Chapter 1

The Real Numbers

In this chapter, we review some properties of the real numbers $\mathbb{R}$ and its subsets. We don't give proofs for most of the results stated here.

1.1. Completeness of $\mathbb{R}$

Intuitively, unlike the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ form a continuum with no 'gaps.' There are two main ways to state this completeness, one in terms of the existence of suprema and the other in terms of the convergence of Cauchy sequences.

1.1.1. Suprema and infima.

Definition 1.1. Let $A \subset \mathbb{R}$ be a set of real numbers. A real number $M \in \mathbb{R}$ is an upper bound of A if $x \leq M$ for every $x \in A$, and $m \in \mathbb{R}$ is a lower bound of A if $x \geq m$ for every $x \in A$. A set is bounded from above if it has an upper bound, bounded from below if it has a lower bound, and bounded if it has both an upper and a lower bound

An equivalent condition for A to be bounded is that there exists $R \in \mathbb{R}$ such that $|x| \leq R$ for every $x \in A$.

Example 1.2. The set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

is bounded from below by any $m \in \mathbb{R}$ with $m \leq 1$. It is not bounded from above, so $\mathbb{N}$ is unbounded.

Definition 1.3. Suppose that $A \subset \mathbb{R}$ is a set of real numbers. If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every upper bound M' of A, then M is called the supremum or least upper bound of A, denoted

$$M = \sup A.$$

If $m \in \mathbb{R}$ is a lower bound of A such that $m \geq m'$ for every lower bound m' of A, then m is called the infimum or greatest lower bound of A, denoted

$$m = \inf A.$$

The supremum or infimum of a set may or may not belong to the set. If $\sup A \in A$ does belong to A, then we also denote it by $\max A$ and refer to it as the maximum of A; if $\inf A \in A$ then we also denote it by $\min A$ and refer to it as the minimum of A.

Example 1.4. Every finite set of real numbers

$$A = \{x_1, x_2, \dots, x_n\}$$

is bounded. Its supremum is the greatest element,

$$\sup A = \max\{x_1, x_2, \dots, x_n\},$$

and its infimum is the smallest element,

$$\inf A = \min\{x_1, x_2, \dots, x_n\}.$$

Both the supremum and infimum of a finite set belong to the set.

Example 1.5. Let

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

be the set of reciprocals of the natural numbers. Then $\sup A = 1$, which belongs to A, and $\inf A = 0$, which does not belong to A.

Example 1.6. For $A = (0, 1)$, we have

$$\sup(0, 1) = 1, \qquad \inf(0, 1) = 0.$$

In this case, neither $\sup A$ nor $\inf A$ belongs to A. The closed interval $B = [0, 1]$, and the half-open interval $C = (0, 1]$ have the same supremum and infimum as A. Both $\sup B$ and $\inf B$ belong to B, while only $\sup C$ belongs to C.

The completeness of $\mathbb{R}$ may be expressed in terms of the existence of suprema.

Theorem 1.7. Every nonempty set of real numbers that is bounded from above has a supremum.

Since $\inf A = -\sup(-A)$, it follows immediately that every nonempty set of real numbers that is bounded from below has an infimum.

Example 1.8. The supremum of the set of real numbers

$$A = \left\{x \in \mathbb{R} : x < \sqrt{2}\right\}$$

is $\sup A = \sqrt{2}$. By contrast, since $\sqrt{2}$ is irrational, the set of rational numbers

$$B = \left\{x \in \mathbb{Q} : x < \sqrt{2}\right\}$$

has no supremum in $\mathbb{Q}$. (If $M \in \mathbb{Q}$ is an upper bound of B, then there exists $M' \in \mathbb{Q}$ with $\sqrt{2} < M' < M$, so M is not a least upper bound.)

1.1.2. Cauchy sequences. We assume familiarity with the convergence of real sequences, but we recall the definition of Cauchy sequences and their relation with the completeness of $\mathbb{R}$.

Definition 1.9. A sequence (x_n) of real numbers is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \epsilon \qquad \text{for all } m, n > N.$$

Every convergent sequence is Cauchy. Conversely, it follows from Theorem 1.7 that every Cauchy sequence of real numbers has a limit.

Theorem 1.10. A sequence of real numbers converges if and only if it is a Cauchy sequence.

The fact that real Cauchy sequences have a limit is an equivalent way to formulate the completeness of $\mathbb{R}$. By contrast, the rational numbers $\mathbb{Q}$ are not complete.

Example 1.11. Let (x_n) be a sequence of rational numbers such that $x_n \to \sqrt{2}$ as $n \to \infty$. Then (x_n) is Cauchy in $\mathbb{Q}$ but (x_n) does not have a limit in $\mathbb{Q}$.

1.2. Open sets

Open sets are among the most important subsets of $\mathbb{R}$. A collection of open sets is called a topology, and any property (such as compactness or continuity) that can be defined entirely in terms of open sets is called a topological property.

Definition 1.12. A set $G \subset \mathbb{R}$ is open in $\mathbb{R}$ if for every $x \in G$ there exists a $\delta > 0$ such that $G \supset (x - \delta, x + \delta)$.

Another way to state this definition is in terms of interior points.

Definition 1.13. Let $A \subset \mathbb{R}$ be a subset of $\mathbb{R}$. A point $x \in A$ is an interior point of A a if there is a $\delta > 0$ such that $A \supset (x - \delta, x + \delta)$. A point $x \in \mathbb{R}$ is a boundary point of A if every interval $(x - \delta, x + \delta)$ contains points in A and points not in A.

Thus, a set is open if and only if every point in the set is an interior point.

Example 1.14. The open interval $I = (0, 1)$ is open. If $x \in I$ then I contains an open interval about x,

$$I \supset \left(\frac{x}{2}, \frac{1+x}{2}\right), \qquad x \in \left(\frac{x}{2}, \frac{1+x}{2}\right),$$

and, for example, $I \supset (x - \delta, x + \delta)$ if

$$\delta = \min\left(\frac{x}{2}, \frac{1-x}{2}\right) > 0.$$

Similarly, every finite or infinite open interval (a, b), $(-\infty, b)$, (a, ∞) is open.

An arbitrary union of open sets is open; one can prove that every open set in $\mathbb{R}$ is a countable union of disjoint open intervals. A *finite* intersection of open sets is open, but an intersection of infinitely many open sets needn't be open.

Example 1.15. The interval

$$I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is open for every $n \in \mathbb{N}$, but

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

is not open.

Instead of using intervals to define open sets, we can use neighborhoods, and it is frequently simpler to refer to neighborhoods instead of open intervals of radius $\delta > 0$.

Definition 1.16. A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if

$$U \supset (x - \delta, x + \delta)$$

for some $\delta > 0$. The open interval $(x - \delta, x + \delta)$ is called a δ-neighborhood of x.

A neighborhood of x needn't be an open interval about x, it just has to contain one. Sometimes a neighborhood is also required to be an open set, but we don't do this and will specify that a neighborhood is open when it is needed.

Example 1.17. If $a < x < b$ then the closed interval $[a, b]$ is a neighborhood of x, since it contains the interval $(x - \delta, x + \delta)$ for sufficiently small $\delta > 0$. On the other hand, $[a, b]$ is not a neighborhood of the endpoints a, b since no open interval about a or b is contained in $[a, b]$.

We can restate Definition 1.12 in terms of neighborhoods as follows.

Definition 1.18. A set $G \subset \mathbb{R}$ is open if every $x \in G$ has a neighborhood U such that $G \supset U$.

We define relatively open sets by restricting open sets in $\mathbb{R}$ to a subset.

Definition 1.19. If $A \subset \mathbb{R}$ then $B \subset A$ is relatively open in A, or open in A, if $B = A \cap U$ where U is open in $\mathbb{R}$.

Example 1.20. Let $A = [0, 1]$. Then the half-open intervals $(a, 1]$ and $[0, b)$ are open in A for every $0 \le a < 1$ and $0 < b \le 1$, since

$$(a, 1] = [0, 1] \cap (a, 2), \qquad [0, b) = [0, 1] \cap (-1, b)$$

and $(a, 2)$, $(-1, b)$ are open in $\mathbb{R}$. By contrast, neither $(a, 1]$ nor $[0, b)$ is open in $\mathbb{R}$.

The neighborhood definition of open sets generalizes to relatively open sets.

Definition 1.21. If $A \subset \mathbb{R}$ then a relative neighborhood of $x \in A$ is a set $C = A \cap V$ where V is a neighborhood of x in $\mathbb{R}$.

As for open sets in $\mathbb{R}$, a set is relatively open if and only if it contains a relative neighborhood of every point. Since we use this fact at one point later on, we give a proof.

Proposition 1.22. A set $B \subset A$ is relatively open in A if and only if every $x \in B$ has a relative neighborhood C such that $B \supset C$.

Proof. Assume that $B = A \cap U$ is open in A, where U is open in $\mathbb{R}$. If $x \in B$, then $x \in U$. Since U is open, there is a neighborhood V of x in $\mathbb{R}$ such that $U \supset V$. Then $C = A \cap V$ is a relative neighborhood of x with $B \supset C$. (Alternatively, we could observe that B itself is a relative neighborhood of every $x \in B$.)

Conversely, assume that every point $x \in B$ has a relative neighborhood $C_x = A \cap V_x$ such that $C_x \subset B$. Then, since V_x is a neighborhood of x in $\mathbb{R}$, there is an open neighborhood $U_x \subset V_x$ of x, for example a δ-neighborhood. We claim that that $B = A \cap U$ where

$$U = \bigcup_{x \in B} U_x.$$

To prove this claim, we show that $B \subset A \cap U$ and $B \supset A \cap U$. First, $B \subset A \cap U$ since $x \in A \cap U_x \subset A \cap U$ for every $x \in B$. Second, $A \cap U_x \subset A \cap V_x \subset B$ for every $x \in B$. Taking the union over $x \in B$, we get that $A \cap U \subset B$. Finally, U is open since it's a union of open sets, so $B = A \cap U$ is relatively open in A. □

1.3. Closed sets

Closed sets are complements of open sets.

Definition 1.23. A set $F \subset \mathbb{R}$ is closed if $F^c = \{x \in \mathbb{R} : x \notin F\}$ is open.

Closed sets can also be characterized in terms of sequences.

Definition 1.24. A set $F \subset \mathbb{R}$ is sequentially closed if the limit of every convergent sequence in F belongs to F.

A subset of $\mathbb{R}$ is closed if and only if it is sequentially closed, so we can use either definition, and we don't distinguish between closed and sequentially closed sets.

Example 1.25. The closed interval $[0, 1]$ is closed. To verify this from Definition 1.23, note that

$$[0, 1]^c = (-\infty, 0) \cup (1, \infty)$$

is open. To verify this from Definition 1.24, note that if (x_n) is a convergent sequence in $[0, 1]$, then $0 \le x_n \le 1$ for all $n \in \mathbb{N}$. Since limits preserve (non-strict) inequalities, we have

$$0 \le \lim_{n \to \infty} x_n \le 1,$$

meaning that the limit belongs to $[0, 1]$. Similarly, every finite or infinite closed interval $[a, b]$, $(-\infty, b]$, $[a, \infty)$ is closed.

An arbitrary intersection of closed sets is closed and a *finite* union of closed sets is closed. A union of infinitely many closed sets needn't be closed.

Example 1.26. If I_n is the closed interval

$$I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right],$$

then the union of the I_n is an open interval

$$\bigcup_{n=1}^{\infty} I_n = (0, 1).$$

The only sets that are both open and closed are the real numbers $\mathbb{R}$ and the empty set $\varnothing$. In general, sets are neither open nor closed.

Example 1.27. The half-open interval $I = (0, 1]$ is neither open nor closed. It's not open since I doesn't contain any neighborhood of the point $1 \in I$. It's not closed since $(1/n)$ is a convergent sequence in I whose limit 0 doesn't belong to I.

1.4. Accumulation points and isolated points

An accumulation point of a set A is a point in $\mathbb{R}$ that has points in A arbitrarily close to it.

Definition 1.28. A point $x \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$ if for every $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point in A that is different from x.

Accumulation points are also called limit points or cluster points. By taking smaller and smaller intervals about x, we see that if x is an accumulation point of A then every neighborhood of x contains infinitely many points in A. This leads to an equivalent sequential definition.

Definition 1.29. A point $x \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$ if there is a sequence (x_n) in A with $x_n \neq x$ for every $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$.

An accumulation point of a set may or may not belong to the set (a set is closed if and only if all its accumulation points belong to the set), and a point that belongs to the set may or may not be an accumulation point.

Example 1.30. The set $\mathbb{N}$ of natural numbers has no accumulation points.

Example 1.31. If

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

then 0 is an accumulation point of A since every open interval about 0 contains $1/n$ for sufficiently large n. Alternatively, the sequence $(1/n)$ in A converges to 0 as $n \to \infty$. In this case, the accumulation point 0 does not belong to A. Moreover, 0 is the only accumulation point of A; in particular, none of the points in A are accumulation points of A.

Example 1.32. The set of accumulation points of a bounded, open interval $I = (a, b)$ is the closed interval $[a, b]$. Every point in I is an accumulation point of I. In addition, the endpoints a, b are accumulation points of I that do not belong to I. The set of accumulation points of the closed interval $[a, b]$ is again the closed interval $[a, b]$.

Example 1.33. Let $a < c < b$ and suppose that

$$A = (a, c) \cup (c, b)$$

is an open interval punctured at c. Then the set of accumulation points of A is the closed interval $[a, b]$. The points a, b, c are accumulation points of A that do not belong to A.

An isolated point of a set is a point in the set that does not have other points in the set arbitrarily close to it.

Definition 1.34. Let $A \subset \mathbb{R}$. A point $x \in A$ is an isolated point of A if there exists $\delta > 0$ such that x is the only point belonging to A in the interval $(x - \delta, x + \delta)$.

Unlike accumulation points, isolated points are required to belong to the set. Every point $x \in A$ is either an accumulation point of A (if every neighborhood contains other points in A) or an isolated point of A (if some neighborhood contains no other points in A).

Example 1.35. If

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

then every point $1/n \in A$ is an isolated point of A since the interval $(1/n-\delta, 1/n+\delta)$ does not contain any points $1/m$ with $m \in \mathbb{N}$ and $m \neq n$ when $\delta > 0$ is sufficiently small.

Example 1.36. An interval has no isolated points (excluding the trivial case of closed intervals of zero length that consist of a single point $[a, a] = \{a\}$).

1.5. Compact sets

Compactness is not as obvious a property of sets as being open, but it plays a central role in analysis. One motivation for the property is obtained by turning around the Bolzano-Weierstrass and Heine-Borel theorems and taking their conclusions as a definition.

We will give two equivalent definitions of compactness, one based on sequences (every sequence has a convergent subsequence) and the other based on open covers (every open cover has a finite subcover). A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded, in which case it has both of these properties. For example, every closed, bounded interval $[a, b]$ is compact. There are also other, more exotic, examples of compact sets, such as the Cantor set.

1.5.1. Sequential compactness. Intuitively, a compact set confines every infinite sequence of points in the set so much that the sequence must accumulate at some point of the set. This implies that a subsequence converges to the accumulation point and leads to the following definition.

Definition 1.37. A set $K \subset \mathbb{R}$ is sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.

Note that we require that the subsequence converges to a point in K, not to a point outside K.

Example 1.38. The open interval $I = (0, 1)$ is not sequentially compact. The sequence $(1/n)$ in I converges to 0, so every subsequence also converges to $0 \notin I$. Therefore, $(1/n)$ has no convergent subsequence whose limit belongs to I.

Example 1.39. The set $\mathbb{N}$ is closed, but it is not sequentially compact since the sequence (n) in $\mathbb{N}$ has no convergent subsequence. (Every subsequence diverges to infinity.)

As these examples illustrate, a sequentially compact set must be closed and bounded. Conversely, the Bolzano-Weierstrass theorem implies that that every closed, bounded subset of $\mathbb{R}$ is sequentially compact.

Theorem 1.40. A set $K \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Proof. First, assume that K is sequentially compact. Let (x_n) be any sequence in K that converges to $x \in \mathbb{R}$. Then every subsequence of K also converges to x, so the compactness of K implies that $x \in K$, meaning that K is closed.

Suppose for contradiction that K is unbounded. Then there is a sequence (x_n) in K such that $|x_n| \to \infty$ as $n \to \infty$. Every subsequence of (x_n) is unbounded and therefore diverges, so (x_n) has no convergent subsequence. This contradicts the assumption that K is sequentially compact, so K is bounded.

Conversely, assume that K is closed and bounded. Let (x_n) be a sequence in K. Then (x_n) is bounded since K is bounded, and the Bolzano-Weierstrass theorem implies that (x_n) has a convergent subsequence. Since K is closed the limit of this subsequence belongs to K, so K is sequentially compact. □

For later use, we explicitly state and prove one other property of compact sets.

Proposition 1.41. If $K \subset \mathbb{R}$ is sequentially compact, then K has a maximum and minimum.

Proof. Since K is sequentially compact it is bounded and, by the completeness of $\mathbb{R}$, it has a (finite) supremum $M = \sup K$. From the definition of the supremum, for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$M - \frac{1}{n} < x_n \leq M.$$

It follows (from the 'sandwich' theorem) that $x_n \to M$ as $n \to \infty$. Since K is closed, $M \in K$, which proves that K has a maximum. A similar argument shows that $m = \inf K$ belongs to K, so K has a minimum. □

1.5.2. Compactness. Next, we give a topological definition of compactness in terms of open sets. If A is a subset of $\mathbb{R}$, an open cover of A is a collection of open sets

$$\{G_i \subset \mathbb{R} : i \in \mathcal{I}\}$$

whose union contains A,

$$\bigcup_{i \in \mathcal{I}} G_i \supset A.$$

A finite subcover of this open cover is a finite collection of sets in the cover

$$\{G_{i_1}, G_{i_2}, \dots, G_{i_N}\}$$

whose union still contains A,

$$\bigcup_{n=1}^{N} G_{i_n} \supset A.$$

Definition 1.42. A set $K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover.

We illustrate the definition with several examples.

Example 1.43. The collection of open intervals

$$\{I_n : n \in \mathbb{N}\}, \qquad I_n = (n-1, n+1)$$

is an open cover of the natural numbers $\mathbb{N}$ since

$$\bigcup_{n=1}^{\infty} I_n = (0, \infty) \supset \mathbb{N}.$$

However, no finite subcollection $\{I_1, I_2, \dots, I_N\}$ of intervals covers $\mathbb{N}$ since their union

$$\bigcup_{n=1}^{N} I_n = (0, N+1)$$

does not contain sufficiently large integers with $n \geq N+1$. (A finite subcover that omits some of the intervals I_i for $1 \leq i \leq N$ would have an even smaller union.) Thus, $\mathbb{N}$ is not compact. A similar argument, using the intervals $I_n = (-n, n)$, shows that a compact set must be bounded.

Example 1.44. The collection of open intervals (which get smaller as they get closer to 0)

$$\{I_n : n = 0, 1, 2, 3, \dots\}, \qquad I_n = \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}, \frac{1}{2^n} + \frac{1}{2^{n+1}}\right)$$

is an open cover of the open interval $(0,1)$; in fact

$$\bigcup_{n=0}^{\infty} I_n = \left(0, \frac{3}{2}\right) \supset (0,1).$$

However, no finite subcollection $\{I_0, I_1, I_2, \dots, I_N\}$ of intervals covers $(0,1)$ since their union

$$\bigcup_{n=0}^{N} I_n = \left(\frac{1}{2^N} - \frac{1}{2^{N+1}}, \frac{3}{2}\right),$$

does not contain points in $(0,1)$ that are sufficiently close to 0. Thus, $(0,1)$ is not compact.

Example 1.45. The collection of open intervals $\{I_n\}$ in Example 1.44 isn't an open cover of the closed interval $[0,1]$ since 0 doesn't belong to their union. We can get an open cover $\{I_n, J\}$ of $[0,1]$ by adding to the I_n an open interval $J = (-\delta, \delta)$ about zero, where $\delta > 0$ can be arbitrarily small. In that case, if we choose $N \in \mathbb{N}$ sufficiently large that

$$\frac{1}{2^N} - \frac{1}{2^{N+1}} < \delta,$$

then $\{I_0, I_1, I_2, \dots, I_N, J\}$ is a finite subcover of $[0,1]$ since

$$\bigcup_{n=0}^{N} I_n \cup J = \left(-\delta, \frac{3}{2}\right) \supset [0,1].$$

Points sufficiently close to 0 belong to J, while points further away belong to I_i for some $0 \leq i \leq N$. As this example illustrates, $[0,1]$ is compact and every open cover of $[0,1]$ has a finite subcover.

Theorem 1.46. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

This result follows from the Heine-Borel theorem, that every open cover of a closed, bounded interval has a finite subcover, but we omit a detailed proof.

It follows that a subset of $\mathbb{R}$ is sequentially compact if and only if it is compact, since the subset is closed and bounded in either case. We therefore refer to any such set simply as a compact set. We will use the sequential definition of compactness in our proofs.