

An Introduction to Real Analysis

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ABSTRACT. These are some notes on introductory real analysis. They cover the properties of the real numbers, sequences and series of real numbers, limits of functions, continuity, differentiability, sequences and series of functions, and Riemann integration. They don't include multi-variable calculus or contain any problem sets. Optional sections are starred.

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