Math 118: PDE

HW 3 Solutions

1.3.3

• By the law of conservation of energy, we have

\[
\text{rate of change of thermal energy} = \text{heat flux in} - \text{heat flux out}. \quad (1)
\]

• Let \( u(x, t) \) denote the temperature of the rod at position \( x \) and time \( t \). The thermal energy density \( e(x, t) \) per volume is given by \( c \rho u(x, t) \), where \( c \) is the specific heat capacity of the rod, and \( \rho \) the density.

• The heat flux \( q(x, t) \) per cross sectional area is proportional to the negative gradient of temperature, i.e. \( q(x, t) = -ku_x \) for some \( k > 0 \).

• The additional heat loss to the outside through the lateral sides of the rod is given by \( h(x, t)Pdx \) where \( h(x, t) = \mu[u(x, t) - T_0] \) for some \( \mu > 0 \), where \( T_0 \) is the ambient temperature, based on Newton’s law of cooling.

• By (1), we have

\[
\frac{d}{dt}\int_a^b e(x, t)A \, dx = \left[q(a, t)A - q(b, t)A\right] - \int_a^b h(x, t)P \, dx, \quad (2)
\]

which can be simplified to

\[
\int_a^b c \rho u_t \, dx - Akx + \mu P[u - T_0] \, dx = 0. \quad (3)
\]

• Since \( a, b \) are arbitrary, it follows that

\[
u_t = \frac{k}{c \rho}u_{xx} - \frac{\mu P}{c \rho A}[u - T_0]. \quad (4)
\]

1.3.5

• By the conservation of mass, we have

\[
\text{the rate of change in the fluid mass} = \text{change in flux based on diffusion (i.e., flow in - flow out)} + \text{change in flux based on advection (i.e., move in - move out)}. \quad (5)
\]
• Let $u(x, t)$ denote the mass of fluid particles at position $x$ and time $t$. Then (5) gives
\[
\frac{d}{dt} \int_a^b u(x, t) \, dx = -\int_a^b q_x(x, t) \, dx - \int_a^b V u_x(x, t) \, dx,
\] (6)

• Since $q = -k u_t$ and $a, b$ are chosen arbitrarily, we have
\[
u_t = k u_{xx} - V u_x.
\] (7)

2.2.6

• (a) Substitute the following derivative to the PDE
\[
u_t = \alpha f'(t - \beta)
\]
\[
u_{tt} = \alpha f''(t - \beta)
\]
\[
u_r = \alpha' f(t - \beta) - \alpha \beta f'(t - \beta)
\]
\[
u_{rr} = \alpha'' f(t - \beta) - 2 \alpha' \beta f'(t - \beta) - \alpha \beta'' f'(t - \beta) + \alpha(\beta')^2 f''(t - \beta)
\]
we get
\[
c^2(\alpha'' + \frac{n-1}{r} \alpha') f - c^2(2 \alpha' \beta' + \alpha \beta'' + \frac{n-1}{r} \alpha \beta') f' + (c^2 \alpha (\beta')^2 - \alpha) f'' = 0
\] (8)

• (b) Setting the coefficients of $f''$, $f'$, and $f$ equal to zero, we obtain
\[
c^2(\alpha'' + \frac{n-1}{r} \alpha') = 0
\] (9)
\[
c^2(2 \alpha' \beta' + \alpha \beta'' + \frac{n-1}{r} \alpha \beta') = 0
\] (10)
\[
c^2 \alpha (\beta')^2 - \alpha = 0
\] (11)

• (c) Suppose that $\alpha \neq 0$ and $c \neq 0$, then (11) gives $\beta' = \pm 1/c$ and thus $\beta'' = 0$. Plug these results to (20), we obtain
\[
2 \alpha' + \frac{n-1}{r} \alpha = 0
\] (12)

• The equation (9) gives
\[
\alpha'' + \frac{n-1}{r} \alpha' = 0.
\] (13)

Solving this ODE, we obtain
\[
\alpha' = r^{1-n}, \quad \alpha = \frac{1}{2-n} r^{2-n}
\]
Plugging them into (12), we get
\[ r^{1-n} \left( 2 + \frac{n-1}{2-n} \right) = 0. \] (14)

It follows that \( n = 1 \) or \( n = 3 \).

• (d) If \( n = 1 \), \( \alpha(r) = r^0 = 1 \) is a constant.

2.3.1
- Maximum Principle tells that the max or min of \( u(x, t) \) occurs on the boundaries, i.e., \( t = 0, T, x = 0, 1 \).

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( u(x, 0) = 1 - x^2 )</th>
<th>max</th>
<th>( u(x, 0) ) at ( x = 0 )</th>
<th>min</th>
<th>( u(x, 0) ) at ( x = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( u(0, t) = 1 - 2kt )</td>
<td>max</td>
<td>( u(0, t) ) at ( t = 0 )</td>
<td>min</td>
<td>( u(0, t) ) at ( t = T )</td>
</tr>
<tr>
<td>( x = 1 )</td>
<td>( u(1, t) = -2kt )</td>
<td>min</td>
<td>( u(1, t) ) at ( t = 0 )</td>
<td>max</td>
<td>( u(1, t) ) at ( t = T )</td>
</tr>
</tbody>
</table>

• Thus, the global max of \( u(x, t) \) is 1 at \((0, 0)\), and the global min is \(-2kT \) at \((1, T)\).

2.4.6
- Let \( I = \int_0^\infty e^{-x^2} \, dx \), then
\[
I^2 = \int_0^\infty e^{-x^2} \, dx \int_0^\infty e^{-y^2} \, dy
= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dx \, dy
= \int_0^{\pi/4} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta
= \pi/4.
\]

Thus, \( I = \sqrt{\pi}/2 \).

2.4.7
- \( \int_{-\infty}^\infty e^{-p^2} \, dp = 2I = \sqrt{\pi} \).
- Let \( p = x/\sqrt{4kt} \), then \( dp = dx/\sqrt{4kt} \), and
\[
\int_{-\infty}^\infty S(x, t) \, dx = \int_{-\infty}^\infty \frac{e^{-p^2}}{\sqrt{\pi}} \, dp = 1.
\]
2.4.9

- Differentiating both sides of the diffusion equation thrice with respect to x, we have
  \[(u_{xxx})_t = k(u_{xxx})_{xx},\]  
  due to the continuity of partial derivatives.

- Differentiating \(u(x,0) = x^2\) thrice with respect to x, we have the initial condition
  \[u_{xxx}(x,0) = 0.\]  

- By the uniqueness of solutions, \(u_{xxx} = 0\) is the solution of the IVP \((15)\) and \((16)\).

- Integrating the result thrice,
  \[u(x,t) = A(t)x^2 + B(t)x + C(t).\]  

- The initial condition \(u(x,0) = x^2\) implies
  \[A(0) = 1, \quad B(0) = C(0) = 0.\]

- Differentiating \((17)\) with respect to t,
  \[u_t = A'x^2 + B'x + C'.\]

- Differentiating \((17)\) with respect to x twice,
  \[u_{xx} = 2A.\]

- Plugging \((18)\) and \((19)\) into the original diffusion equation, we obtain
  \[A'(t) = B'(t) = 0, \quad C'(t) = 2kA(t).\]

- It follows that \(A = 1, \quad B = 0, \quad C = 2k,\) and the solution of the original problem is
  \[u(x,t) = x^2 + 2kt.\]