Chapter 2

Limits of Functions

In this chapter, we define limits of functions and describe some of their properties.

2.1. Limits

We begin with the ϵ-δ definition of the limit of a function.

Definition 2.1. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

We also denote limits by the 'arrow' notation $f(x) \to L$ as $x \to c$, and often leave it to be implicitly understood that $x \in A$ is restricted to the domain of f. Note that we exclude $x = c$, so the function need not be defined at c for the limit as $x \to c$ to exist. Also note that it follows directly from the definition that

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c} |f(x) - L| = 0.$$

Example 2.2. Let $A = [0, \infty) \setminus \{9\}$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x - 9}{\sqrt{x} - 3}.$$

We claim that

$$\lim_{x \to 9} f(x) = 6.$$

To prove this, let $\epsilon > 0$ be given. For $x \in A$, we have from the difference of two squares that $f(x) = \sqrt{x} + 3$, and

$$|f(x) - 6| = \left|\sqrt{x} - 3\right| = \left|\frac{x - 9}{\sqrt{x} + 3}\right| \le \frac{1}{3}|x - 9|.$$

Thus, if $\delta = 3\epsilon$, then $|x - 9| < \delta$ and $x \in A$ implies that $|f(x) - 6| < \epsilon$.

We can rephrase the ϵ-δ definition of limits in terms of neighborhoods. Recall from Definition 1.16 that a set $V \subset \mathbb{R}$ is a neighborhood of $c \in \mathbb{R}$ if $V \supset (c-\delta, c+\delta)$ for some $\delta > 0$, and $(c-\delta, c+\delta)$ is called a δ-neighborhood of c. A set U is a punctured (or deleted) neighborhood of c if $U \supset (c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$, and $(c-\delta, c) \cup (c, c+\delta)$ is called a punctured (or deleted) δ-neighborhood of c. That is, a punctured neighborhood of c is a neighborhood of c with the point c itself removed.

Definition 2.3. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x\to c} f(x) = L$$

if and only if for every neighborhood V of L, there is a punctured neighborhood U of c such that

$$x \in A \cap U \text{ implies that } f(x) \in V.$$

This is essentially a rewording of the ϵ-δ definition. If Definition 2.1 holds and V is a neighborhood of L, then V contains an ϵ-neighborhood of L, so there is a punctured δ-neighborhood U of c that maps into V, which verifies Definition 2.3. Conversely, if Definition 2.3 holds and $\epsilon > 0$, let $V = (L-\epsilon, L+\epsilon)$ be an ϵ-neighborhood of L. Then there is a punctured neighborhood U of c that maps into V and U contains a punctured δ-neighborhood of c, which verifies Definition 2.1.

The next theorem gives an equivalent sequential characterization of the limit.

Theorem 2.4. Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x\to c} f(x) = L$$

if and only if

$$\lim_{n\to\infty} f(x_n) = L.$$

for every sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} x_n = c.$$

Proof. First assume that the limit exists. Suppose that (x_n) is any sequence in A with $x_n \neq c$ that converges to c, and let $\epsilon > 0$ be given. From Definition 2.1, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$, and since $x_n \to c$ there exists $N \in \mathbb{N}$ such that $0 < |x_n - c| < \delta$ for all $n > N$. It follows that $|f(x_n) - L| < \epsilon$ whenever $n > N$, so $f(x_n) \to L$ as $n \to \infty$.

To prove the converse, assume that the limit does not exist. Then there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a point $x \in A$ with $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon_0$. Therefore, for every $n \in \mathbb{N}$ there is an $x_n \in A$ such that

$$0 < |x_n - c| < \frac{1}{n}, \qquad |f(x_n) - L| \geq \epsilon_0.$$

It follows that $x_n \neq c$ and $x_n \to c$, but $f(x_n) \not\to L$, so the sequential condition does not hold. This proves the result. □

This theorem gives a way to show that a limit of a function does not exist.

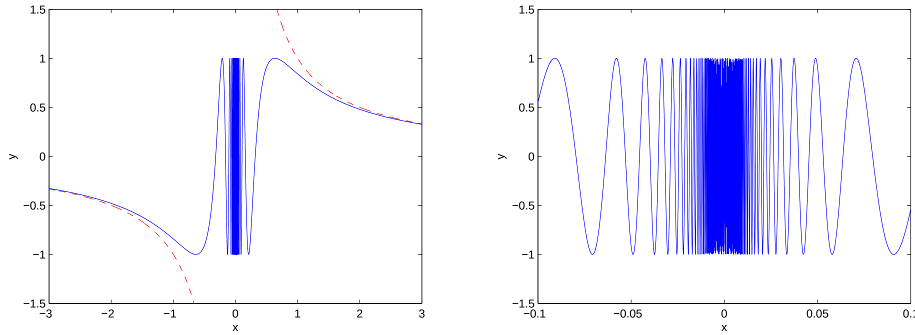

Figure 1. A plot of the function $y = \sin(1/x)$, with the hyperbola $y = 1/x$ shown in red, and a detail near the origin.

Corollary 2.5. Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of A. Then $\lim_{x\to c} f(x)$ does not exist if either of the following conditions holds:

(1) There are sequences (x_n), (y_n) in A with $x_n, y_n \neq c$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c, \quad \text{but} \quad \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n).$$

(2) There is a sequence (x_n) in A with $x_n \neq c$ such that $\lim_{n\to\infty} x_n = c$ but the sequence $(f(x_n))$ does not converge.

Example 2.6. Define the sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ by

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

Then the limit

$$\lim_{x\to 0} \operatorname{sgn} x$$

doesn't exist. To prove this, note that $(1/n)$ is a non-zero sequence such that $1/n \to 0$ and $\operatorname{sgn}(1/n) \to 1$ as $n \to \infty$, while $(-1/n)$ is a non-zero sequence such that $-1/n \to 0$ and $\operatorname{sgn}(-1/n) \to -1$ as $n \to \infty$. Since the sequences of sgn-values have different limits, Corollary 2.5 implies that the limit does not exist.

Example 2.7. The limit

$$\lim_{x\to 0} \frac{1}{x},$$

corresponding to the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$, doesn't exist. For example, consider the non-zero sequence (x_n) given by $x_n = 1/n$. Then $1/n \to 0$ but the sequence of values (n) doesn't converge.

Example 2.8. The limit

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right),$$

corresponding to the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \sin(1/x)$, doesn't exist. (See Figure 1.) For example, the non-zero sequences (x_n), (y_n) defined by

$$x_n = \frac{1}{2\pi n}, \qquad y_n = \frac{1}{2\pi n + \pi/2}$$

both converge to zero as $n \to \infty$, but the limits

$$\lim_{n\to\infty} f(x_n) = 0, \qquad \lim_{n\to\infty} f(y_n) = 1$$

are different.

2.2. Left, right, and infinite limits

We can define other kinds of limits in an obvious way. We list some of them here and give examples, whose proofs are left as an exercise. All these definitions can be combined in various ways and have obvious equivalent sequential characterizations.

Definition 2.9 (Right and left limits)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then (right limit)

$$\lim_{x\to c^+} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c < x < c + \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon,$$

and (left limit)

$$\lim_{x\to c^-} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$c - \delta < x < c \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Example 2.10. For the sign function in Example 2.6, we have

$$\lim_{x\to 0^+} \operatorname{sgn} x = 1, \qquad \lim_{x\to 0^-} \operatorname{sgn} x = -1.$$

Next we introduce some convenient definitions for various kinds of limits involving infinity. We emphasize that ∞ and $-\infty$ are not real numbers (what is $\sin\infty$, for example?) and all these definition have precise translations into statements that involve only real numbers.

Definition 2.11 (Limits as $x \to \pm\infty$)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. If A is not bounded from above, then

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists an $M \in \mathbb{R}$ such that

$$x > M \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

If A is not bounded from below, then

$$\lim_{x\to-\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists an $m \in \mathbb{R}$ such that

$$x < m \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

Sometimes we write $+\infty$ instead of ∞ to indicate that it denotes arbitrarily large, positive values, while $-\infty$ denotes arbitrarily large, negative values. It follows from this definition that

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right), \qquad \lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f\left(\frac{1}{t}\right),$$

and it is often useful to convert one of these limits into the other.

Example 2.12. We have

$$\lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = 1, \qquad \lim_{x\to-\infty} \frac{x}{\sqrt{1+x^2}} = -1.$$

Definition 2.13 (Divergence to $\pm\infty$)**.** Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then

$$\lim_{x\to c} f(x) = \infty$$

if for every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) > M,$$

and

$$\lim_{x\to c} f(x) = -\infty$$

if for every $m \in \mathbb{R}$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } f(x) < m.$$

The notation $\lim_{x\to c} f(x) = \pm\infty$ is simply shorthand for the property stated in this definition; it does not mean that the limit exists, and we say that f diverges to $\pm\infty$.

Example 2.14. We have

$$\lim_{x\to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x\to\infty} \frac{1}{x^2} = 0.$$

Example 2.15. We have

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

How would you define these statements precisely? Note that

$$\lim_{x\to 0} \frac{1}{x} \neq \pm\infty,$$

since $1/x$ takes arbitrarily large positive (if $x > 0$) and negative (if $x < 0$) values in every two-sided neighborhood of 0.

Example 2.16. None of the limits

$$\lim_{x\to 0^+} \frac{1}{x}\sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^-} \frac{1}{x}\sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0} \frac{1}{x}\sin\left(\frac{1}{x}\right)$$

is ∞ or $-\infty$, since $(1/x)\sin(1/x)$ oscillates between arbitrarily large positive and negative values in every one-sided or two-sided neighborhood of 0.

Example 2.17. We have

$$\lim_{x\to\infty}\left(\frac{1}{x} - x^3\right) = -\infty, \qquad \lim_{x\to-\infty}\left(\frac{1}{x} - x^3\right) = \infty.$$

How would you define these statements precisely and prove them?

2.3. Properties of limits

The properties of limits of functions follow immediately from the corresponding properties of sequences and the sequential characterization of the limit in Theorem 2.4. We can also prove them directly from the ϵ-δ definition of the limit, and we shall do so in a few cases below.

2.3.1. Uniqueness and boundedness.

The following result might be taken for-granted, but it requires proof.

Proposition 2.18. The limit of a function is unique if it exists.

Proof. Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of $A \subset \mathbb{R}$. Assume that

$$\lim_{x\to c} f(x) = L_1, \qquad \lim_{x\to c} f(x) = L_2$$

where $L_1, L_2 \in \mathbb{R}$. Then for every $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L_1| < \epsilon/2,$$
$$0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |f(x) - L_2| < \epsilon/2.$$

Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x - c| < \delta$. It follows that

$$|L_1 - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \epsilon.$$

Since this holds for arbitrary $\epsilon > 0$, we must have $L_1 = L_2$. □

Note that in this proof we used the requirement in the definition of a limit that c is an accumulation point of A. The limit definition would be vacuous if it was applied to a non-accumulation point, and in that case every $L \in \mathbb{R}$ would be a limit.

Definition 2.19. A function $f : A \to \mathbb{R}$ is bounded on $B \subset A$ if there exists $M \ge 0$ such that

$$|f(x)| \le M \text{ for every } x \in B.$$

A function is bounded if it is bounded on its domain.

Equivalently, f is bounded on B if $f(B)$ is a bounded subset of $\mathbb{R}$.

Example 2.20. The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is unbounded, but it is bounded on any interval $[\delta, 1]$ with $0 < \delta < 1$. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is unbounded, but is it bounded on any finite interval $[a, b]$.

If a function has a limit as $x \to c$, it must be locally bounded at c, as stated in the next proposition.

Proposition 2.21. Suppose that $f : A \to \mathbb{R}$ and c is an accumulation point of A. If $\lim_{x\to c} f(x)$ exists, then there is a punctured neighborhood U of c such that f is bounded on $A \cap U$.

Proof. Suppose that $f(x) \to L$ as $x \to c$. Taking $\epsilon = 1$ in the definition of the limit, we get that there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < 1.$$

Let $U = (c - \delta, c) \cup (c, c + \delta)$, which is a punctured neighborhood of c. Then for $x \in A \cap U$, we have

$$|f(x)| \le |f(x) - L| + |L| < 1 + |L|,$$

so f is bounded on $A \cap U$. □

2.3.2. Algebraic properties.

Limits of functions respect algebraic operations.

Theorem 2.22. Suppose that $f, g : A \to \mathbb{R}$, c is an accumulation point of A, and the limits

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M$$

exist. Then

$$\begin{aligned}
&\lim_{x\to c} kf(x) = kL && \text{for every } k \in \mathbb{R},\\
&\lim_{x\to c} [f(x) + g(x)] = L + M,\\
&\lim_{x\to c} [f(x)g(x)] = LM,\\
&\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M} && \text{if } M \ne 0.
\end{aligned}$$

Proof. We prove the results for sums and products from the definition of the limit, and leave the remaining proofs as an exercise. All of the results also follow from the corresponding results for sequences.

First, we consider the limit of $f + g$. Given $\epsilon > 0$, choose δ_1, δ_2 such that

$$\begin{aligned}
&0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/2,\\
&0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/2,
\end{aligned}$$

and let $\delta = \min(\delta_1, \delta_2) > 0$. Then $0 < |x - c| < \delta$ implies that

$$|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M| < \epsilon,$$

which proves that $\lim(f + g) = \lim f + \lim g$.

To prove the result for the limit of the product, first note that from the local boundedness of functions with a limit (Proposition 2.21) there exists $\delta_0 > 0$ and $K > 0$ such that $|g(x)| \le K$ for all $x \in A$ with $0 < |x - c| < \delta_0$. Choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
&0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon/(2K),\\
&0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \epsilon/(2|L| + 1).
\end{aligned}$$

Let $\delta = \min(\delta_0, \delta_1, \delta_2) > 0$. Then for $0 < |x - c| < \delta$ and $x \in A$,

$$\begin{aligned} |f(x)g(x) - LM| &= |(f(x) - L)\, g(x) + L\,(g(x) - M)| \\ &\leq |f(x) - L|\; |g(x)| + |L|\, |g(x) - M| \\ &< \frac{\epsilon}{2K} \cdot K + |L| \cdot \frac{\epsilon}{2|L| + 1} \\ &< \epsilon, \end{aligned}$$

which proves that $\lim(fg) = \lim f \lim g$. □

2.3.3. Order properties. As for limits of sequences, limits of functions preserve (non-strict) inequalities.

Theorem 2.23. Suppose that $f, g : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \leq g(x) \qquad \text{for all } x \in A,$$

and $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Proof. Let

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M.$$

Suppose for contradiction that $L > M$, and let

$$\epsilon = \frac{1}{2}(L - M) > 0.$$

From the definition of the limit, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |f(x) - L| < \epsilon \qquad &\text{if } x \in A \text{ and } 0 < |x - c| < \delta_1, \\ |g(x) - M| < \epsilon \qquad &\text{if } x \in A \text{ and } 0 < |x - c| < \delta_2. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$. Since c is an accumulation point of A, there exists $x \in A$ such that $0 < |x - a| < \delta$, and it follows that

$$\begin{aligned} f(x) - g(x) &= [f(x) - L] + L - M + [M - g(x)] \\ &> L - M - 2\epsilon \\ &> 0, \end{aligned}$$

which contradicts the assumption that $f(x) \leq g(x)$. □

Finally, we state a useful "sandwich" or "squeeze" criterion for the existence of a limit.

Theorem 2.24. Suppose that $f, g, h : A \to \mathbb{R}$ and c is an accumulation point of A. If

$$f(x) \leq g(x) \leq h(x) \qquad \text{for all } x \in A$$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then the limit of $g(x)$ as $x \to c$ exists and

$$\lim_{x\to c} g(x) = L.$$

We leave the proof as an exercise. We often use this result, without comment, in the following way: If

$$0 \leq f(x) \leq g(x) \quad \text{or} \quad |f(x)| \leq g(x)$$

and $g(x) \to 0$ as $x \to c$, then $f(x) \to 0$ as $x \to c$.

It is essential for the bounding functions f, h in Theorem 2.24 to have the same limit.

Example 2.25. We have

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \qquad \text{for all } x \neq 0$$

and

$$\lim_{x\to 0}(-1) = -1, \qquad \lim_{x\to 0} 1 = 1,$$

but

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist.}$$