4.1.1

- A violin string is modeled exactly by the problem (1), (2), and (3).
- Its general solution is given as a Fourier expansion in (9).
- We find that the frequencies (or note produced by the violin string) are
  \[ \frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \]
  for \( n = 1, 2, 3, \ldots \),
  which depends on length of string \( l \) and tension force \( T \).
- The frequencies double when \( l \) is decreased by half.
- The frequencies are also proportional to \( \sqrt{T} \). Thus, the frequencies increase, i.e., the note rises, when \( T \) increases, i.e., when the string is tightened.

4.1.2

- The problem is given by
  \[
  \begin{align*}
  \text{DE} & : u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty) \\
  \text{BC} & : u(0, t) = u(l, t) = 0 \\
  \text{IC} & : u(x, 0) = 1.
  \end{align*}
  \]
- We have
  \[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2kt} \sin \frac{n\pi x}{l} \]
  is the solution provided that
  \[ 1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \]
• Assuming the infinite series expansion

\[ 1 = \frac{4}{\pi} \left( \sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \frac{1}{5} \sin \frac{5\pi x}{l} + \cdots \right), \]

we obtain

\[ A_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \]

• The formula for the temperature \( u(x, t) \) at later times is

\[ u(x, t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-\frac{(n\pi/l)^2}{2}kt} \sin \frac{n\pi x}{l} \]

4.1.3

• Plugging the form \( u(x, t) = X(x)T(t) \) into the Schrodinger’s equation, we get

\[ X(x)T'(t) = iX''(x)T(t), \]

or dividing by \( iXT \),

\[ \frac{T'}{iT} = \frac{X''}{X} = -\lambda. \]

• The quantity \( \lambda \) must be a positive constant. (This will be shown at the end of the section.) We get a pair of separate ODE:

\[ X'' + \lambda X = 0, \quad T' + i\lambda T = 0. \]

• Using (8), we see that

\[ \lambda_n = \left( \frac{n\pi}{l} \right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l} \quad (n = 1, 2, 3, \ldots) \]

are distinct solutions that satisfy the Dirichlet conditions.

• Solving for \( T \), we obtain

\[ T_n(t) = e^{-i\lambda_n t}. \]

• The general solution is given by

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-i(n\pi/l)^2 t}. \]
Plugging the form \( u(x,t) = X(x)T(t) \) into the equation, we get

\[
 tX(x)T'(t) = X''(x)T(t) + 2X(x)T(t),
\]
or

\[
 tX(x)T''(t) - 2X(x)T(t) = X''(x)T(t).
\]

Dividing by \( XT \), we obtain

\[
 \frac{tT'}{T} - 2 = \frac{X''}{X} = -\lambda.
\]

Using (8) with \( l = \pi \), we see that

\[
 \lambda_n = n^2, \quad X_n(x) = \sin(nx) \quad (n = 1, 2, 3, \ldots)
\]

are distinct solutions that satisfy the given boundary conditions.

Solving for \( T \), we get

\[
 T_n(t) = t^{2-\lambda_n}.
\]

The general solution is

\[
 u(x,t) = \sum_{1}^{\infty} A_n \sin(nx) t^{2-n^2},
\]

which satisfies the initial condition \( u(x,0) = 0 \) for all \( A_n \). Thus, the solution is not unique.

Let \( x = \pi/4 \), we have

\[
 \sin nx = \begin{cases} 
 \sqrt{2}/2 & \text{if } n = 8k + 1 \text{ or } 8k + 3, \\
 -\sqrt{2}/2 & \text{if } n = 8k - 1 \text{ or } 8k - 3, \\
 0 & \text{otherwise.}
\end{cases}
\]

where \( k \) is an integer.
• The expansion \( 1 = \sum_{n \text{ odd}} (4/n\pi) \sin nx \) can be rewritten as
\[
1 = \frac{4\sqrt{2}}{\pi} \left( \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots \right).
\]

• The sum \( 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots \) is thus given by \( \pi\sqrt{2}/4 \).

• Read the hint.

5.1.2(a)

• Its Fourier since series has coefficients
\[
A_m = 2 \int_0^1 x^2 \sin m\pi x \, dx
\]
\[
= -\frac{2x^2}{m\pi} \cos m\pi x \bigg|_0^1 + \frac{4}{m\pi} \int_0^1 x \cos m\pi x \, dx
\]
\[
= -\frac{2x^2}{m\pi} \cos m\pi x \bigg|_0^1 + \frac{4}{m\pi} \left( \frac{x}{m\pi} \sin m\pi x + \frac{1}{m^2\pi^2} \cos m\pi x \right) \bigg|_0^1
\]
\[
= -\frac{2}{m\pi} \cos m\pi + \frac{4}{m^3\pi^3} (\cos m\pi - 1)
\]
\[
= -\frac{2}{m\pi} (-1)^m + \frac{4}{m^3\pi^3}((-1)^m - 1)
\]
\[
= \begin{cases} 
\frac{2}{m\pi} & \text{for } n \text{ even} \\
-\frac{4}{m^3\pi^3} & \text{for } n \text{ odd}
\end{cases}.
\]

5.1.3(a)
5.1.8

- The problem is modeled by
  
  \[
  \text{DE} : u_t = ku_{xx} \quad (0 < x < 1, 0 < t < \infty)
  \]
  \[
  \text{BC} : u(0, t) = 0, \quad u(1, t) = 0
  \]
  \[
  \text{IC} : u(x, 0) = \phi(x),
  \]

  where

  \[
  \phi(x) = \begin{cases} 
  \frac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\
  3 - 2x & \text{for } \frac{2}{3} < x < 1.
  \end{cases}
  \]

- The equilibrium solution of this problem is given by \( U(x) = x \).

- Let \( v(x, t) = u(x, t) - x \), then \( v(x, t) \) is the solution of the problem
  
  \[
  \text{DE} : v_t = kv_{xx} \quad (0 < x < 1, 0 < t < \infty)
  \]
  \[
  \text{BC} : v(0, t) = v(1, t) = 0
  \]
  \[
  \text{IC} : v(x, 0) = \phi(x) - x = \psi(x),
  \]
where
\[
\psi(x) = \begin{cases} 
3x/2 & \text{for } 0 < x < 2/3 \\
3 - 3x & \text{for } 2/3 < x < 1.
\end{cases}
\]

• We have
\[
v(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2kt} \sin \frac{n\pi x}{l}
\]
is the solution provided that
\[
\psi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.
\]

• Using the Fourier sine series for 1 and \(x\) in the interval \((0, l)\) on Page 104-105, we can simply get
\[
A_n = \begin{cases} 
(-1)^{m+1} \frac{3l}{m\pi} & \text{for } 0 < x < 2/3 \\
\frac{6}{m\pi} \left[1 - (-1)^m - (-1)^{m+1}l\right] & \text{for } 2/3 < x < 1.
\end{cases}
\]