ADVANCED ANALYSIS

Math 121, Fall 2004

Solutions, Midterm 2

1. [20%] Use Laplace transforms to find the solution $y(t)$ of the following initial value problem

$$y'' + 9y = f(t),$$
$$y(t) = 0, \quad y'(0) = 0,$$

where $f(t)$ is an arbitrary function. Express your answer as a convolution.

Solution.

- Let $Y(p) = \mathcal{L}[y(t)]$ and $F(p) = \mathcal{L}[f(t)]$, where $\mathcal{L}$ denotes the Laplace transform. Then
$$p^2Y + 9Y = F(p),$$
so
$$Y(p) = \frac{F(p)}{p^2+9}.$$
From the tables,
$$\mathcal{L}^{-1}\left[\frac{1}{p^2+9}\right] = \frac{1}{3}\sin(3t).$$
Therefore, using the convolution theorem, we have
$$y(t) = \frac{1}{3}\int_0^t \sin[3(t-s)]f(s)\,ds.$$

2. [20%] (a) Say what jump conditions the solution $y(t)$ of the following initial value problem satisfies at $t = 0$, and find the solution directly (do not use Laplace transforms):

$$\begin{aligned} y' + 2y &= \delta(t), \\ y(t) &= 0 \quad \text{for } t < 0. \end{aligned}$$

(b) Write the solution of the following initial value problem, where $f(t)$ is an arbitrary function, as a convolution (you don't need to derive your answer):

$$\begin{aligned} y' + 2y &= f(t), \\ y(0) &= 0. \end{aligned}$$

Solution.

- (a) The solution $y(t)$ has a jump discontinuity of size one at $t = 0$, so that $y(t)$ for $t > 0$ satisfies the IVP

  $$\begin{aligned} y' + 2y &= 0, \\ y(0) &= 1. \end{aligned}$$

  The solution of the ODE is

  $$y(t) = ce^{-2t}$$

  where c is a constant of integration, and the initial condition implies that $c = 1$. Hence,

  $$y(t) = \begin{cases} e^{-2t} & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

- (b) The solution is given by

  $$y(t) = \int_0^t e^{-2(t-s)} f(s)\, ds.$$

3. [20%] Use Fourier series to find the solution $u(x, y)$ of the following boundary value problem for Laplace's equation in the strip $0 < x < 1$, $y > 0$:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \\
&\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(1, y) = 0, \\
&u(x, 0) = x, \qquad u(x, y) \text{ is bounded as } y \to \infty.
\end{aligned}$$

What does the solution $u(x, y)$ approach as $y \to \infty$?

Solution.

- The appropriate linear combination of separated solutions of Laplace's equation that satisfy the boundary conditions at $x = 0, 1$ and as $y \to \infty$ is

$$u(x, y) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-n\pi y},$$

where the a_n are constants. Imposing the boundary condition at $y = 0$, we obtain that

$$x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

so

$$a_n = 2\int_0^1 x \cos(n\pi x)\, dx.$$

Evaluating this integral, we find that $a_0 = 1$, and for $n \geq 1$

$$a_n = \begin{cases} -4/(n\pi)^2 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}. \end{cases}$$

Hence the solution is

$$\begin{aligned}
u(x, y) &= \frac{1}{2} - \frac{4}{\pi^2}\left\{\cos(\pi x)e^{-\pi y} + \frac{1}{3^2}\cos(3\pi x)e^{-3\pi y}\right. \\
&\qquad \left. + \frac{1}{5^2}\cos(5\pi x)e^{-5\pi y} + \dots\right\}.
\end{aligned}$$

It follows that $u(x, y) \to 1/2$ as $y \to \infty$. Thus, the temperature approaches a constant value equal to the mean value of the boundary data at $y = 0$.

4. [20%] Reduce the boundary conditions to homogeneous boundary conditions and use Fourier series to find the solution $u(x,t)$ of the following initial-boundary value problem for the heat equation in $0 < x < 1$ and $t > 0$:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \\ &u(0,t) = 0, \qquad u(1,t) = 1, \\ &u(x,0) = 0. \end{aligned}$$

What does the solution $u(x,t)$ approach as $t \to \infty$?

Solution.

- The steady-state solution $u = x$ satisfies the non-homogeneous boundary conditions and the PDE. We write

$$u(x,t) = x + v(x,t).$$

Then $v(x,t)$ satisfies

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \\ &v(0,t) = 0, \qquad v(1,t) = 0, \\ &v(x,0) = -x. \end{aligned}$$

We look for a solution for v of the form

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}.$$

This function satisfies the PDE and the boundary conditions at $x = 0, 1$ for any choice of the constants b_n. Imposing the initial condition at $t = 0$ we get that

$$-x = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Using the expression for the Fourier sine coefficients, we get

$$\begin{aligned} b_n &= -2\int_0^1 x \sin(n\pi x)\,dx \\ &= \frac{2(-1)^n}{n\pi}. \end{aligned}$$

The solution of the original problem is therefore

$$u(x,t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) e^{-n^2\pi^2 t}.$$

As $t \to \infty$, we have $u(x,t) \to x$, so the temperature approaches the steady-state temperature distribution associated with the given non-homogeneous boundary conditions.

5. [20%] (a) Consider the following initial value problem for $u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3},$$
$$u(x,0) = f(x),$$

where $-\infty < x < \infty$, $t > 0$. Solve for the Fourier transform of $u(x,t)$ with respect to x, and write the solution for $u(x,t)$ as a Fourier integral, but do not attempt to invert the transform explicitly.

(b) State Parseval's theorem for the Fourier transform. Use your solution from (a) to show that

$$\int_{-\infty}^{\infty} u^2(x,t)\,dx = \int_{-\infty}^{\infty} f^2(x)\,dx$$

for all times t. (This result expresses 'conservation of energy'.)

Solution.

- (a) Let

$$\widehat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx$$

be the Fourier transform of $u(x,t)$ with respect to x. Then, Fourier tranforming the PDE, we find that

$$\frac{\partial \widehat{u}}{\partial t} = (ik)^3\widehat{u},$$
$$\widehat{u}(k,0) = \widehat{f}(k),$$

where $\widehat{f}$ is the Fourier transform of f. Solving this ODE, we get

$$\widehat{u}(k,t) = \widehat{f}(k)e^{(ik)^3 t}.$$

Using this expression in the Fourier inversion formula,

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)e^{ikx}\,dk,$$

and writing $(ik)^3 = -ik^3$, we obtain the solution

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx-ik^3 t}\,dk.$$

- (b) Parseval's theorem states that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} \left|\widehat{f}(k)\right|^2\,dk.$$

Since $|e^{i\theta}| = 1$ for any real number θ, we have that

$$\left|e^{-ik^3t}\right| = 1.$$

Hence, for any $t > 0$, we have

$$|\widehat{u}(k,t)| = \left|\widehat{f}(k)e^{-ik^3t}\right| = \left|\widehat{f}(k)\right|.$$

It then follows from Parseval's theorem and this equation that

$$\begin{aligned}\int_{-\infty}^{\infty} u^2(x,t)\,dx &= 2\pi\int_{-\infty}^{\infty} |\widehat{u}(k,t)|^2\,dk\\ &= 2\pi\int_{-\infty}^{\infty} \left|\widehat{f}(k)\right|^2\,dk\\ &= \int_{-\infty}^{\infty} f^2(x)\,dx.\end{aligned}$$