1. Prove that
\[ \frac{1}{2} \leq \int_0^1 \frac{2x}{\sqrt{x^{2013} + 2x + 1}} \, dx \leq 1. \]

Solution.

- For \( 0 \leq x \leq 1 \), we have
  \[ 1 \leq \sqrt{x^{2013} + 2x + 1} \leq 2, \]
  so
  \[ x \leq \frac{2x}{\sqrt{x^{2013} + 2x + 1}} \leq 2x. \]

By the monotonicity of the integral,
\[
\frac{1}{2} = \int_0^1 x \, dx \leq \int_0^1 \frac{2x}{\sqrt{x^{2013} + 2x + 1}} \, dx \leq \int_0^1 2x \, dx = 1.
\]
2. Prove or disprove: if $E$ is a subset of $\mathbb{R}^2$, then the closure of the interior of $E$ is necessarily the same as the closure of $E$.

Solution.

- This statement is false.
- For example, if $E = \{0\}$ consists of a single point, then $E^\circ = \emptyset$ and $\overline{E^\circ} = \emptyset$, but $\overline{E} = \{0\}$.
- Or, for another example, if $E = \mathbb{Q}^2$, then $E^\circ = \emptyset$ and $\overline{E^\circ} = \emptyset$, but $\overline{E} = \mathbb{R}^2$. 

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3. Evaluate the limit
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2}.
\]

Solution.

- Consider the integral
  \[
  \int_{0}^{1} f(x) \, dx, \quad f(x) = \frac{x}{1 + x^2}.
  \]
The function \( f \) is integrable on \([0, 1]\) since it’s continuous (or, alternatively, since it’s monotone).

- The function \( f \) is increasing on \([0, 1]\) since
  \[
  f'(x) = \frac{1-x^2}{(1+x^2)^2} \geq 0.
  \]
Therefore, in an upper Riemann sum \( U(f; P) \) we evaluate \( f \) at the right endpoints and in a lower Riemann sum \( L(f; P) \), we evaluate \( f \) at the left endpoints.

- Let \( P \) be the partition of \([0, 1]\) into \( n \) intervals of equal length \( 1/n \) with endpoints \( x_k = k/n \), where \( k = 0, 1, \ldots, n \).

- For this partition,
  \[
  U(f; P) = \frac{1}{n} \sum_{k=1}^{n} \frac{k/n}{1 + (k/n)^2} = \sum_{k=1}^{n} \frac{k}{n^2 + k^2},
  \]
  \[
  L(f; P) = \frac{1}{n} \sum_{k=1}^{n} \frac{(k-1)/n}{1 + ((k-1)/n)^2} = \sum_{k=1}^{n} \frac{k}{n^2 + k^2} - \frac{1}{2n}.
  \]

- Since \( L(f; P) \leq \int_{0}^{1} f \leq U(f; P) \), it follows that
  \[
  \int_{0}^{1} \frac{x}{1+x^2} \, dx \leq \sum_{k=1}^{n} \frac{k}{n^2 + k^2} \leq \int_{0}^{1} \frac{x}{1+x^2} \, dx + \frac{1}{2n},
  \]
and the “squeeze” theorem implies that
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2} = \int_{0}^{1} \frac{x}{1+x^2} \, dx = \left[ \frac{1}{2} \ln (1+x^2) \right]_{0}^{1} = \frac{1}{2} \ln 2.
\]
4. Suppose that $f : [0, \pi] \to \mathbb{R}$ is a continuously differentiable function. Prove that
\[
\lim_{n \to \infty} \int_{0}^{\pi} f(x) \sin(nx) \, dx = 0.
\]
HINT. Integrate by parts.

Solution.
- Since both $f$ and $\sin nx$ are continuously differentiable on $[0, \pi]$, the integration by parts formula applies, and
\[
\int_{0}^{\pi} f(x) \sin(nx) \, dx = \left[ -\frac{\cos(nx)}{n} \cdot f(x) \right]_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} f'(x) \cos(nx) \, dx
\]
\[
= \frac{f(0)}{n} + (-1)^{n+1} \frac{f(\pi)}{n} + \frac{1}{n} \int_{0}^{\pi} f'(x) \cos(nx) \, dx.
\]
- The limit as $n \to \infty$ of the constant term proportional to $1/n$ is zero.
- For the integral term, either observe that
\[
\frac{1}{n} f'(x) \cos(nx) \to 0 \quad \text{uniformly on } [0, 1]
\]
since $f'$ is continuous and therefore bounded, and use the fact that we can exchange the order of uniform limits and integration, or estimate the integral directly:
\[
\left| \frac{1}{n} \int_{0}^{\pi} f'(x) \cos(nx) \, dx \right| \leq \frac{1}{n} \int_{0}^{1} |f'(x)| \, dx
\]
\[
\leq \frac{1}{n} \sup_{[0,1]} |f'| \to 0 \quad \text{as } n \to \infty.
\]

Remark. This result says that the Fourier sine coefficients
\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx
\]
of a continuously differentiable function $f$ approach zero as $n \to \infty$. It’s a special case of the Riemann-Lebesgue lemma. The general result for a Lebesgue integrable function $f$ such that
\[
\int_{0}^{\pi} |f(x)| \, dx < \infty
\]
follows by approximating $f$ with smooth functions and using the proof above.
5. Let \( p > 0 \). Define the following improper Riemann integral as a limit of Riemann integrals:

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^p} \, dx.
\]

For what values of \( p \) does this integral converge? HINT. Use the substitution \( u = \ln x \).

Solution.

- The improper integral is defined by

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^p} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x(\ln x)^p} \, dx.
\]

The Riemann integral on the right-hand side exists since the integrand is continuous on \([2, b]\).

- The substitution formula with \( u = \ln x \) and \( du = dx/x \) gives

\[
\int_{2}^{b} \frac{1}{x(\ln x)^p} \, dx = \int_{2}^{ln b} \frac{1}{u^p} \, du,
\]

\[
= \left[ \frac{u^{1-p}}{1-p} \right]_{ln 2}^{ln b}
\]

\[
= \frac{(ln b)^{1-p} - (ln 2)^{1-p}}{1-p}
\]

where we assume that \( p \neq 1 \).

- Since \( \ln b \to \infty \) as \( b \to \infty \), this integral diverges if \( 0 < p < 1 \), and converges if \( p > 1 \) to

\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^p} \, dx = \frac{1}{(p-1)(\ln 2)^{p-1}}.
\]

- The integral also diverges (very, very slowly) if \( p = 1 \) since

\[
\int_{2}^{b} \frac{1}{x \ln x} \, dx = \int_{\ln 2}^{\ln b} \frac{1}{u} \, du = \ln \ln b - \ln \ln 2 \to \infty.
\]
6. Suppose that $f : [a, b] \to \mathbb{R}$ is a nonzero, Riemann integrable function such that $1/f : [a, b] \to \mathbb{R}$ is bounded. Prove that $1/f$ is Riemann integrable.

Solution.

- Suppose that 
  \[ \frac{1}{|f(x)|} \leq M \quad \text{for } a \leq x \leq b. \]

- Then for all $x, y \in [a, b]$, we have 
  \[ \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \frac{|f(x) - f(y)|}{|f(x)f(y)|} \leq M^2 |f(x) - f(y)|. \]

  This inequality implies that (see Proposition 2.19 in the class notes on sups and infs) for every subset $I \subset [a, b]$ we have 
  \[ \sup_I \frac{1}{f} - \inf_I \frac{1}{f} \leq M^2 \left( \sup_I f - \inf_I f \right). \]

- For every partition $P = \{ I_1, I_2, \ldots, I_n \}$ of $[a, b]$, we have 
  \[ U \left( \frac{1}{f}; P \right) - L \left( \frac{1}{f}; P \right) = \sum_{k=1}^{n} \left( \sup_{I_k} \frac{1}{f} - \inf_{I_k} \frac{1}{f} \right) |I_k| \]
  \[ \leq M^2 \sum_{k=1}^{n} \left( \sup_{I_k} f - \inf_{I_k} f \right) |I_k|. \]

  Therefore, $1/f$ satisfies the Cauchy criterion for integrability if $f$ does, and it follows that $1/f$ is integrable if $f$ is nonzero and integrable and $1/f$ is bounded.
7. Define \( f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
f_1(x_1, x_2) = e^{x_1} \cos x_2, \quad f_2(x_1, x_2) = e^{x_1} \sin x_2.
\]
(a) Why is \( f \) differentiable on \( \mathbb{R}^2 \)? Compute the differential matrix of \( f \).
(b) Evaluate the directional derivative \( D_{(3/5, 4/5)} f(0, \pi/2) \) of \( f \) at \( P = (0, \pi/2) \) in the direction \( e = (3/5, 4/5) \). Which component \( f_1, f_2 \) is increasing at \( P \) in the direction \( e \)?
(c) What does the implicit function theorem say about the existence of local inverses of \( f \)? Does \( f \) has a global inverse \( f^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \)?

Solution.

- (a) The functions \( f_1, f_2 \) have continuous partial derivatives on \( \mathbb{R}^2 \), so \( f \) is differentiable on \( \mathbb{R}^2 \). The differential matrix is
\[
[df] = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
= \begin{pmatrix}
e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\
e^{x_1} \sin x_2 & e^{x_1} \cos x_2
\end{pmatrix}.
\]

- (b) The differential matrix of \( f \) at \((0, \pi/2)\) is
\[
[df(0, \pi/2)] = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
so the directional derivative is
\[
D_{(3/5, 4/5)} f(0, \pi/2) = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
3/5 \\
4/5
\end{pmatrix}
= \begin{pmatrix}
-4/5 \\
3/5
\end{pmatrix}.
\]
The component \( f_2 \) is increasing in the direction \( e \) since its directional derivative is positive.

- (c) The derivative of \( f \) has determinant
\[
\det df = e^{x_1} \cos^2 x_2 + e^{x_1} \sin^2 x_2 = e^{x_1} > 0
\]
so \( df(x_1, x_2) \) is invertible at every \((x_1, x_2) \in \mathbb{R}^2\), and \( f \) is \( C^1 \). The inverse function theorem implies that there are open neighborhoods \( U \) of \((x_1, x_2)\) and \( V \) of \((f_1(x_1, x_2), f_2(x_1, x_2))\) such that \( f : U \to V \) is one-to-one and onto with \( C^1 \)-inverse \( f^{-1} : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2 \).
• Although $f$ is locally invertible at every point is is not globally invertible since it is not one-to-one:

$$f(x_1, x_2 + 2n\pi) = f(x_1, x_2)$$

for every $n \in \mathbb{Z}$.

**Remark.** This function correspond to the complex exponential function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$. The local inverse is a branch of the complex logarithm $f^{-1}(z) = \log z$, but the logarithm can’t be extended to a single-valued, differentiable function on $\mathbb{C}$. 

8. Suppose that 0 < a < b and 0 < δ < π/2. Let A be the region

\[ A = \{(x, y) \in \mathbb{R}^2 : a^2 \leq x^2 + y^2 \leq b^2 \text{ and } 0 \leq \tan^{-1}(y/x) \leq 2\pi - \delta\}, \]

and consider the integral

\[ I = \int_A e^{-(x^2+y^2)} \, dx \, dy. \]

Make the change of coordinates

\[ x = r \cos \theta, \quad y = r \sin \theta \]

in this integral and evaluate it. Justify your steps.

Solution.

• The change of coordinates is a $C^1$-diffeomorphism in an open neighborhood $U$ of $A$ that doesn’t contain the origin e.g.,

\[ U = \{(x, y) \in \mathbb{R}^2 : a^2 - \epsilon < x^2 + y^2 < b^2 + \epsilon, \]

\[ \text{and } -\epsilon < \tan^{-1}(y/x) < 2\pi - \delta + \epsilon \} \]

for sufficiently small $\epsilon > 0$, so we can apply the change of variables theorem.

• The Jacobian determinant is

\[ \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \]

so formally $dx \, dy = r \, dr \, d\theta$ and

\[ \int_A e^{-(x^2+y^2)} \, dx \, dy = \int_R e^{-r^2} \, r \, dr \, d\theta \]

where $A$ is the image of the rectangle

\[ R = \{(r, \theta) : a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi - \delta\}. \]

• Since it’s continuous, the function $re^{-r^2}$ is integrable on the rectangle, as are its iterated integrals. Fubini’s theorem implies that

\[ \int_R e^{-r^2} \, r \, dr \, d\theta = \int_0^{2\pi-\delta} \left( \int_a^b re^{-r^2} \, dr \right) \, d\theta. \]
Making the substitution $u = r^2$, $du = 2r dr$, we get
\[ \int_a^b re^{-r^2} dr = \frac{1}{2} \int_{a^2}^{b^2} e^{-u} du = \frac{1}{2} \left[-e^{-u}\right]_{a^2}^{b^2} = \frac{1}{2} \left(e^{-a^2} - e^{-b^2}\right). \]

Thus,
\[ \int_A e^{-(x^2+y^2)} dxdy = \frac{1}{2} \int_0^{2\pi-\delta} \left(e^{-a^2} - e^{-b^2}\right) d\theta \]
\[ = \frac{1}{2} (2\pi - \delta) \left(e^{-a^2} - e^{-b^2}\right). \]

**Remark.** By considering the improper integral with $a \to 0^+$, $b \to \infty$, and $\delta \to 0^+$, we get that
\[ \int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_0^{2\pi} \int_0^{\infty} re^{-r^2} dr = \pi. \]

This is the classic trick (apparently due to Laplace) for evaluating the definite Gaussian integral
\[ \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \]