Solutions to Sample Midterm 2
Math 121, Fall 2004

1. Use Fourier series to find the solution $u(x, y)$ of the following boundary value problem for Laplace's equation in the semi-infinite strip $0 < x < 1$, $y > 0$:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\ &u(0, y) = u(1, y) = 0, \\ &u(x, 0) = 1, \\ &u(x, y) \to 0 \quad \text{as } y \to \infty. \end{aligned}$$

Solution.

- The separated solutions of Laplace's equation that satisfy the boundary conditions at $x = 0, 1$ and as $y \to \infty$ are $\sin(n\pi x)e^{-n\pi y}$, where n is a positive integer. We therefore look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)e^{-n\pi y}.$$

Imposing the boundary condition at $y = 0$, we obtain

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) = 1,$$

so

$$\begin{aligned} b_n &= 2\int_0^1 \sin(n\pi x)\, dx \\ &= 2\left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1 \\ &= 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi}\right]_0^1 \\ &= \begin{cases} 4/(n\pi) & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

The solution is therefore

$$u(x, y) = \frac{4}{\pi} \left\{ \sin(\pi x) e^{-\pi y} + \frac{1}{3} \sin(3\pi x) e^{-3\pi y} + \frac{1}{5} \sin(5\pi x) e^{-5\pi y} + \dots \right\}.$$

2. Use Fourier series to find the solution $u(x,t)$ of the following initial-boundary value problem for the wave equation in $0 < x < 1$ and $t > 0$:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,\\
&u(x,0) = 0,\\
&\frac{\partial u}{\partial t}(x,0) = x.
\end{aligned}$$

Solution.

- The separated solutions of the wave equation that are zero at $t = 0$ and satisfy the boundary conditions at $x = 0, 1$ are t and $\cos(n\pi x)\sin(n\pi t)$, where $n = 1, 2, \ldots$. We therefore look for a solution of the form

$$u(x,t) = \frac{1}{2}a_0 t + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\sin(n\pi t).$$

Differentiating this series with respect to t, we find that

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} n\pi a_n \cos(n\pi x)\cos(n\pi t).$$

Imposing the initial condition for $\partial u/\partial t$ at $t = 0$, we get the Fourier cosine expansion:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} n\pi a_n \cos(n\pi x) = x.$$

Hence, for $n \geq 1$ we have

$$\begin{aligned}
n\pi a_n &= 2\int_0^1 x\cos(n\pi x)\,dx\\
&= 2\left[\frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}\right]_0^1\\
&= 2\left[\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right]_0^1\\
&= \begin{cases} -4/(n\pi)^2 & \text{for } n \text{ odd,}\\ 0 & \text{for } n \text{ even,}\end{cases}
\end{aligned}$$

and

$$a_n = \begin{cases} -4/(n\pi)^3 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

For $n = 0$, we get

$$\begin{aligned} a_0 &= 2\int_0^1 x\,dx \\ &= 2\left[\frac{1}{2}x^2\right]_0^1 \\ &= 1. \end{aligned}$$

Hence, the solution is

$$u(x,t) = \frac{1}{2}t - \frac{4}{\pi^3}\left\{\cos(\pi x) + \frac{1}{3^3}\cos(3\pi x) + \frac{1}{5^3}\cos(5\pi x) + \dots\right\}.$$

3. Use Fourier transforms to solve the following initial value problem for $u(x,t)$ in $-\infty < x < \infty$, $t > 0$:

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4},$$
$$u(x,0) = f(x).$$

Write the solution for $u(x,t)$ as a convolution, but do not compute any inverse transforms explicitly. How smooth is the solution for $t > 0$?

Solution.

- Let

$$\widehat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx$$

be the Fourier transform of u with respect to x. Then, taking the Fourier transform of the initial value problem, we get

$$\frac{\partial \widehat{u}}{\partial t} = -(-ik)^4\widehat{u},$$
$$\widehat{u}(k,0) = \widehat{f}(k),$$

where $\widehat{f}$ is the Fourier transform of f. It follows that

$$\frac{\partial \widehat{u}}{\partial t} = -k^4\widehat{u},$$
$$\widehat{u}(k,0) = \widehat{f}(k),$$

which has the solution

$$\widehat{u}(k,t) = \widehat{f}(k)e^{-k^4 t}.$$

According to the convolution theorem, if f, g have Fourier transforms $\widehat{f}$, $\widehat{g}$ respectively then $\widehat{f}\cdot\widehat{g}$ is the Fourier transform of $\frac{1}{2\pi}f * g$. It follows that

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy$$

where

$$\widehat{G}(k,t) = \frac{1}{2\pi}e^{-k^4 t}.$$

The solution is smooth (infinitely differentiable with respect to x) for $t > 0$ since its Fourier transform decays exponentially quickly as $k \to \infty$ (assuming, for example, that $\widehat{f}(k)$ is a bounded function of k).

4. (a) Give the formulas for the Fourier transform $\widehat{f}(k)$ of a function $f(x)$ and the inverse Fourier transform.
(b) Compute the Fourier transform of $e^{-|x|}$.
(c) State Parseval's theorem, and use it to evaluate

$$\int_0^\infty \frac{1}{(1+k^2)^2}\,dk.$$

Solution.

- (a) A function $f(x)$ and its Fourier transform $\widehat{f}(k)$ are related by

$$\begin{aligned}\widehat{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx,\\ f(x) &= \int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx}\,dk,\end{aligned}$$

- (b) If $f(x) = e^{-|x|}$, then using

$$|x| = \begin{cases} x & \text{for } x \geq 0,\\ -x & \text{for } x \leq 0,\end{cases}$$

and changing $x \to -x$ in the integral for $-\infty < x < 0$, we find that

$$\begin{aligned}\widehat{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-ikx}\,dx,\\ &= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{(1-ik)x}\,dx + \int_0^\infty e^{-(1+ik)x}\,dx\right\}\\ &= \frac{1}{2\pi}\int_0^\infty \left\{e^{-(1-ik)x} + e^{-(1+ik)x}\right\}\,dx\\ &= -\frac{1}{2\pi}\left[\frac{e^{-(1-ik)x}}{1-ik} + \frac{e^{-(1+ik)x}}{1+ik}\right]_0^\infty\\ &= \frac{1}{2\pi}\left[\frac{1}{1-ik} + \frac{1}{1+ik}\right]_0^\infty\\ &= \frac{1}{\pi}\frac{1}{1+k^2}.\end{aligned}$$

- (c) Parseval's theorem states that

$$\int_{-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 \, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

  For $f(x) = e^{-|x|}$, we compute that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2 \, dx &= 2\int_0^{\infty} e^{-2x} \, dx \\ &= -\left[e^{-2x}\right]_0^{\infty} \\ &= 1.\end{aligned}$$

  It follows from Parsevals theorem and (b) that

$$\frac{2}{\pi^2} \int_0^{\infty} \frac{1}{(1+k^2)^2} \, dk = \frac{1}{2\pi} \cdot 1,$$

  so

$$\int_0^{\infty} \frac{1}{(1+k^2)^2} \, dk = \frac{\pi}{4}.$$

- **Remark.** The integral in (c) can also be evaluated directly by use of the substitution $k = \tan\theta$, which gives

$$\begin{aligned}\int_0^{\infty} \frac{1}{(1+k^2)^2} \, dk &= \int_0^{\pi/2} \frac{1}{(1+\tan^2\theta)^2} \sec^2\theta \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sec^4\theta} \sec^2\theta \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sec^2\theta} \, d\theta \\ &= \int_0^{\pi/2} \cos^2\theta \, d\theta \\ &= \frac{\pi}{2} \cdot \frac{1}{2} \\ &= \frac{\pi}{4},\end{aligned}$$

  which verifies Parseval's theorem explicitly in this case.

5. Use Laplace transforms to solve the following initial value problem:

$$y'' + 2y' + 2y = 1,$$
$$y(t) = 0, \quad y'(0) = 1.$$

Solution.

- Let $Y(p)$ be the Laplace transform of $y(t)$. Then, taking the Laplace transform of the ODE and using the initial conditions, we get that

$$p^2 Y - 1 + 2pY + 2Y = \frac{1}{p}.$$

Solving for Y, we get

$$Y(p) = \frac{1}{p^2 + 2p + 2} + \frac{1}{p(p^2 + 2p + 2)}.$$

We have $p^2 + 2p + 2 = (p+1)^2 + 1$, so (from L13 of the table)

$$L^{-1}\left[\frac{1}{p^2 + 2p + 2}\right] = e^{-t}\sin t.$$

Also

$$\begin{aligned} \frac{1}{p(p^2 + 2p + 2)} &= \frac{1}{2}\left[\frac{1}{p} - \frac{p+2}{p^2 + 2p + 2}\right] \\ &= \frac{1}{2}\left[\frac{1}{p} - \frac{p+1}{(p+1)^2 + 1} - \frac{1}{(p+1)^2 + 1}\right]. \end{aligned}$$

So (from L1, L13, L14) we have

$$L^{-1}\left[\frac{1}{p(p^2 + 2p + 2)}\right] = \frac{1}{2}\left[1 - e^{-t}\cos t - e^{-t}\sin t\right]$$

Hence, combining these inverse transforms, we get

$$y(t) = \frac{1}{2}\left[1 - e^{-t}\cos t + e^{-t}\sin t\right].$$

6. (a) Say what jump conditions the solution of $y(t)$ of the following initial value problem satisfies at $t = 0$, and find the solution directly (do not use Laplace transforms):

$$\begin{aligned} &y'' - 4y = \delta(t), \\ &y(t) = 0 \quad \text{for } t < 0. \end{aligned}$$

(b) Write the solution of the following initial value problem, where $f(t)$ is an arbitrary function, as a convolution (you don't need to derive your answer):

$$\begin{aligned} &y'' - 4y = f(t), \\ &y(0) = y'(0) = 0. \end{aligned}$$

Solution.

- (a) The derivative of y has a jump discontinuity of size one at $t = 0$. The solution is therefore

$$y(t) = \begin{cases} y_+(t) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

where

$$\begin{aligned} &y_+'' - 4y_+ = 0 \qquad \text{for } t > 0, \\ &y_+(0) = 0, \qquad y_+'(0) = 1. \end{aligned}$$

The general solution of the ODE is $y_+(t) = a \cosh 2t + b \sinh 2t$, and the initial conditions imply that $a = 0$ and $b = 1/2$. Hence,

$$y(t) = \frac{1}{2} \sinh 2t \qquad \text{for } t \geq 0.$$

- (b) The solution is

$$y(t) = \frac{1}{2} \int_0^t \sinh 2(t - s) f(s)\, ds.$$