Advanced Analysis

Math 121, Fall 2004

Final

NAME..

I.D. NUMBER..

No books, notes, or calculators. Show all your work.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	25	
6	25	
7	20	
8	25	
9	25	
Total	200	

1. [20 pts.] Use Laplace transforms to solve the following initial value problem for $y(t)$,

$$\begin{aligned} &y'' - 6y' + 9y = te^{3t}, \\ &y(0) = 1, \quad y'(0) = 3. \end{aligned}$$

2. [20 pts.] (a) Find the Green's function $G(t)$ that satisfies

$$\begin{aligned} &G'' + \omega^2 G = \delta(t), \\ &G(t) = 0 \qquad t < 0, \end{aligned}$$

where ω is a nonzero constant. (You can use any method you like to find G.)
(b) Express the solution $y(t)$ of

$$\begin{aligned} &y'' + \omega^2 y = f(t), \\ &y(0) = y'(0) = 0, \end{aligned}$$

where $f(t)$ is an arbitrary function in terms of $G(t)$ and $f(t)$. (You don't have to justify your answer.)

3. [20 pts.] Compute the coefficients c_0, c_1, c_2, c_3 in the expansion

$$|x| = \sum_{n=0}^{\infty} c_n P_n(x) \qquad -1 \le x \le 1.$$

of $|x|$ with respect to the Legendre polynomials $P_n(x)$ on the interval $[-1, 1]$.
HINT. You can use the formulas:

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 2/(2n+1)) & \text{for } n = m, \\ 0 & \text{for } n \neq m, \end{cases}$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

4. [20 pts.] If $f(x)$ is a 2π periodic function with Fourier series expansion $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, we define the Hilbert transform $g(x)$ of $f(x)$ by

$$g(x) = -i \sum_{n=-\infty}^{-1} c_n e^{inx} + i \sum_{n=1}^{\infty} c_n e^{inx}.$$

(That is, we get $g(x)$ by multipling the Fourier coefficient c_n of $f(x)$ by $-i$ if $n < 0$, by 0 if $n = 0$, and by i if $n > 0$.)

(a) Compute the Hilbert transform $g(x)$ of $f(x) = \cos(Nx)$, where $N > 0$ is a positive integer.

(b) Show that if $g(x)$ is the Hilbert transform of $f(x)$ and $\int_0^{2\pi} f(x)\,dx = 0$, then

$$\int_0^{2\pi} f^2(x)\,dx = \int_0^{2\pi} g^2(x)\,dx.$$

5. [25 pts.] Consider the function $f(x) = e^x$ for $0 < x < 1$.

(a) Sketch the graphs of the even and odd periodic extensions of $f(x)$ (with period 2) for $-3 < x < 3$.

(b) Write out Fourier sine and cosine series that represent $f(x)$ on the interval $0 < x < 1$. Give expressions for the coefficients as integrals, but do not evaluate the integrals.

(c) What do the Fourier cosine and sine series converge to in $-3 < x < 3$? Which one converges faster?

6. [25 pts.] Use Fourier series to find the solution $u(x, y)$ of the following boundary value problem for Laplace's equation in the strip $0 < x < 1$, $y > 0$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(1, y) = 0,$$
$$u(x, 0) = 1 - x, \quad u(x, y) \text{ is bounded as } y \to \infty.$$

What does the solution approach as $y \to \infty$?

7. [20 pts.] Use Fourier transforms to find the solution $u(x,t)$ of the the following initial value problem for the heat equation in $-\infty < x < \infty$, $t > 0$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \\ u(x,0) = f(x),$$

where $f(x)$ is a given function and α is a non-zero constant. Express your answer as a convolution. (You can use the table of Fourier transforms provided.)

8. [25 pts.] Use Laplace transforms in time to find the solution $u(x,t)$ of the following initial-boundary value problem for the wave equation in $x > 0$, $t > 0$

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}, \\
&u(0,t) = f(t), \\
&u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,
\end{aligned}
$$

where $f(t)$ is a given function defined for $t > 0$, and c is a non-zero constant. Give your answer as explicitly as possible. (You can use the table of Laplace transforms provided.)

9. [25 pts.] Consider the following initial-boundary value problem for the heat equation with a source in $0 < x < 1$, $t > 0$

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + ku, \\
&u(0,t) = u(1,t) = 0, \\
&u(x,0) = f(x),
\end{aligned}$$

where $f(x)$ is a given function and α, k are non-zero constants.

(a) Find the separated solutions of the partial differential equation that staisfy the boundary conditions at $x = 0, 1$.

(b) Use Fourier series to solve the problem.

(c) How does the solution behave as $t \to \infty$? Does this behavior depend on the value of the constant k?