Advanced Analysis
Math 121, Fall 2004
Solutions, Midterm 1

1. Answer the following questions. Give a brief justification of your answer.
(a) What is the (smallest) period of the function $\cos(5\pi x)$?
(b) What is the value of

$$\int_0^{10} \cos^2(5\pi x)\ dx?$$

(c) What is the value of

$$\int_0^{10} \cos(5\pi x)\cos(30\pi x)\ dx?$$

Solution.

- (a) The function $\cos(kx)$ has period $2\pi/k$, so $\cos(5\pi x)$ has period $2/5$.
- (b) The integration interval is an integer multiple of the period, since $10 = 25 \cdot (2/5)$. Therefore, since the average value of $\cos^2 x$ over a period is $1/2$, we have

$$\int_0^{10} \cos^2(5\pi x)\ dx = 10 \cdot \frac{1}{2} = 5.$$

- (c) The functions $\cos(5\pi x)$ and $\cos(30\pi x)$ are orthogonal on any period, so

$$\int_0^{10} \cos(5\pi x)\cos(30\pi x)\ dx = 0.$$

2. Suppose that $f(x)$ is defined by

$$f(x) = x^2 \qquad \text{for } 0 < x < \pi.$$

Sketch graphs of the following extensions of $f(x)$ for $-3\pi < x < 3\pi$:
(a) the periodic extension $f_{\mathrm{p}}(x)$ with period π;
(b) the even periodic extension $f_{\mathrm{e}}(x)$ with period 2π;
(c) the odd periodic extension $f_{\mathrm{o}}(x)$ with period 2π.

Solution.

- Graphs omitted.

3. Suppose that $f(x)$ is the 2π-periodic function defined by

$$f(x) = x \qquad \text{for } -\pi < x < \pi.$$

(a) What are the Fourier cosine coefficients a_n of $f(x)$? Explain how you know the answer without evaluating any integrals.
(b) What does the Fourier series of $f(x)$ converge to at $x = \pi/2$, and $x = \pi$? Why?
(c) Compute the Fourier sine coefficients b_n of $f(x)$, and write out the Fourier series expansion of $f(x)$.

Solution.

- (a) Since f is odd, the Fourier cosine coefficients are zero, so $a_n = 0$ for all $n \geq 0$.

- (b) The function f is piecewise smooth. According to Dirichlet's theorem, its Fourier series converges to $f(x)$ at points x where f is continuous, and to the average of the left and right hand limits of f at points where f has a jump discontinuity. Since $f(x)$ is continuous at $x = \pi/2$ and $f(\pi/2) = \pi/2$, the Fourier series converges to $\pi/2$ at $x = \pi/2$. At $x = \pi$, the function f has a jump discontinuity with left-hand limit equal to π and right-hand limit equal to $-\pi$. Hence the Fourier series converges to 0.

- (c) The Fourier sine coefficients of f are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n}\right] \\ &= \frac{2}{n}(-1)^{n+1} \end{aligned}$$

Thus,

$$f(x) = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)$$

4. (a) State the orthogonality relations for the functions $\sin(n\pi x)$ on the interval $0 < x < 1$, where $n = 1, 2, 3, \ldots$.
(b) Use these orthogonality relations to *derive* an expression for the coefficients b_n in the Fourier sine expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

of a function defined in $0 < x < 1$.

Solution.

- (a) Since $\sin m\pi x$ and $\sin n\pi x$ are orthogonal for $n \neq m$, and the average value of $\sin^2 x$ over a period is $1/2$, the orthogonality relations are
$$\int_0^1 \sin m\pi x \sin n\pi x \, dx = \begin{cases} 0 & \text{for } m \neq n, \\ 1/2 & \text{for } m = n. \end{cases}$$
- (b) Replacing the summation variable n by m, we have
$$f(x) = \sum_{m=1}^{\infty} b_m \sin m\pi x.$$
Multiplying this equation by $\sin n\pi x$, where n is a fixed but arbitrary positive integer, integrating the result over $0 < x < 1$, and using the orthogonality relations, we get
$$\begin{aligned} \int_0^1 f(x) \sin n\pi x \, dx &= \int_0^1 \left(\sum_{m=1}^{\infty} b_m \sin m\pi x \right) \sin n\pi x \, dx \\ &= \sum_{m=1}^{\infty} b_m \left(\int_0^1 \sin m\pi x \sin n\pi x \, dx \right) \\ &= \frac{1}{2} b_n. \end{aligned}$$
Hence,
$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx.$$

5. The 2π-periodic function defined by

$$f(x) = |x| \qquad -\pi < x < \pi$$

has the Fourier series expansion

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \dots\right).$$

Apply Parseval's theorem to this function, and use the result to deduce the sum of the infinite series

$$\sum_{n \text{ odd}} \frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots.$$

Solution.

- According to Parseval's theorem, if

  $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

  then

  $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2.$$

  We have

  $$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^2\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} \\ &= \frac{\pi^2}{3}. \end{aligned}$$

  From the Fourier series expansion of f, we have

  $$a_0 = \pi, \quad a_n = \begin{cases} -4/(\pi n^2) & \text{for } n \geq 1 \text{ odd}, \\ 0 & \text{for } n \geq 2 \text{ even}. \end{cases}$$

It follows that

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^4}.$$

Hence,

$$\sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

6. Suppose that a function $f(x)$ has the real Fourier series expansion

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2} \{\cos nx + n \sin nx\}.$$

(a) Find the complex Fourier series expansion of f, of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

(b) Does this Fourier series converge slowly or rapidly? How smooth do you think $f(x)$ is? That is, how many orders of derivatives of $f(x)$ do you expect to be continuous? Explain your answer briefly. (No proofs required.)

Solution.

- Using Euler's equation, we get

$$\begin{aligned} f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2} \{\cos nx + n \sin nx\} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2} \left\{ \left(\frac{e^{inx} + e^{-inx}}{2} \right) + n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right\} \\ &= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} e^{-n^2} \left\{ (1 - in) e^{inx} + (1 + in) e^{-inx} \right\} \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (1 - in) e^{-n^2} e^{inx}. \end{aligned}$$

  Hence, the complex Fourier coefficients of f are given by

$$c_n = \frac{1}{2} (1 - in) e^{-n^2}. \tag{1}$$

- Alternative proof. We have

$$c_n = \frac{1}{2} (a_n - ib_n) \qquad \text{for } n \geq 0,$$

  where a_n and b_n are the Fourier cosine and sine coefficients of f, respectively.

From the Fourier series, we have

$$a_n = e^{-n^2}, \qquad b_n = n e^{-n^2}.$$

Note that $a_0 = 1$, so the expression for a_n also holds when $n = 0$. Thus,

$$c_n = \frac{1}{2}(1 - in)\, e^{-n^2} \qquad \text{for } n \geq 0.$$

Also,

$$\begin{aligned} c_{-n} &= \overline{c_n} \\ &= \frac{1}{2}(1 + in)\, e^{-n^2}, \end{aligned}$$

so the same formula holds for $n < 0$, and we get (1).

- The Fourier coefficients of f tend to zero exponentially quickly as $n \to \infty$, so the the Fourier series converges very rapidly. Since $n^p e^{-n^2}$ converges to zero rapidly as $n \to \infty$ for every $p = 1, 2, 3 \ldots$, the series obtained from the one for f by term-by-term differentiation converges uniformly, and we expect that f has continuous derivatives of all orders (as, in fact, it does).