Homework 5 Solutions

2.3.2 (a) increases or stays the same. (b) decreases or stays the same.

- Consider the two rectangles $R_1 := \{0 \leq x \leq l, \ 0 \leq t \leq T_1\}$ and $R_2 := \{0 \leq x \leq l, \ 0 \leq t \leq T_2\}$ with $T_1 < T_2$. We have $R_1 \subset R_2$. Thus, $M(T_1) \leq M(T_2)$ and $m(T_1) \geq m(T_2)$.

2.3.5

- Given $u = -2xt - x^2$, we have $u_t = -2x = xu_{xx}$, so it is a solution.
- On $x = -2$: $u = 4t - 4$. The max occurs at $t = 1$ with $u(-2,1) = 0$.
- On $x = 2$: $u = -4t - 4$. The max occurs at $t = 0$ with $u(0,0) = -4$.
- On $t = 0$: $u = -x^2$. The max occurs at $x = 0$ with $u(0,0) = 0$.
- On $t = 1$: $u = -2x - x^2 = 1 - (x + 1)^2$. The max occurs at $x = -1$ with $u(-1,1) = 1$.
- The maximum is assumed on the top instead of on the bottom or the lateral sides, which contradicts the Maximum Principle.

2.3.6

- Consider $\omega = u - v$, then $\omega \leq 0$ for $t = 0, x = 0$ and $x = l$.
- By the Maximum Principle, the maximum of $\omega(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$). Denote the maximum by $\omega_{\text{max}}$. We then have $\omega_{\text{max}} \leq 0$.
- It follows that $\omega(x, t) \leq \omega_{\text{max}} \leq 0$, i.e., $u \leq v$, for $0 \leq t < \infty, 0 \leq x \leq l$.

2.3.8

- Multiplying the diffusion equation by $u$, we obtain
  
  $uu_t = ku_{xx}u \iff \frac{1}{2}u^2_t = (kuu_x)_x - ku^2_x$.

  - Upon integrating over the interval $0 < x < l$, we get
  
    $\frac{1}{2} \left( \frac{d}{dt} \int_0^l u^2 \, dx \right) = \int_0^l \frac{1}{2}u^2_t \, dx = (kuu_x)_{x=x=l}^x_{x=x=0} - k \int_0^l u^2_x \, dx$.

  - Plugging in the Robin boundary conditions, we have
  
    $\frac{1}{2} \left( \frac{d}{dt} \int_0^l u^2 \, dx \right) = -a_t u^2(l, t) - a_0 u^2(0, t) - k \int_0^l u^2_x \, dx$.

    - Since $a_0 > 0$ and $a_t > 0$, the endpoints contribute to the decrease of $\int_0^l u^2(x, t) \, dx$.
    - It’s easy to verify that the right hand side is indeed negative.

2.5.1

- Consider the “hammer blow” example, problem 4 and 5 from section 2.4, as a counter example.