converges to a point in K, not to a point outside K.

We usually abbreviate "sequentially compact" to "compact," but sometimes we need to distinguish explicitly between the sequential definition of compactness given above and the topological definition given in Definition 5.52 below.

Example 5.33. The open interval $I = (0, 1)$ is not compact. The sequence $(1/n)$ in I converges to 0, so every subsequence also converges to $0 \notin I$. Therefore, $(1/n)$ has no convergent subsequence whose limit belongs to I.

Example 5.34. The set $A = \mathbb{Q} \cap [0, 1]$ of rational numbers in $[0, 1]$ is not compact. If (r_n) is a sequence of rational numbers $0 \leq r_n \leq 1$ that converges to $1/\sqrt{2}$, then every subsequence also converges to $1/\sqrt{2} \notin A$, so (r_n) has no subsequence that converges to a point in A.

Example 5.35. The set $\mathbb{N}$ is closed, but it is not compact. The sequence (n) in $\mathbb{N}$ has no convergent subsequence since every subsequence diverges to infinity.

As these examples illustrate, a compact set must be closed and bounded. Conversely, the Bolzano-Weierstrass theorem implies that that every closed, bounded subset of $\mathbb{R}$ is compact. This fact may be taken as an alternative statement of the theorem.

Theorem 5.36 (Bolzano-Weierstrass)**.** A subset of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Proof. First, assume that $K \subset \mathbb{R}$ is sequentially compact. Let (x_n) be a sequence in K that converges to $x \in \mathbb{R}$. Then every subsequence of K also converges to x, so the compactness of K implies that $x \in K$. It follows from Proposition 5.18 that K is closed. Next, suppose for contradiction that K is unbounded. Then there is a sequence (x_n) in K such that $|x_n| \to \infty$ as $n \to \infty$. Every subsequence of (x_n) is also unbounded and therefore diverges, so (x_n) has no convergent subsequence. This contradicts the assumption that K is sequentially compact, so K is bounded.

Conversely, assume that $K \subset \mathbb{R}$ is closed and bounded. Let (x_n) be a sequence in K. Then (x_n) is bounded since K is bounded, and Theorem 3.57 implies that (x_n) has a convergent subsequence. Since K is closed the limit of this subsequence belongs to K, so K is sequentially compact. □

Example 5.37. Every closed, bounded interval $[a, b]$ is compact.

Example 5.38. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then A is not compact, since it isn't closed. However, the set $K = A \cup \{0\}$ is closed and bounded, so it is compact.

Example 5.39. The Cantor set defined in Section 5.5 is compact.

For later use, we prove a useful property of compact sets in $\mathbb{R}$ which follows from Theorem 5.36.

Proposition 5.40. If $K \subset \mathbb{R}$ is compact, then K has a maximum and minimum.

Proof. Since K is compact it is bounded and therefore it has a (finite) supremum $M = \sup K$. From the definition of the supremum, for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$M - \frac{1}{n} < x_n \leq M.$$

It follows from the 'squeeze' theorem that $x_n \to M$ as $n \to \infty$. Since K is closed, $M \in K$, which proves that K has a maximum. A similar argument shows that $m = \inf K$ belongs to K, so K has a minimum. □

Example 5.41. The bounded closed interval $[0, 1]$ is compact and its maximum 1 and minimum 0 belong to the set, while the open interval $(0, 1)$ is not compact and its supremum 1 and infimum 0 do not belong to the set. The unbounded, closed interval $[0, \infty)$ is not compact, and it has no maximum.

Example 5.42. The set A in Example 5.38 is not compact and its infimum 0 does not belong to the set, but the compact set K has 0 as a minimum value.

Compact sets have the following nonempty intersection property.

Theorem 5.43. Let $\{K_n : n \in \mathbb{N}\}$ be a decreasing sequence of nonempty compact sets of real numbers, meaning that

$$K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} \supset \ldots,$$

and $K_n \neq \varnothing$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Moreover, if $\operatorname{diam} K_n \to 0$ as $n \to \infty$, then the intersection consists of a single point.

Proof. For each $n \in \mathbb{N}$, choose $x_n \in K_n$. Since (x_n) is a sequence in the compact set K_1, it has a convergent subsequence (x_{n_k}) with $x_{n_k} \to x$ as $k \to \infty$. Then $x_{n_k} \in K_n$ for all k sufficiently large that $n_k \geq n$. Since a "tail" of the subsequence belongs to K_n and K_n is closed, we have $x \in K_n$ for every $n \in \mathbb{N}$. Hence, $x \in \bigcap K_n$, and the intersection is nonempty.

If $x, y \in \bigcap K_n$, then $x, y \in K_n$ for every $n \in \mathbb{N}$, so $|x - y| \leq \operatorname{diam} K_n$. If $\operatorname{diam} K_n \to 0$ as $n \to \infty$, then $|x - y| = 0$, so $x = y$ and $\bigcap K_n$ consists of a single point. □

We refer to a decreasing sequence of sets as a nested sequence. In the case when each $K_n = [a_n, b_n]$ is a compact interval, the preceding result is called the nested interval theorem.

Example 5.44. The nested compact intervals $[0, 1 + 1/n]$ have nonempty intersection $[0, 1]$. Here, $\operatorname{diam}[0, 1 + 1/n] \to 1$ as $n \to \infty$, and the intersection consists of an interval. The nested compact intervals $[0, 1/n]$ have nonempty intersection $\{0\}$, which consists of a single point since $\operatorname{diam}[0, 1/n] \to 0$ as $n \to \infty$. On the other hand, the nested half-open intervals $(0, 1/n]$ have empty intersection, as do the nested unbounded, closed intervals $[n, \infty)$. In particular, Theorem 5.43 doesn't hold if we replace "compact" by "closed."

Example 5.45. Define a nested sequence $A_1 \supset A_2 \supset \dots$ of non-compact sets by

$$A_n = \left\{ \frac{1}{k} : k = n, n+1, n+2, \dots \right\},$$

so $A_1 = A$ where A is the set considered in Example 5.38. Then

$$\bigcap_{n=1}^{\infty} A_n = \varnothing.$$

If we add 0 to the A_n to make them compact and define $K_n = A_n \cup \{0\}$, then the intersection

$$\bigcap_{n=1}^{\infty} K_n = \{0\}$$

is nonempty.

5.3.2. Topological definition.

To give a topological definition of compactness in terms of open sets, we introduce the notion of an open cover of a set.

Definition 5.46. Let $A \subset \mathbb{R}$. A cover of A is a collection of sets $\{A_i \subset \mathbb{R} : i \in I\}$ whose union contains A,

$$\bigcup_{i \in I} A_i \supset A.$$

An open cover of A is a cover such that A_i is open for every $i \in I$.

Example 5.47. Let $A_i = (1/i, 2)$. Then $\mathcal{C} = \{A_i : i \in \mathbb{N}\}$ is an open cover of $(0, 1]$, since

$$\bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 2 \right) = (0, 2) \supset (0, 1].$$

On the other hand, $\mathcal{C}$ is not a cover of $[0, 1]$ since its union does not contain 0. If, for any $\delta > 0$, we add the interval $B = (-\delta, \delta)$ to $\mathcal{C}$, then

$$\bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 2 \right) \cup B = (-\delta, 2) \supset [0, 1],$$

so $\mathcal{C}' = \mathcal{C} \cup \{B\}$ is an open cover of $[0, 1]$.

Example 5.48. If $A_i = (i-1, i+1)$, then $\{A_i : i \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$. On the other hand, if $B_i = (i, i+1)$, then $\{B_i : i \in \mathbb{Z}\}$ is not open cover of $\mathbb{R}$, since its union doesn't contain any of the integers. Finally, if $C_i = [i, i+1)$, then $\{C_i : i \in \mathbb{Z}\}$ is a cover of $\mathbb{R}$ by disjoint, half-open intervals, but it isn't an open cover. Thus, to get an open cover, we need the intervals to "overlap".

Example 5.49. Let $\{r_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers $r_i \in [0,1]$, and fix $\epsilon > 0$. Define $A_i = (r_i - \epsilon, r_i + \epsilon)$. Then $\{A_i : i \in \mathbb{N}\}$ is an open cover of $[0,1]$ since every irrational number $x \in [0,1]$ can be approximated to within ϵ by some rational number. Similarly, if $I = [0,1] \setminus \mathbb{Q}$ denotes the set of irrational numbers in $[0,1]$, then $\{(x-\epsilon, x+\epsilon) : x \in I\}$ is an open cover of $[0,1]$. In this case, the cover consists of uncountably many sets.

Next, we define subcovers.

Definition 5.50. Suppose that $\mathcal{C} = \{A_i \subset \mathbb{R} : i \in I\}$ is a cover of $A \subset \mathbb{R}$. A subcover $\mathcal{S}$ of $\mathcal{C}$ is a sub-collection $\mathcal{S} \subset \mathcal{C}$ that covers A, meaning that

$$\mathcal{S} = \{A_{i_k} \in \mathcal{C} : k \in J\}, \qquad \bigcup_{k \in J} A_{i_k} \supset A.$$

A finite subcover is a subcover $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ that consists of finitely many sets.

Example 5.51. Consider the cover $\mathcal{C} = \{A_i : i \in \mathbb{N}\}$ of $(0,1]$ in Example 5.47, where $A_i = (1/i, 2)$. Then $\{A_{2j} : j \in \mathbb{N}\}$ is a subcover. There is, however, no finite subcover $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ since if $N = \max\{i_1, i_2, \ldots, i_n\}$ then

$$\bigcup_{k=1}^{n} A_{k_i} = \left(\frac{1}{N}, 2\right),$$

which does not contain the points $x \in (0,1)$ with $0 < x \leq 1/N$. On the other hand, the cover $\mathcal{C}' = \mathcal{C} \cup \{(-\delta, \delta)\}$ of $[0,1]$ does have a finite subcover. For example, if $N \in \mathbb{N}$ is such that $1/N < \delta$, then

$$\left\{\left(\frac{1}{N}, 2\right), (-\delta, \delta)\right\}$$

is a finite subcover of $[0,1]$ consisting of two sets (whose union is the same as the original cover).

Having introduced this terminology, we give a topological definition of compact sets.

Definition 5.52. A set $K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover.

First, we illustrate Definition 5.52 with several examples.

Example 5.53. The collection of open intervals

$$\{A_i : i \in \mathbb{N}\}, \qquad A_i = (i-1, i+1)$$

is an open cover of the natural numbers $\mathbb{N}$, since

$$\bigcup_{i=1}^{\infty} A_i = (0, \infty) \supset \mathbb{N}.$$

However, no finite sub-collection

$$\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$$

covers $\mathbb{N}$, since if $N = \max\{i_1, i_2, \dots, i_n\}$, then

$$\bigcup_{k=1}^{n} A_{i_k} \subset (0, N+1)$$

so its union does not contain sufficiently large integers with $n \geq N+1$. Thus, $\mathbb{N}$ is not compact.

Example 5.54. Consider the open intervals

$$A_i = \left(\frac{1}{2^i} - \frac{1}{2^{i+1}}, \frac{1}{2^i} + \frac{1}{2^{i+1}}\right),$$

which get smaller as they get closer to 0. Then $\{A_i : i = 0, 1, 2, \dots\}$ is an open cover of the open interval $(0, 1)$; in fact

$$\bigcup_{i=0}^{\infty} A_i = \left(0, \frac{3}{2}\right) \supset (0, 1).$$

However, no finite sub-collection

$$\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$$

of intervals covers $(0, 1)$, since if $N = \max\{i_1, i_2, \dots, i_n\}$, then

$$\bigcup_{k=1}^{n} A_{i_k} \subset \left(\frac{1}{2^N} - \frac{1}{2^{N+1}}, \frac{3}{2}\right),$$

so it does not contain the points in $(0, 1)$ that are sufficiently close to 0. Thus, $(0, 1)$ is not compact. Example 5.51 gives another example of an open cover of $(0, 1)$ with no finite subcover.

Example 5.55. The collection of open intervals $\{A_0, A_1, A_2, \dots\}$ in Example 5.54 isn't an open cover of the closed interval $[0, 1]$ since 0 doesn't belong to their union. We can get an open cover $\{A_0, A_1, A_2, \dots, B\}$ of $[0, 1]$ by adding to the A_i an open interval $B = (-\delta, \delta)$, where $\delta > 0$ is arbitrarily small. In that case, if we choose $n \in \mathbb{N}$ sufficiently large that

$$\frac{1}{2^n} - \frac{1}{2^{n+1}} < \delta,$$

then $\{A_0, A_1, A_2, \dots, A_n, B\}$ is a finite subcover of $[0, 1]$ since

$$\bigcup_{i=0}^{n} A_i \cup B = \left(-\delta, \frac{3}{2}\right) \supset [0, 1].$$

Points sufficiently close to 0 belong to B, while points further away belong to A_i for some $0 \leq i \leq n$. The open cover of $[0, 1]$ in Example 5.51 is similar.

As the previous example suggests, and as follows from the next theorem, every open cover of $[0, 1]$ has a finite subcover, and $[0, 1]$ is compact.

Theorem 5.56 (Heine-Borel)**.** *A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. The most important direction of the theorem is that a closed, bounded set is compact.

First, we prove that a closed, bounded interval $K = [a, b]$ is compact. Suppose that $\mathcal{C} = \{A_i : i \in I\}$ is an open cover of $[a, b]$, and let

$$B = \{x \in [a, b] : [a, x] \text{ has a finite subcover } \mathcal{S} \subset \mathcal{C}\}.$$

We claim that $\sup B = b$. The idea of the proof is that any open cover of $[a, x]$ must cover a larger interval since the open set that contains x extends past x.

Since $\mathcal{C}$ covers $[a, b]$, there exists a set $A_i \in \mathcal{C}$ with $a \in A_i$, so $[a, a] = \{a\}$ has a subcover consisting of a single set, and $a \in B$. Thus, B is non-empty and bounded from above by b, so $c = \sup B \le b$ exists. Assume for contradiction that $c < b$. Then $[a, c]$ has a finite subcover

$$\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\},$$

with $c \in A_{i_k}$ for some $1 \le k \le n$. Since A_{i_k} is open and $a \le c < b$, there exists $\delta > 0$ such that $[c, c+\delta) \subset A_{i_k} \cap [a, b]$. Then $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$ is a finite subcover of $[a, x]$ for $c < x < c + \delta$, contradicting the definition of c, so $\sup B = b$. Moreover, the following argument shows that, in fact, $b = \max B$.

Since $\mathcal{C}$ covers $[a, b]$, there is an open set $A_{i_0} \in \mathcal{C}$ such that $b \in A_{i_0}$. Then $(b - \delta, b + \delta) \subset A_{i_0}$ for some $\delta > 0$, and since $\sup B = b$ there exists $c \in B$ such that $b - \delta < c \le b$. Let $\{A_{i_1}, \ldots, A_{i_n}\}$ be a finite subcover of $[a, c]$. Then $\{A_{i_0}, A_{i_1}, \ldots, A_{i_n}\}$ is a finite subcover of $[a, b]$, which proves that $[a, b]$ is compact.

Now suppose that $K \subset \mathbb{R}$ is a closed, bounded set, and let $\mathcal{C} = \{A_i : i \in I\}$ be an open cover of K. Since K is bounded, $K \subset [a, b]$ for some closed bounded interval $[a, b]$, and, since K is closed, $\mathcal{C}' = \mathcal{C} \cup \{K^c\}$ is an open cover of $[a, b]$. From what we have just proved, $[a, b]$ has a finite subcover that is included in $\mathcal{C}'$. Omitting K^c from this subcover, if necessary, we get a finite subcover of K that is included in the original cover $\mathcal{C}$.

To prove the converse, suppose that $K \subset \mathbb{R}$ is compact. Let $A_i = (-i, i)$. Then

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{R} \supset K,$$

so $\{A_i : i \in \mathbb{N}\}$ is an open cover of K, which has a finite subcover $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$. Let $N = \max\{i_1, i_2, \ldots, i_n\}$. Then

$$K \subset \bigcup_{k=1}^{n} A_{i_k} = (-N, N),$$

so K is bounded.

To prove that K is closed, we prove that K^c is open. Suppose that $x \in K^c$. For $i \in \mathbb{N}$, let

$$A_i = \left[x - \frac{1}{i}, x + \frac{1}{i}\right]^c = \left(-\infty, x - \frac{1}{i}\right) \cup \left(x + \frac{1}{i}, \infty\right).$$

Then $\{A_i : i \in \mathbb{N}\}$ is an open cover of K, since

$$\bigcup_{i=1}^{\infty} A_i = (-\infty, x) \cup (x, \infty) \supset K.$$

Since K is compact, there is a finite subcover $\{A_{i_1}, A_{i_2}, \ldots, A_{i_n}\}$. Let $N = \max\{i_1, i_2, \ldots, i_n\}$. Then

$$K \subset \bigcup_{k=1}^{n} A_{i_k} = \left(-\infty, x - \frac{1}{N}\right) \cup \left(x + \frac{1}{N}, \infty\right),$$

which implies that $(x - 1/N, x + 1/N) \subset K^c$. This proves that K^c is open and K is closed. □

The following corollary is an immediate consequence of what we have proved.

Corollary 5.57. A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact.

Proof. By Theorem 5.36 and Theorem 5.56, a subset of $\mathbb{R}$ is compact or sequentially compact if and only if it is closed and bounded. □

Corollary 5.57 generalizes to an arbitrary metric space, where a set is compact if and only if it is sequentially compact, although a different proof is required. By contrast, Theorem 5.36 and Theorem 5.56 do not hold in an arbitrary metric space, where a closed, bounded set need not be compact.

5.4. Connected sets

A connected set is, roughly speaking, a set that cannot be divided into "separated" parts. The formal definition is as follows.

Definition 5.58. A set of real numbers $A \subset \mathbb{R}$ is disconnected if there are disjoint open sets $U, V \subset \mathbb{R}$ such that $A \cap U$ and $A \cap V$ are nonempty and

$$A = (A \cap U) \cup (A \cap V).$$

A set is connected if it not disconnected.

The condition $A = (A \cap U) \cup (A \cap V)$ is equivalent to $U \cup V \supset A$. If A is disconnected as in the definition, then we say that the open sets U, V separate A.

It is easy to give examples of disconnected sets. As the following examples illustrate, any set of real numbers that is "missing" a point is disconnected.

Example 5.59. The set $\{0, 1\}$ consisting of two points is disconnected. For example, let $U = (-1/2, 1/2)$ and $V = (1/2, 3/2)$. Then U, V are open and $U \cap V = \varnothing$. Furthermore, $A \cap U = \{0\}$ and $A \cap V = \{1\}$ are nonempty, and $A = (A \cap U) \cup (A \cap V)$. Similarly, the union of half-open intervals $[0, 1/2) \cup (1/2, 1]$ is disconnected.

Example 5.60. The set $\mathbb{R} \setminus \{0\}$ is disconnected since $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

Example 5.61. The set $\mathbb{Q}$ of rational numbers is disconnected. For example, let $U = (-\infty, \sqrt{2})$ and $V = (\sqrt{2}, \infty)$. Then U, V are disjoint open sets, $\mathbb{Q} \cap U$ and $\mathbb{Q} \cap V$ are nonempty, and $U \cup V = \mathbb{R} \setminus \{\sqrt{2}\} \supset \mathbb{Q}$.

In general, it is harder to prove that a set is connected than disconnected, because one has to show that there is no way to separate it by open sets. However, the ordering properties of $\mathbb{R}$ enable us to characterize its connected sets: they are exactly the intervals.

First, we give a precise definition of an interval.

Definition 5.62. A set of real numbers $I \subset \mathbb{R}$ is an interval if $x, y \in I$ and $x < y$ implies that $z \in I$ for every $x < z < y$.

That is, an interval is a set with the property that it contains all the points between any two points in the set.

We claim that, according to this definition, an interval contains all the points that lie between its infimum and supremum. The infimum and supremum may be finite or infinite, and they may or may not belong to the interval. Depending on which of these these possibilities occur, we see that an interval is any open, closed, half-open, bounded, or unbounded interval of the form

$$\varnothing, \quad (a,b), \quad [a,b], \quad [a,b), \quad (a,b], \quad (a,\infty), \quad [a,\infty), \quad (-\infty,b), \quad (-\infty,b], \quad \mathbb{R},$$

where $a, b \in \mathbb{R}$ and $a \leq b$. If $a = b$, then $[a,a] = \{a\}$ is an interval that consists of a single point, which — like the empty set — satisfies the definition vacuously. Thus, Definition 5.62 is consistent with the usual definition of an interval.

To prove the previous claim, suppose that I is an interval and let $a = \inf I$, $b = \sup I$ where $-\infty \leq a, b \leq \infty$. If $a > b$, then $I = \varnothing$, and if $a = b$, then I consists of a single point $\{a\}$. Otherwise, $-\infty \leq a < b \leq \infty$. In that case, the definition of the infimum and supremum implies that for every $a', b' \in \mathbb{R}$ with $a < a' < b' < b$, there exist $x, y \in I$ such that $a \leq x < a'$ and $b' < y \leq b$. Since I is an interval, it follows that $I \supset [x,y] \supset [a',b']$, and since $a' > a$, $b' < b$ are arbitrary, it follows that $I \supset (a,b)$. Moreover, since $a = \inf I$ and $b = \sup I$, the interval I cannot contain any points $x \in \mathbb{R}$ such that $x < a$ or $x > b$.

The slightly tricky part of the following theorem is the proof that every interval is connected.

Theorem 5.63. A set of real numbers is connected if and only if it is an interval.

Proof. First, suppose that $A \subset \mathbb{R}$ is not an interval. Then there are $a, b \in A$ and $c \notin A$ such that $a < c < b$. If $U = (-\infty, c)$ and $V = (c, \infty)$, then $a \in A \cap U$, $b \in A \cap V$, and $A = (A \cap U) \cup (A \cap V)$, so A is disconnected. It follows that every connected set is an interval.

To prove the converse, suppose that $I \subset \mathbb{R}$ is not connected. We will show that I is not an interval. Let U, V be open sets that separate I. Choose $a \in I \cap U$ and $b \in I \cap V$, where we can assume without loss of generality that $a < b$. Let

$$c = \sup\left(U \cap [a,b]\right).$$

We will prove that $a < c < b$ and $c \notin I$, meaning that I is not an interval. If $a \leq x < b$ and $x \in U$, then $U \supset [x, x+\delta)$ for some $\delta > 0$, so $x \neq \sup\left(U \cap [a,b]\right)$. Thus, $c \neq a$ and if $a < c < b$, then $c \notin U$. If $a < y \leq b$ and $y \in V$, then $V \supset (y - \delta, y]$ for some $\delta > 0$, and therefore $(y-\delta, y]$ is disjoint from U, which implies that $y \neq \sup\left(U \cap [a,b]\right)$. It follows that $c \neq b$ and $c \notin U \cap V$, so $a < c < b$ and $c \notin I$, which completes the proof. □

Figure 1. An illustration of the removal of middle-thirds from an interval in the construction of the Cantor set. The figure shows the interval $[0, 1]$ and the first four sets F_1, F_2, F_3, F_4, going from top to bottom.

5.5. * The Cantor set

One of the most interesting examples of a compact set is the Cantor set, which is obtained by "removing middle-thirds" from closed intervals in $[0, 1]$, as illustrated in Figure 1.

We define a nested sequence (F_n) of sets $F_n \subset [0, 1]$ as follows. First, we remove the middle-third from $[0, 1]$ to get $F_1 = [0, 1] \setminus (1/3, 2/3)$, or

$$F_1 = I_0 \cup I_1, \qquad I_0 = \left[0, \frac{1}{3}\right], \quad I_1 = \left[\frac{2}{3}, 1\right].$$

Next, we remove middle-thirds from I_0 and I_1, which splits $I_0 \setminus (1/9, 2/9)$ into $I_{00} \cup I_{01}$ and $I_1 \setminus (7/9, 8/9)$ into $I_{10} \cup I_{11}$, to get

$$\begin{aligned} &F_2 = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}, \\ &I_{00} = \left[0, \frac{1}{9}\right], \quad I_{01} = \left[\frac{2}{9}, \frac{1}{3}\right], \qquad I_{10} = \left[\frac{2}{3}, \frac{7}{9}\right], \quad I_{11} = \left[\frac{8}{9}, 1\right]. \end{aligned}$$

Then we remove middle-thirds from I_{00}, I_{01}, I_{10}, and I_{11} to get

$$\begin{aligned} &F_3 = I_{000} \cup I_{001} \cup I_{010} \cup I_{011} \cup I_{100} \cup I_{101} \cup I_{110} \cup I_{111}, \\ &I_{000} = \left[0, \frac{1}{27}\right], \quad I_{010} = \left[\frac{2}{27}, \frac{1}{9}\right], \qquad I_{010} = \left[\frac{2}{9}, \frac{7}{27}\right], \quad I_{011} = \left[\frac{8}{27}, \frac{1}{3}\right], \\ &I_{100} = \left[\frac{2}{3}, \frac{19}{27}\right], \quad I_{101} = \left[\frac{20}{27}, \frac{7}{9}\right], \qquad I_{110} = \left[\frac{8}{9}, \frac{25}{27}\right], \quad I_{111} = \left[\frac{26}{27}, 1\right]. \end{aligned}$$

Continuing in this way, we get at the nth stage a set of the form

$$F_n = \bigcup_{\mathbf{s}\in\Sigma_n} I_{\mathbf{s}},$$

where $\Sigma_n = \{(s_1, s_2, \dots, s_n) : s_n = 0, 1\}$ is the set of binary n-tuples. Furthermore, each $I_{\mathbf{s}} = [a_{\mathbf{s}}, b_{\mathbf{s}}]$ is a closed interval, and if $\mathbf{s} = (s_1, s_2, \dots, s_n)$, then

$$a_{\mathbf{s}} = \sum_{k=1}^{n} \frac{2s_k}{3^k}, \qquad b_{\mathbf{s}} = a_{\mathbf{s}} + \frac{1}{3^n}.$$

In other words, the left endpoints $a_{\mathbf{s}}$ are the points in $[0, 1]$ that have a finite base three expansion consisting entirely of 0's and 2's.

We can verify this formula for the endpoints $a_{\mathbf{s}}$, $b_{\mathbf{s}}$ of the intervals in F_n by induction. It holds when $n = 1$. Assume that it holds for some $n \in \mathbb{N}$. If we remove the middle-third of length $1/3^{n+1}$ from the interval $[a_{\mathbf{s}}, b_{\mathbf{s}}]$ of length $1/3^n$ with $\mathbf{s} \in \Sigma_n$, then we get the original left endpoint, which may be written as

$$a_{\mathbf{s}'} = \sum_{k=1}^{n+1} \frac{2s'_k}{3^k},$$

where $\mathbf{s}' = (s'_1, \dots, s'_{n+1}) \in \Sigma_{n+1}$ is given by $s'_k = s_k$ for $k = 1, \dots, n$ and $s'_{n+1} = 0$. We also get a new left endpoint $a_{\mathbf{s}'} = a_{\mathbf{s}} + 2/3^{n+1}$, which may be written as

$$a_{\mathbf{s}'} = \sum_{k=1}^{n+1} \frac{2s'_k}{3^k},$$

where $s'_k = s_k$ for $k = 1, \dots, n$ and $s'_{n+1} = 1$. Moreover, $b_{\mathbf{s}'} = a_{\mathbf{s}'} + 1/3^{n+1}$, which proves that the formula for the endpoints holds for $n + 1$.

Definition 5.64. The Cantor set C is the intersection

$$C = \bigcap_{n=1}^{\infty} F_n,$$

of the nested sequence of sets (F_n) defined above.

The compactness of C follows immediately from its definition.

Theorem 5.65. The Cantor set is compact.

Proof. The Cantor set C is bounded, since it is a subset of $[0, 1]$. All the sets F_n are closed, because they are a finite union of closed intervals, so, from Proposition 5.20, their intersection is closed. It follows that C is closed and bounded, and Theorem 5.36 implies that it is compact. □

The Cantor set C is clearly nonempty since the endpoints $a_{\mathbf{s}}$, $b_{\mathbf{s}}$ of $I_{\mathbf{s}}$ are contained in F_n for every finite binary sequence $\mathbf{s}$ and every $n \in \mathbb{N}$. These endpoints form a countably infinite set. What may be initially surprising is that there are uncountably many other points in C that are not endpoints. For example, $1/4$ has the infinite base three expansion $1/4 = 0.020202\dots$, so it is not one of the endpoints, but, as we will show, it belongs to C because it has a base three expansion consisting entirely of 0's and 2's.

Let Σ be the set of binary sequences in Definition 1.29. The idea of the next theorem is that each binary sequence picks out a unique point of the Cantor set by telling us whether to choose the left or the right interval at each stage of the "middle-thirds" construction. For example, $1/4$ corresponds to the sequence $(0, 1, 0, 1, 0, 1, \dots)$, and we get it by alternately choosing left and right intervals.

Theorem 5.66. The Cantor set has the same cardinality as Σ.

Proof. We use the same notation as above. Let $\mathbf{s} = (s_1, s_2, \dots, s_k, \dots) \in \Sigma$, and define $\mathbf{s}_n = (s_1, s_2, \dots, s_n) \in \Sigma_n$. Then $(I_{\mathbf{s}_n})$ is a nested sequence of intervals such that $\operatorname{diam} I_{\mathbf{s}_n} = 1/3^n \to 0$ as $n \to \infty$. Since each $I_{\mathbf{s}_n}$ is a compact interval, Theorem 5.43 implies that there is a unique point

$$x \in \bigcap_{n=1}^{\infty} I_{\mathbf{s}_n} \subset C.$$

Thus, $\mathbf{s} \mapsto x$ defines a function $f : \Sigma \to C$. Furthermore, this function is one-to-one: if two sequences differ in the nth place, say, then the corresponding points in C belong to different intervals $I_{\mathbf{s}_n}$ at the nth stage of the construction, and therefore the points are different since the intervals are disjoint.

Conversely, if $x \in C$, then $x \in F_n$ for every $n \in \mathbb{N}$ and there is a unique $\mathbf{s}_n \in \Sigma_n$ such that $x \in I_{\mathbf{s}_n}$. The intervals $(I_{\mathbf{s}_n})$ are nested, so there is a unique sequence $\mathbf{s} = (s_1, s_2, \dots, s_k, \dots) \in \Sigma$, such that $\mathbf{s}_n = (s_1, s_2, \dots, s_n)$. It follows that $f : \Sigma \to C$ is onto, which proves the result. $\square$

The argument also shows that $x \in C$ if and only if it is a limit of left endpoints $a_{\mathbf{s}}$, meaning that

$$x = \sum_{k=1}^{\infty} \frac{2s_k}{3^k}, \qquad s_k = 0, 1.$$

In other words, $x \in C$ if and only if it has a base 3 expansion consisting entirely of 0's and 2's. Note that this condition does not exclude 1, which corresponds to the sequence $(1, 1, 1, 1, \dots)$ or "always pick the right interval," and

$$1 = 0.2222\cdots = \sum_{k=1}^{\infty} \frac{2}{3^k}.$$

We may use Theorem 5.66, together with the Schröder-Bernstein theorem, to prove that Σ, $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same uncountable cardinality of the continuum. It follows, in particular, that the Cantor set has the same cardinality as $\mathbb{R}$, even though it appears, at first sight, to be a very sparse subset.

Theorem 5.67. The set $\mathbb{R}$ of real numbers has the same cardinality as $\mathcal{P}(\mathbb{N})$.

Proof. The inclusion map $f : C \to \mathbb{R}$, where $f(x) = x$, is one-to-one, so $C \lesssim \mathbb{R}$. From Theorem 5.66 and Corollary 1.48, we have $C \approx \Sigma \approx \mathcal{P}(\mathbb{N})$, so $\mathcal{P}(\mathbb{N}) \lesssim \mathbb{R}$.

Conversely, the map from real numbers to their Dedekind cuts, given by

$$g : \mathbb{R} \to \mathcal{P}(\mathbb{Q}), \qquad g : x \mapsto \{r \in \mathbb{Q} : r < x\},$$

is one-to-one, so $\mathbb{R} \lesssim \mathcal{P}(\mathbb{Q})$. Since $\mathbb{Q}$ is countably infinite, $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q})$, so $\mathbb{R} \lesssim \mathcal{P}(\mathbb{N})$. The conclusion then follows from Theorem 1.40. $\square$

Another proof of this theorem, which doesn't require the Schröder-Bernstein theorem, can be given by associating binary sequences in Σ with binary expansions of real numbers in $[0, 1]$:

$$h : (s_1, s_2, \dots, s_k, \dots) \mapsto \sum_{k=1}^{\infty} \frac{s_k}{2^k}.$$

Some real numbers, however, have two distinct binary expansion; e.g.,

$$\frac{1}{2} = 0.10000\dots = 0.01111\dots.$$

There are only countably many such numbers, so they do not affect the cardinality of $[0, 1]$, but they complicate the explicit construction of a one-to-one, onto map $f : \Sigma \to \mathbb{R}$ by this approach. An alternative method is to represent real numbers by continued fractions instead of binary expansions, but we won't describe these proofs in more detail here.