1.3.1

- Applying Newton’s Law $F = ma$ in both $x$- and $y$-directions. Refer to class notes for assumptions.

- For $x$-direction, no motion implies no resistance force. We still have as in class

$$T(b, t) \cos(\theta(b, t)) - T(a, t) \cos(\theta(a, t)) = 0$$

$$\implies T(b, t) - T(a, t) = 0$$

$$\implies T = T_0 = \text{constant}.$$  

- For $y$-direction we have

$$T(b, t) \sin(\theta(b, t)) - T(a, t) \sin(\theta(a, t)) - \int_a^b \alpha u_t \, dx = \int_a^b \rho_0 u_{tt} \, dx$$

$$T_0 (u_x(b, t) - u_x(a, t)) - \int_a^b \alpha u_t \, dx = \int_a^b \rho_0 u_{tt} \, dx$$

$$\int_a^b T_0 u_{xx}(x, t) \, dx - \int_a^b \alpha u_t \, dx = \int_a^b \rho_0 u_{tt} \, dx$$

$$\int_a^b T_0 u_{xx}(x, t) - \alpha u_t - \rho_0 u_{tt} \, dx = 0,$$

for arbitrary points $a, b$. Thus

$$T_0 u_{xx}(x, t) - \alpha u_t - \rho_0 u_{tt} = 0,$$

or

$$u_{tt} - c^2 u_{xx} + ru_t = 0,$$

where $c^2 = \frac{T_0}{\rho_0}, r = \frac{\alpha}{\rho_0} > 0$.

- Resistance force is again the motion, so $r > 0$.

1.5.1

- Characteristic equation is $r^2 + 1 = 0$ which has roots $r = \pm i$.

- General solution has the form $u(x) = A \cos x + B \sin x$.

- Applying boundary condition $u(0) = 0$ leads to $A = 0$, and $u(x) = B \sin x$.

- Another boundary condition $u(L) = 0$ implies $B \sin L = 0$. There are two possibilities:
  - if $L \neq k\pi$ for $k \in \mathbb{Z}$, $B$ has to be zero, and $u(x) = 0$ is the unique solution.
  - if $L = k\pi$ for $k \in \mathbb{Z}$, then $u(x) = B \sin L$ are all solutions.
2.1.1

- The general solution is
  \[ u(x,t) = f(x - ct) + g(x - ct). \]
- Applying the initial conditions lead to
  \[ f(x) + g(x) = e^x, \quad \text{and} \quad -cf'(x) + cg'(x) = \sin x. \]
- Solving for \( f \) and \( g \), we obtain
  \[ f(x) = \frac{1}{2}(e^x + \frac{1}{c} \cos x), \quad g(x) = \frac{1}{2}(e^x - \frac{1}{c} \cos x). \]
- Solution of the IVP is
  \[ u(x,t) = \frac{1}{2}e^x(e^{-ct} + e^{ct}) + \frac{1}{2c}(\cos(x - ct) - \cos(x + ct)) \]
  \[ = e^x \cosh(ct) + \frac{1}{c} \sin(x) \sin(ct). \]

2.1.5

- At \( t = a/2c \),
  \[ u(x,a/2c) = \begin{cases} 0 & \text{if } |x| > \frac{3a}{2} \\ \frac{1}{2c}(\frac{3a}{2} - |x|) & \text{if } \frac{a}{2} \leq |x| \leq \frac{3a}{2} \\ \frac{a}{2c} & \text{if } |x| < \frac{a}{2} \end{cases} \]
- At \( t = a/c \),
  \[ u(x,a/c) = \begin{cases} 0 & \text{if } |x| > 2a \\ \frac{1}{2c}(2a - |x|) & \text{if } |x| \leq 2a \end{cases} \]
- At \( t = 3a/2c \),
  \[ u(x,3a/2c) = \begin{cases} 0 & \text{if } |x| > \frac{5a}{2} \\ \frac{1}{2c}(\frac{5a}{2} - |x|) & \text{if } \frac{a}{2} \leq |x| \leq \frac{5a}{2} \\ \frac{a}{c} & \text{if } |x| < \frac{a}{2} \end{cases} \]
- At \( t = 2a/c \),
  \[ u(x,2a/c) = \begin{cases} 0 & \text{if } |x| > 3a \\ \frac{1}{2c}(3a - |x|) & \text{if } a \leq |x| \leq 3a \\ \frac{a}{c} & \text{if } |x| < a \end{cases} \]
- At \( t = 5a/c \),
  \[ u(x,5a/c) = \begin{cases} 0 & \text{if } |x| > 6a \\ \frac{1}{2c}(6a - |x|) & \text{if } 4a \leq |x| \leq 6a \\ \frac{a}{c} & \text{if } |x| < 4a \end{cases} \]
• Note the pattern. The graph changes slopes at points \( x = \pm (a \pm ct) \). Except the horizontal lines, all slopes have magnitude \( \frac{1}{2c} \).

2.1.6

\[
\max_x u(x, t) = \begin{cases} 
  a/c & \text{for } t \geq a/c \\
  t & \text{for } 0 \leq t < a/c 
\end{cases}
\]

2.2.2

• For part (a), differentiating \( e \) and \( p \) with respect to \( x \) and \( t \), respectively. Then use the wave equation \( u_{tt} - u_{xx} = 0 \) to show.

• For part (b), use the assumptions that the second-partial derivatives are continuous, i.e, \( e_{xt} = e_{tx}, \ p_{xt} = p_{tx} \).