Real Analysis

**Math 125A, Fall 2012**

**Solutions: Midterm 1**

**1.** (a) Suppose that $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$ and $c \in \mathbb{R}$ is an accumulation point of $A$. State the $\epsilon$-$\delta$ definition of $\lim_{x\to c} f(x)$.
(b) Prove *from the definition* that if $f, g : A \to \mathbb{R}$ and $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

**Solution.**

- (a) We have $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ and } x \in A \text{ implies that } |f(x) - L| < \epsilon.$$

- (b) Suppose that

$$\lim_{x\to c} f(x) = L, \qquad \lim_{x\to c} g(x) = M.$$

  Let $\epsilon > 0$ be given. From the definition of the limit for $f$ and $g$, there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \text{ implies that } |f(x) - L| < \frac{\epsilon}{2}$$
$$0 < |x - c| < \delta_2 \text{ and } x \in A \text{ implies that } |g(x) - M| < \frac{\epsilon}{2}$$

  Let $\delta = \min(\delta_1, \delta_2) > 0$. If $0 < |x - c| < \delta$ and $x \in A$, then

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

  which proves that $\lim_{x\to c}[f(x) + g(x)] = L + M$.

**2.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 2x - 1 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ denotes the rational numbers. Determine, with proof, at which points $f$ is continuous and at which points $f$ is discontinuous.

**Solution.**

- The function $f$ is continuous at 1 and discontinuous at every other point. Note that $x^2 = 2x - 1$ if and only if $(x-1)^2 = 0$ or $x = 1$.

- To prove that $f$ is discontinuous at $c \neq 1$, choose sequences $(x_n)$, $(y_n)$ such that $x_n \in \mathbb{Q}$, $y_n \notin \mathbb{Q}$ and $x_n, y_n \to c$ as $n \to \infty$ (possible because both the rational and irrational numbers are dense in $\mathbb{R}$). Then, using the sequential continuity of the polynomial functions $x^2$ and $2x - 1$, we have

  $$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n^2 = c^2, \\ \lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} (2y_n - 1) = 2c - 1. \end{aligned}$$

  Since these limits are different for $c \neq 1$, the sequential definition of continuity implies that $f$ is discontinuous at $c$.

- To prove that $f$ is continuous at 1, where $f(1) = 1$, let $\epsilon > 0$ be given. Choose

  $$\delta = \min\left(1, \frac{\epsilon}{2}\right).$$

  If $|x - 1| < \delta$, then

  $$\begin{aligned} &|x^2 - 1| = |x + 1|\,|x - 1| < 2 \cdot \frac{\epsilon}{2} = \epsilon, \\ &|(2x - 1) - 1| = 2|x - 1| < \epsilon. \end{aligned}$$

  Thus, in either of the cases $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$, we have $|f(x) - f(1)| < \epsilon$, which proves that $f$ is continuous at 1.

**3.** A function $f : A \to \mathbb{R}$ is *locally bounded* on $A \subset \mathbb{R}$ if for every $c \in A$ there exists $\delta > 0$ such that $f$ is bounded on $(c - \delta, c + \delta) \cap A$.

(a) If $f : [0, 1] \to \mathbb{R}$ is locally bounded on the compact interval $[0, 1]$, prove that $f$ is bounded on $[0, 1]$.

(b) Give an example of a function $f : (0, 1) \to \mathbb{R}$ that is locally bounded but not bounded on the open interval $(0, 1)$.

**Solution.**

- (a) Suppose for contradiction that $f$ is not bounded on $[0, 1]$. Then for every $n \in \mathbb{N}$ there exists an $x_n \in [0, 1]$ such that $|f(x_n)| \geq n$. Since $[0, 1]$ is compact, there exists a convergent subsequence $(x_{n_k})$ with limit $x \in [0, 1]$. Then $f$ is unbounded in any neighborhood of $x$ since $x_{n_k}$ belongs to the neighborhood for all sufficiently large $k$ and the sequence $(f(x_{n_k}))$ is unbounded. It follows that $f$ is not locally bounded, and this contradiction shows that $f$ must be bounded.

- (b) The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/x$ is locally bounded but not bounded. If $x \in (0, 1)$ and $0 < \delta < x$, then $f$ is bounded on the neighborhood $(\delta, 1)$ of $x$, so $f$ is locally bounded. On the other hand for every $n \in \mathbb{N}$, we have $f(x) > n$ for $0 < x < 1/n$, so $f$ is unbounded on $(0, 1)$.

**Remark.** One can also prove (a) from the Heine-Borel property of compact sets. For each $x \in [0, 1]$, there is an open neighborhood $I_x$ of $x$ such that $f$ is bounded on $I_x$, meaning that there exists $M_x \geq 0$ such that

$$|f(y)| \leq M_x \qquad \text{for all } y \in I_x \cap [0, 1].$$

The collection of neighborhoods $\{I_x : x \in [0, 1]\}$ is an open cover of $[0, 1]$ (since $x \in I_x$) so since $[0, 1]$ is compact (it's closed and bounded) there is a finite subcover

$$\{I_{x_1}, I_{x_2}, \ldots, I_{x_N}\}$$

of $[0, 1]$. Then $|f(x)| \leq M$ for all $x \in [0, 1]$ where

$$M = \max\left(M_{x_1}, M_{x_2}, \ldots, M_{x_N}\right).$$

Note that we need to extract a finite subcover to ensure that $M$ is finite.

**4.** (a) State the intermediate value theorem.

(b) A *fixed point* of a function $f : [0, 1] \to [0, 1]$ is a point $c \in [0, 1]$ such that $f(c) = c$. Prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (HINT: Note carefully the range of $f$.)

(c) Give an example of a discontinuous function $f : [0, 1] \to [0, 1]$ with no fixed point.

**Solution.**

- (a) Intermediate Value Theorem. If $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval and $f(a) < d < f(b)$, if $f(a) < f(b)$, or $f(b) < d < f(a)$, if $f(a) > f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.

- (b) Let
$$g(x) = f(x) - x.$$
Then $c$ is a fixed point of $f$ if and only if $g(c) = 0$. Since $f(x) \geq 0$, we have
$$g(0) = f(0) \geq 0,$$
and since $f(x) \leq 1$, we have
$$g(1) = f(1) - 1 \leq 0.$$
If $g(0) = 0$ or $g(1) = 0$ then 0 or 1 is a fixed point of $f$. Otherwise $g(0) > 0$ and $g(1) < 0$, so the intermediate value theorem implies that $g(c) = 0$ for some $c \in (0, 1)$, which proves the result.

- (c) For example, define $f : [0, 1] \to [0, 1]$ by
$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ 0 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$
Then the only possible fixed points of $f$ are 0 and 1 (the values of $f$) but $f(0) \neq 0$ and $f(1) \neq 1$, so $f$ has no fixed points.